Lecture 12: Transformations of Functions

In this section, we see how transformations change the shape of the graph of a function. We will also see how we can often use this information to derive the graph of a function by using successive transformations of one of the graphs in the catalogue given at the end of the previous lecture.

Vertical Shifts Let c be a constant with $c > 0$. We can obtain the graph of the equation, $y = f(x) + c$ by shifting the graph of $y = f(x)$ vertically upwards by c units. Likewise, we can obtain the graph of the the function $y = f(x) - c$ by shifting the graph of $y = f(x)$ downwards by c units.

The reason for the above conclusion is because for every point (x_1, y_1) on the graph of $y = f(x)$, the point $(x_1, y_1 + c)$ is on the graph of $y = f(x) + c$ and the point $(x_1, y_1 - c)$ is on the graph of $y = f(x) - c$. The example below demonstrates what is going on.

Example Fill in the tables and sketch the graphs of $y = x^2$, $y = x^2 + 3$ and $y = x^2 - 3$ below.

x	$y = x^2 - 3$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = x^2$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = x^2 + 3$
-3	
-2	
-1	
0	
1	
2	
3	

We can now use this observation to expand the range of graphs we can sketch using the graphs from the previous sections.

Example Sketch the graphs of the following functions:

$$f(x) = |x| + 5, \qquad g(x) = x^3 - 10, \qquad h(x) = \sqrt{x} + 2.$$

Horizontal Shifts Let c be a constant with $c > 0$. We obtain the graph of the equation $y = f(x+c)$ by shifting the graph of $y = f(x)$ to the left by c units. Likewise, we obtain the graph of $y = f(x-c)$ by shifting the graph of $y = f(x)$ to the right by c units.

We can see this by noting that if the point (x_1, y_1) is on the graph of $y = f(x)$, then the point $(x_1 - c, y_1)$ must be on the graph of $y = f(x+c)$ and the point $(x_1 + c, y_1)$ must be on the graph of $y = f(x-c)$. Also consider the example below which demonstrates the horizontal shifts in the case of $f(x) = x^2$ and $c = 3$.

Example Fill in the tables and sketch the graphs of $y = x^2$, $y = (x-3)^2$ and $y = (x+3)^2$ below.

x	$y = (x-3)^2$	*Point*
-2		
-1		
0		
1		
2		
3		
4		
5		
6		

x	$y = x^2$	*Point*
-5		
-4		
-3		
-2		
-1		
0		
1		
2		
3		
4		
5		

x	$y = (x+3)^2$	*Point*
-6		
-5		
-4		
-3		
-2		
-1		
0		
1		
2		

Example Sketch the graphs of the following functions:

$$f(x) = |x+5|, \qquad g(x) = (x-10)^3, \qquad h(x) = \sqrt{x+2} + 3.$$

Combining Vertical and Horizontal Shifts

Example Complete the square and sketch the graphs of the quadratic functions given below:

$$f(x) = x^2 - 6x + 14, \qquad g(x) = x^2 + 10x + 30.$$

Vertical Stretching and Shrinking of Graphs Let c be a constant with $c > 0$.

- If $c > 1$, the graph of $y = cf(x)$ can be derived from the graph of $y = f(x)$ by stretching the graph of $y = f(x)$ vertically by a factor of c.
- If $0 < c < 1$, the graph of $y = cf(x)$ can be derived from the graph of $y = f(x)$ by shrinking the graph of $y = f(x)$ vertically by a factor of c.

We see that the above statements are true by noting that if (x_1, y_1) is on the graph of $y = f(x)$, then (x_1, cy_1) is on the graph of $y = cf(x)$. You can also get a feel for why this is true by experimenting with the graph of $y = x^3$ below.

Example Fill in the tables and sketch the graphs of $y = x^3$, $y = 2(x^3)$ and $y = \frac{1}{2}(x^3)$ below.

x	$y = x^3$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = 2(x^3)$
-3	
-2	
-1	
0	
1	
2	
3	

x	$y = \frac{1}{2}(x^3)$
-3	
-2	
-1	
0	
1	
2	
3	

Example Sketch the graphs of the following functions:

$$f(x) = 10\sqrt{x} \qquad g(x) = \frac{1}{2}x^2 + 5 \qquad h(x) = \frac{1}{2}(x-1)^2 + 5.$$

Horizontal Shrinking and Stretching of Graphs.

Let c be a constant with $c > 0$.

- If $c > 1$, the graph of $y = f(cx)$ can be derived from the graph of $y = f(x)$ by shrinking the graph of $y = f(x)$ horizontally by a factor of $1/c$.
- If $0 < c < 1$, the graph of $y = f(cx)$ can be derived from the graph of $y = f(x)$ by stretching the graph of $y = f(x)$ horizontally by a factor of $1/c$.

To see why, note that if the point (x_1, y_1) is on the graph of $y = f(x)$, then the point $(x_1/c, y_1)$ is on the graph of $y = f(cx)$. The example below will also give you some insight into how this transformation changes the graph.

Example Fill in the tables and sketch the graphs of $y = |x/2|$, $y = |x|$ and $y = |2x|$ below.

x	$y = \|x/2\|$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

x	$y = \|x\|$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

x	$y = \|2x\|$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

Example Sketch the graph of the following function:

$$f(x) = (x^2/9) + 5.$$

Reflecting Graphs

- The graph of $y = f(-x)$ can be obtained by reflecting the graph of $y = f(x)$ in the $y-$ axis.
- The graph of $y = -f(x)$ can be obtained by reflecting the graph of $y = f(x)$ in the $x-$ axis.

To see this, note that if (x_1, y_1) is on the graph of $y = f(x)$, then $(-x_1, y_1)$ is on the graph of $y = f(-x)$ and $(x_1, -y_1)$ is on the graph of $y = -f(x)$.

Example Example Fill in the tables and sketch the graphs of $y = -\sqrt{x}$, $y = \sqrt{x}$ and $y = \sqrt{-x}$ below.

x	$y = -\sqrt{x}$
0	
1	
2	
3	
4	
5	

x	$y = \sqrt{x}$
0	
1	
2	
3	
4	
5	

x	$y = \sqrt{-x}$
-5	
-4	
-3	
-2	
-1	
0	

Example Sketch the graphs of the following functions:

$$f(x) = -(x+2)^2 + 4, \qquad g(x) = \frac{\sqrt{-(x-2)}}{10} + 20.$$

Even and Odd Functions; Symmetry In the section on graphing equations, we saw that if we replace x by $-x$ in an equation relating y and x and the equation remains the same, then the graph of the equation is symmetric with respect to the y-axis. If we replace x by $-x$ and y by $-y$ and the equation remains the same, then the graph is symmetric with respect to the origin. For a function $f(x)$, this amounts to saying:

- If $f(x) = f(-x)$ for all x in the domain of f, then the graph of $y = f(x)$ is symmetric with respect to the y-axis and the function is called an **even** function.
- If $f(x) = -f(-x)$ for all x in the domain of f, then the graph of $y = f(x)$ is symmetric with respect to the origin and the function is called an **odd** function.

Just as symmetry made sketching the grape of equations easier because one only had to draw part of the graph, symmetry also makes it easier to sketch the graphs of functions.

Example Say whether the following functions are even or odd and sketch their graphs:

$$f(x) = x^3 + x, \qquad\qquad g(x) = |x| + x^2.$$