Lecture 13: Composing Functions

In this section, we look at the algebra of functions. Given two functions, we can define or construct new functions in a number of ways. We can add, subtract, multiply and divide functions. Also we can compose functions by applying them consecutively.

Let $f(x)$ and $g(x)$ be functions with domains, Dom. f and Dom g respectively. We define the functions $f+g$, $f-g$, fg and $\dfrac{f}{g}$ as follows:

Function	**Domain**
$(f+g)(x) = f(x) + g(x)$	Dom $f \cap$ Dom g
$(f-g)(x) = f(x) - g(x)$	Dom $f \cap$ Dom g
$(fg)(x) = f(x)g(x)$	Dom $f \cap$ Dom g
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	$\{x \in \text{Dom } f \cap \text{Dom } g \mid g(x) \neq 0\}$

It is not difficult to see that $f+g$ is a function, since given any x for which $f(x)$ and $g(x)$ are defined, $(f+g)(x) = f(x) + g(x)$ gives us a rule by which we can calculate a unique value $f(x) + g(x)$ in the range of $f+g$. Also, given any value of x for which we can calculate $f(x) + g(x)$, we must have that x in both the domain of f and the domain of g. Therefore the domain of the new function $f+g$ is precisely Dom $f \cap$ Dom g. Identical arguments apply to $f-g$ and fg. In the case of $\frac{f}{g}$ we have the added condition on the domain that $g(x) \neq 0$ since we are dealing with a quotient.

Example Let $f(x) = \frac{1}{x-3}$ and $g(x) = \sqrt{x}$.

(a) Find $f+g$, $f-g$, fg and $\frac{f}{g}$ and their domains.

(b) Find $(f+g)(2)$, $(f-g)(2)$, $fg(2)$ and $\frac{f}{g}(2)$.

Composition of Functions We use the term `composition of functions` to indicate the situation where we apply the functions consecutively.

Definition Given two functions f and g, the composite function $f \circ g$ (also referred to as the composition of the functions) is given by

$$(f \circ g)(x) = f(g(x)).$$

Example Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Then $(f \circ g)(x) = f(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1$.

The domain of $f \circ g$ is the set of all numbers x to which we can first apply g and then apply f to the result. In set theoretic language;

$$\text{Dom } f \circ g = \{x \in \text{Dom } g \mid g(x) \in \text{Dom } f\}.$$

Example Find the domain of $f \circ g$ where $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$.

Note that the function $f \circ g$ is not necessarily the same as the function $g \circ f$:

Example Find the function $g \circ f$ and its domain, where f and g are as in the above example.

Example Let $f(x) = \sqrt{\dfrac{1}{x-3}}$ and let $g(x) = \sqrt{x}$.

(a) Find $(f \circ g)(x)$ and its domain.

(b) Find $(f \circ g)(16)$.

(a) Find $(g \circ f)(x)$ and its domain.

(b) Find $(g \circ f)(16)$.

Recognizing a composition of functions:
Example Let $F(x) = (x^3 + 2)^7$. Find two functions f and g such that $F = f \circ g$.

Example Let $F(x) = \sqrt{x+4}$. Find two functions f and g such that $F = f \circ g$.

We can also compose more than two functions:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

and

$$(f_n \circ f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1)(x) = f_n(f_{n-1}(f_{n-2}(\ldots(f_1(x))\ldots))).$$

The **domain** of $(f_n \circ f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1) =$
$\{x \in \text{Dom } f_1 | f_1(x) \in \text{Dom } f_2 \text{ and } f_2(f_1(x)) \in \text{Dom } f_3 \text{ and } \ldots \text{ and } f_{n-1}(f_{n-2}(\ldots(f_1(x))\ldots)) \in \text{Dom } f_n\}$.
These are the x values for which $f_1(x)$ can be evaluated and furthermore each successive evaluation gives us something in the domain of the next function.

Example Let $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$ and $h(x) = \dfrac{1}{x-3}$.

(a) Find $(f \circ g \circ h)(x)$ and find its domain.

(b) Find $(f \circ g \circ h)(12)$.

Example Let $F(x) = 44 + (x^2 + 4)^{100}$ Find functions f, g and h so that $F = f \circ g \circ h$.

Example #59 Section 2.7, Stewart, Redlin, Watson. A spherical weather balloon is being inflated. The radius of the balloon is increasing at the rate of 2 cm/s.

(a) Express the radius of the balloon as a function of time, t (in seconds).

(b) Express the surface area of the balloon as a function of the radius r.

(c) Using a composition of functions, express the surface area of the balloon as a function of time, t.

62. Airplane Trajectory An airplane is flying at a speed of 350 mi/h at an altitude of one mile. The plane passes directly above a radar station at time $t = 0$.

(a) Express the distance s (in miles) between the plane and the radar station as a function of the horizontal distance d (in miles) that the plane has flown.

(b) Express d as a function of the time t (in hours) that the plane has flown.

(c) Use composition to express s as a function of t.