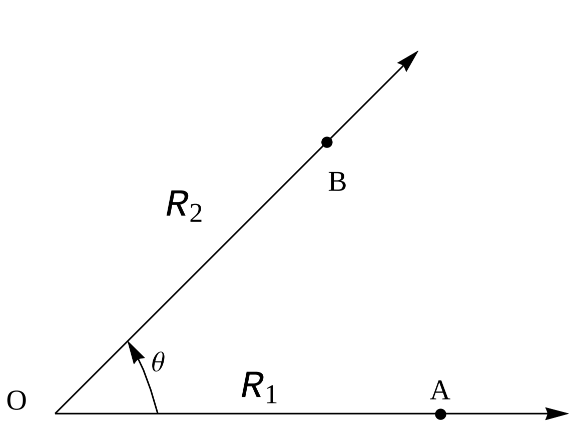
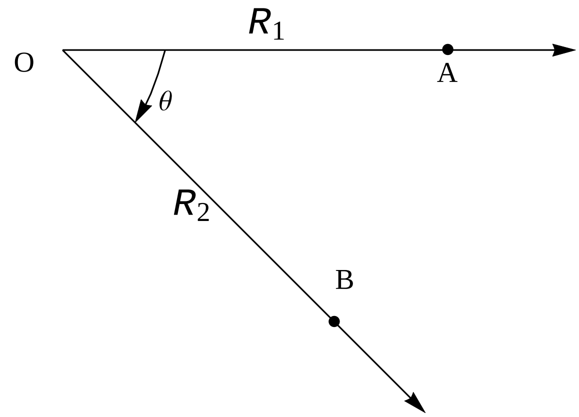
Lecture 14: Trigonometric Functions

Angle measure

An angle AOB consists of two rays R_1 and R_2 with a common vertex O. We can think of the angle as a rotation of the side R_2 about the point O with R_1 remaining fixed. R_1 is called the initial side of the angle and R_2 is called the terminal side of the angle. If the rotation is counterclockwise, the angle is positive as shown on the left below. If the rotation is clockwise, the angle is negative as shown on the right below.

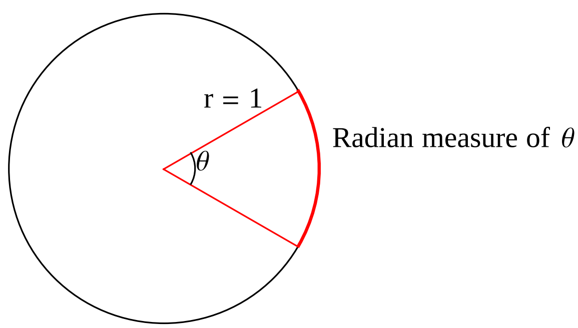




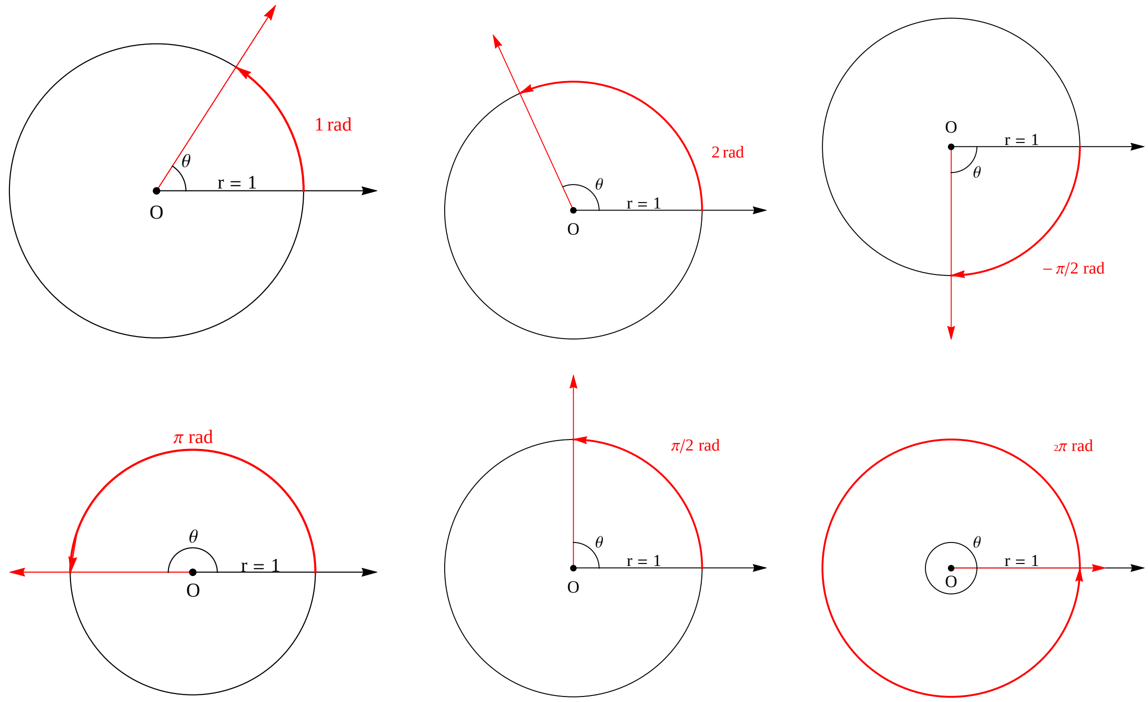
Angle Measure

There are two commonly used measures for angles, degrees and radians. In calculus, we will almost exclusively use radian measure since the trigonometric functions are defined in terms of this measure.

Radian Measure If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this (positive) angle in **radians** is the length of the arc that subtends the the angle (see the picture below).



The circumference of a circle of radius 1 is 2π, so a complete revolution has measure 2π. A straight angle has measure π and a right angle has measure $\pi/2$.



Degrees The degree measure of a full circle is 360^o. Therefore we have that $360^o = 2\pi$ radians. This gives us our conversion formulas.

$$1^o = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians,}$$

Therefore to convert from degrees to radians, we multiply by $\frac{\pi}{180}$.
We also have

$$1 \text{ radian} = \frac{360}{2\pi} = \frac{180}{\pi} \text{ degrees.}$$

Thus to convert from radians to degrees we must multiply by $\frac{180}{\pi}$.

Example Convert the following angles given in degrees to their radian measure:

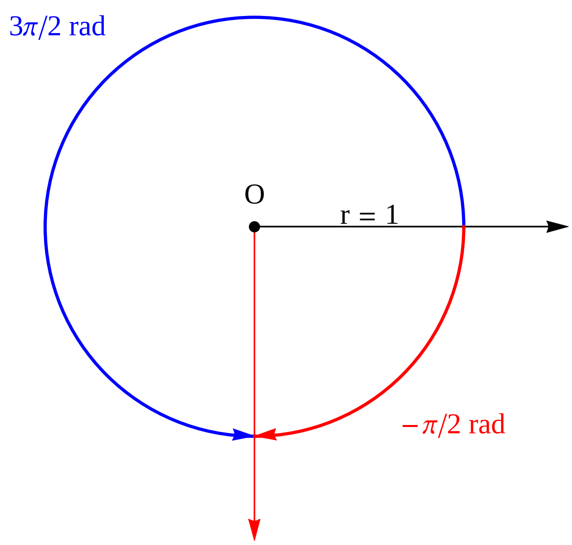
$$45^o, \qquad 60^o, \qquad 30^o, \qquad 180^o.$$

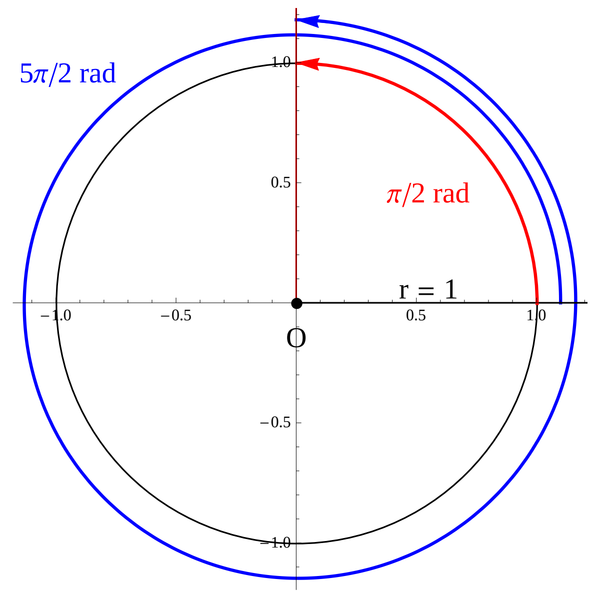
Example Convert the following angles given in radians to their measure in degrees:

$$\frac{\pi}{10} \qquad \frac{3\pi}{4} \qquad \frac{-\pi}{6} \qquad \frac{5\pi}{6}.$$

Angles in Standard Position and Co-terminal angles An angle is said to be in standard position if it is drawn so that its initial side is the positive x-axis and its vertex is the origin. Two different angles may have terminal sides which coincide, in which case we call the angles **coterminal**. This may happen if one of the angles makes more than one revolution about the vertex or if the angles involve revolutions in opposite directions. The angles 1 rad, 2 rad, π rad, $\pi/2$ rad, $-\pi/2$ rad and 2π rad shown above are all in standard position.

Coterminal Angles We see below that the angles $\dfrac{3\pi}{2}$ and $-\dfrac{\pi}{2}$ are co-terminal. Also the angles $\dfrac{5\pi}{2}$ and $\dfrac{\pi}{2}$ are co-terminal.

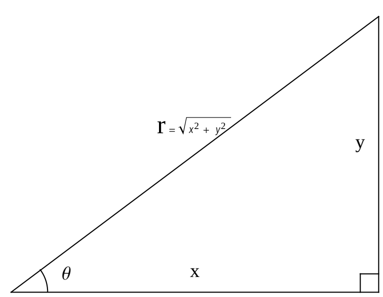




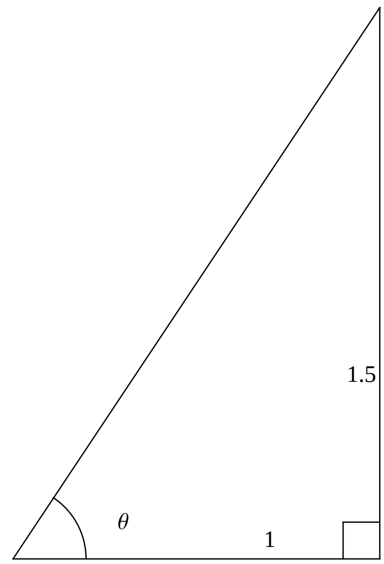
Trigonometry of Right Triangles Recall the definition of $\sin\theta, \cos\theta$ and $\tan\theta$ from trigonometry of right angles. If θ is the angle shown in the right triangle below, we have

$$\sin\theta = \frac{y}{r} = \frac{\text{length of side opposite } \theta}{\text{hypotenuse}}, \quad \cos\theta = \frac{x}{r} = \frac{\text{length of side adjacent to } \theta}{\text{hypotenuse}},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}.$$



Example Find $\cos\theta$, $\sin\theta$ and $\tan\theta$ where θ is shown in the diagram below:

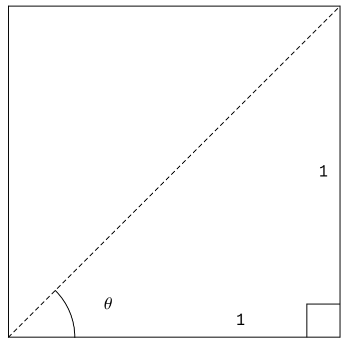


We also have three further trigonometric functions which are often used and referred to in calculus.

$$\sec x = \frac{1}{\cos x}, \quad cscx = \frac{1}{\sin x} \quad \cot x = \frac{1}{\tan x}.$$

Example Find $\csc\theta$, $\sec\theta$ and $\cot\theta$ for the angle θ shown in the diagram in the previous example.

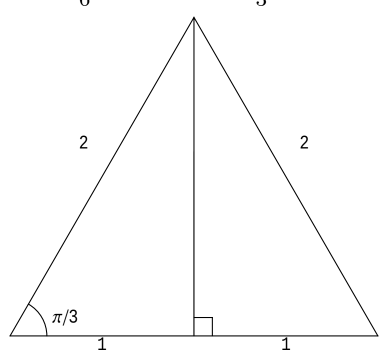
Special Triangles We can derive the cosine, sine and tangent of some basic angles by considering some special right triangles. By drawing a square with sides of length 1, and its diagonal as shown, we can use the Pythagorean theorem to find the length of the diagonal.



Example Use the triangle above to determine:

$$\sin(45^o) = \sin(\pi/4), \qquad \cos(\pi/4), \qquad \tan(\pi/4), \qquad \sec(\pi/4), \qquad \csc(\pi/4), \qquad \cot(\pi/4).$$

By drawing and equilateral triangle with sides of length 2, we know that all angles are equal and add to 180^o or Π radians. Thus each angle is a 60^o angle or has a measure of $\pi/3$ radians. From our Euclidean geometry, we know that the line which bisects the upper angle in this (isosceles) triangle, also bisects the base. Using the Pythagorean theorem we can find the length of the bisector and thus the sine, cosine and tangent of angles of size $30^o = \frac{\pi}{6}$ and $60^o = \frac{\pi}{3}$.



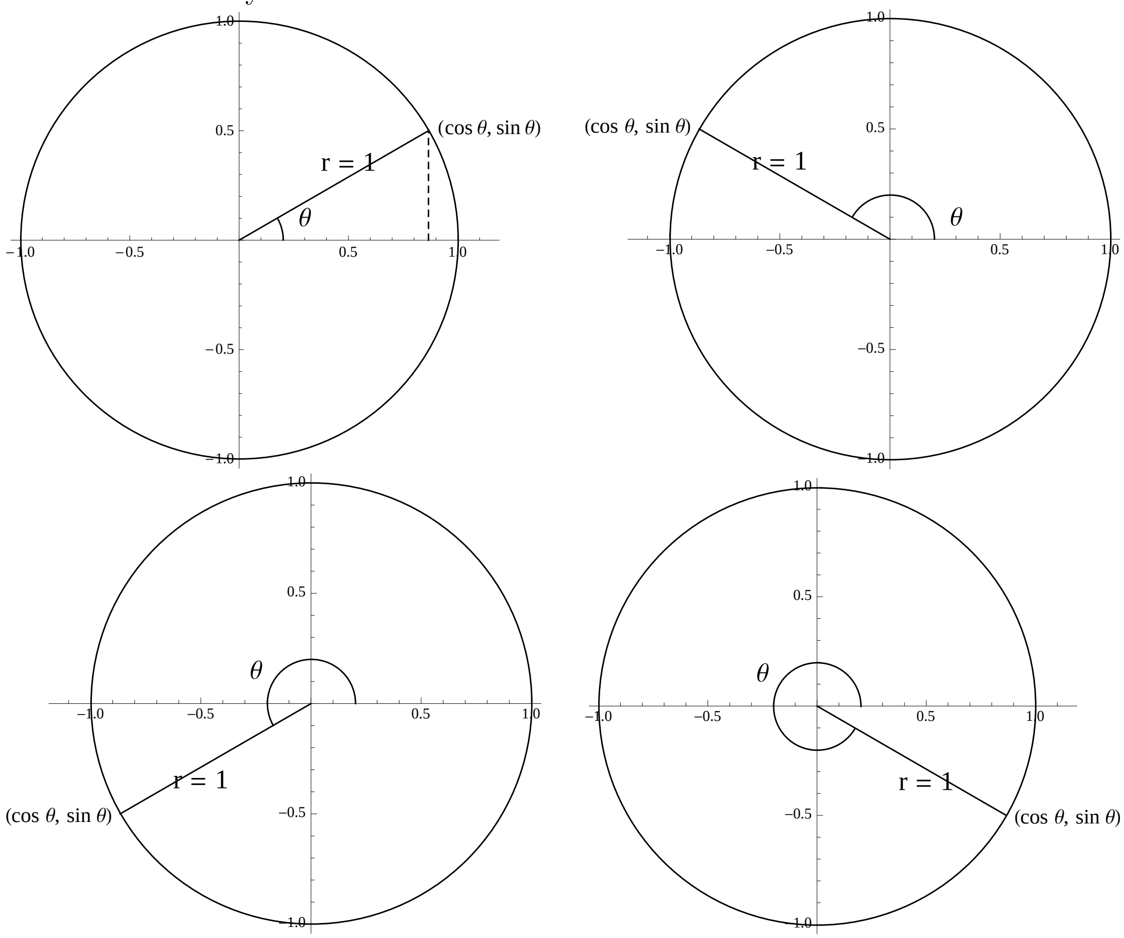
Example Use the triangle above to determine:

$$\sin(\pi/6), \qquad \cos(\pi/6), \qquad \tan(\pi/6), \qquad \sec(\pi/6), \qquad \csc(\pi/6), \qquad \cot(\pi/6).$$

and

$$\sin(\pi/3), \qquad \cos(\pi/3), \qquad \tan(\pi/3), \qquad \sec(\pi/3), \qquad \csc(\pi/3), \qquad \cot(\pi/3).$$

General Trigonometric Functions Note that if we construct a ray joining the origin $(0, 0)$ to a point on the unit circle (radius 1) in the first quadrant, the co-ordinates of the end point of the ray are $(\cos\theta, \sin\theta)$. For angles greater than $\frac{\pi}{2}$, we can extend the definition of $\cos\theta$ and $\sin\theta$ using the unit circle. We define $\cos\theta$ as the x-co-ordinate of the point where the terminal ray of the angle when placed in standard position cuts the unit circle. Similarly, we define $\sin\theta$ as the y-co-ordinate of the point where the terminal ray cuts the unit circle.



We define $\tan\theta$ as:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Angles with terminal rays in all four quadrants are shown above. The diagram below tells us the sign of the trigonometric function for angles in standard position with terminal rays in each of the four quadrants.

cos < 0 sin > 0 tan < 0	cos > 0 sin > 0 tan > 0
cos < 0 sin < 0 tan > 0	cos > 0 sin < 0 tan < 0

Example Find cos, sin and tan of the following angles:

$$0, \qquad \frac{\pi}{2}, \qquad \pi, \qquad \frac{3\pi}{2}, \qquad 2\pi, \qquad -\frac{\pi}{2}, \qquad \frac{5\pi}{2}.$$

Using Symmetry of Unit Circle We can use the symmetry of the unit circle to extend our knowledge of the trigonometric functions for the basic angles given above to a wider range of angles. The following table gives the sin and cos of the basic angles derived above, one can use symmetry of the unit circle to find the sin and cos of other angles greater than $\frac{\pi}{2}$.

Angle θ in rad.	$\cos\theta$	$\sin\theta$	$\tan\theta$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	0	1	Does Not Exist

Example Use the symmetry of the unit circle to find cos, sin and tan of the following angles:

$$\frac{5\pi}{6}, \qquad \frac{4\pi}{3}, \qquad \frac{-\pi}{3}, \qquad \frac{11\pi}{6}, \qquad \frac{5\pi}{4}.$$

Note that since the *cos* and *sin* of an angle depend only on the position of its terminal ray (assuming that it is in standard position), we have $cos(2\pi + \theta) = cos(\theta)$ for any angle theta. Similarly $sin(2\pi + \theta) = sin(\theta)$. This means that we only have to get a picture of the graph of $\sin\theta$ and $\cos\theta$ on the interval $\theta \in [0, 2\pi]$ in order to draw the entire graph, because the graph repeats itself on every subsequent interval of length 2π.

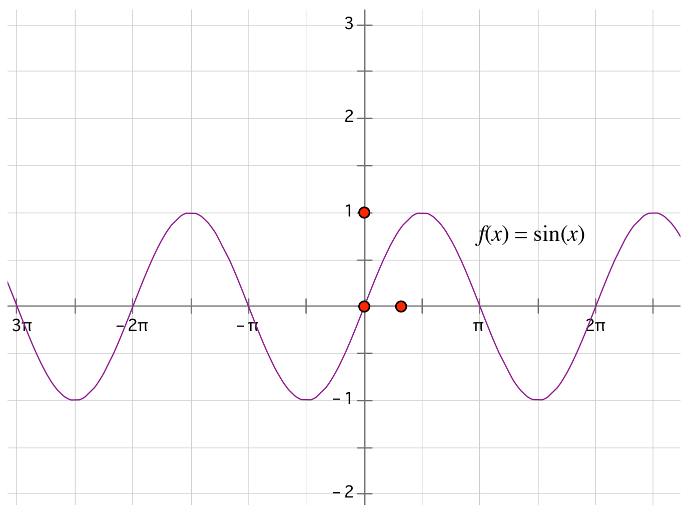
We will look at the following mathematica demonstrations to see what the graphs of the trigonometric functions look like for the angles $0 \leq \theta \leq 2\pi$. You might also wish to use the join-the dots method to sketch the graph on these intervals.

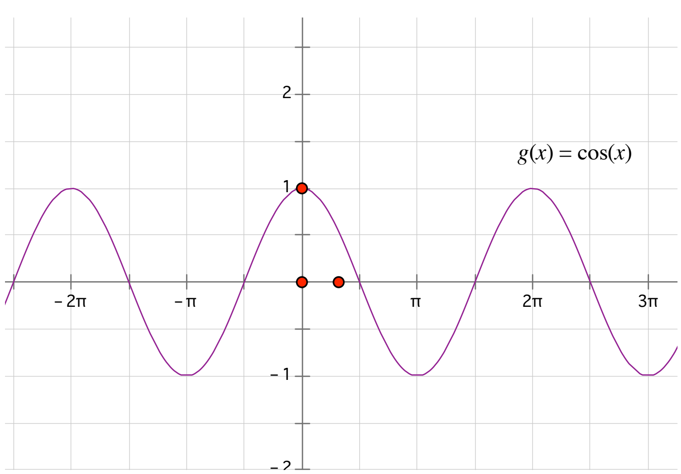
Graph of $\sin(x)$, (note x here is an angle measured in radians).
http://demonstrations.wolfram.com/IllustratingSineWithTheUnitCircle/

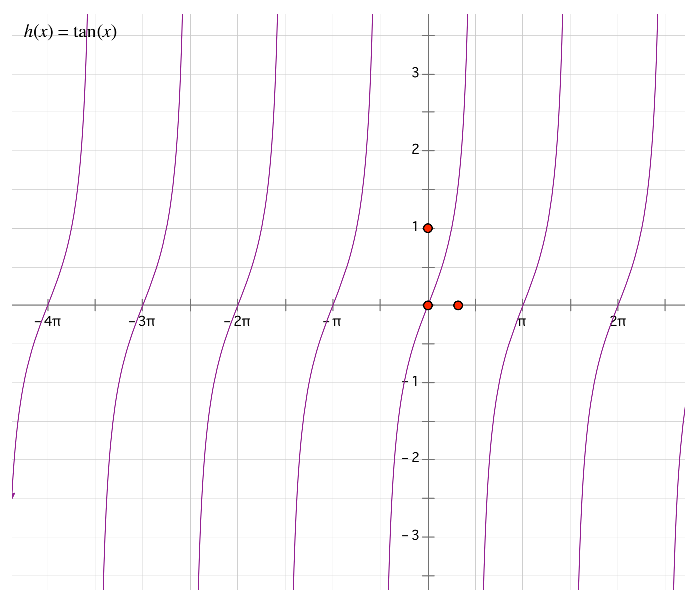
Graph of $\cos(x)$, (note x here is an angle measured in radians).
http://demonstrations.wolfram.com/IllustratingCosineWithTheUnitCircleII/

Graph of $\tan(x)$, (note x here is an angle measured in radians).
http://demonstrations.wolfram.com/IllustratingTangentWithTheUnitCircle/

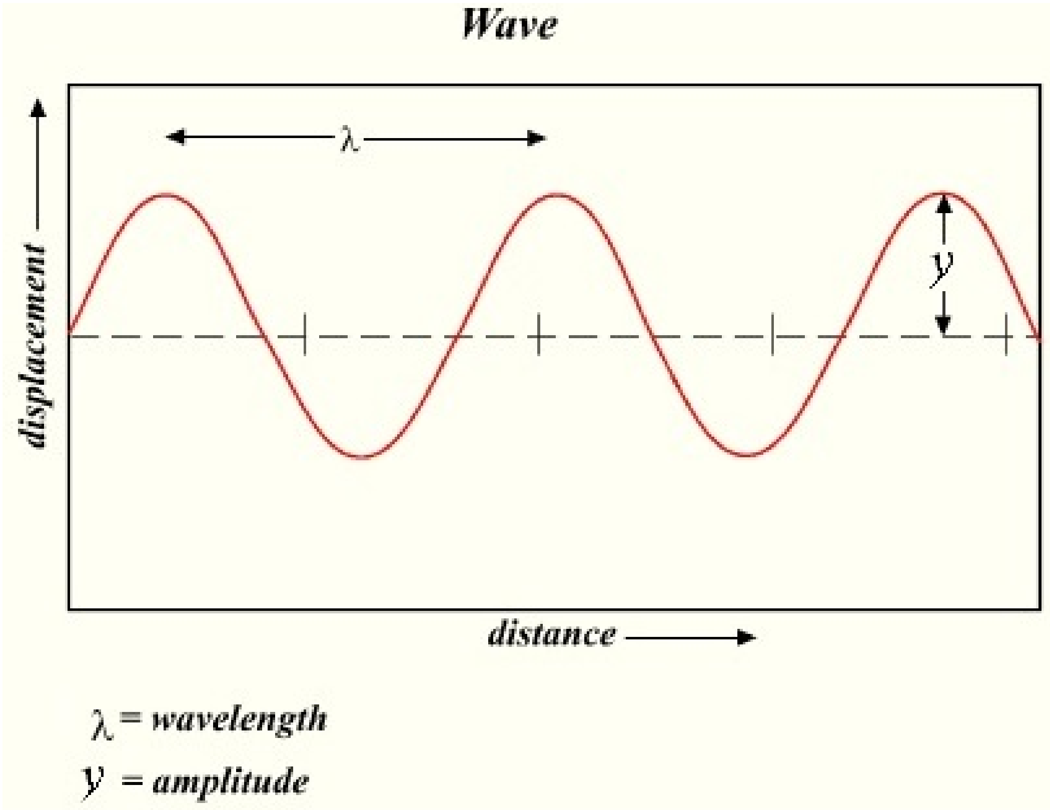
We can extend these graphs by relating the picture on every interval of length 2π and we can now add the following three graphs to our catalog of graphs:





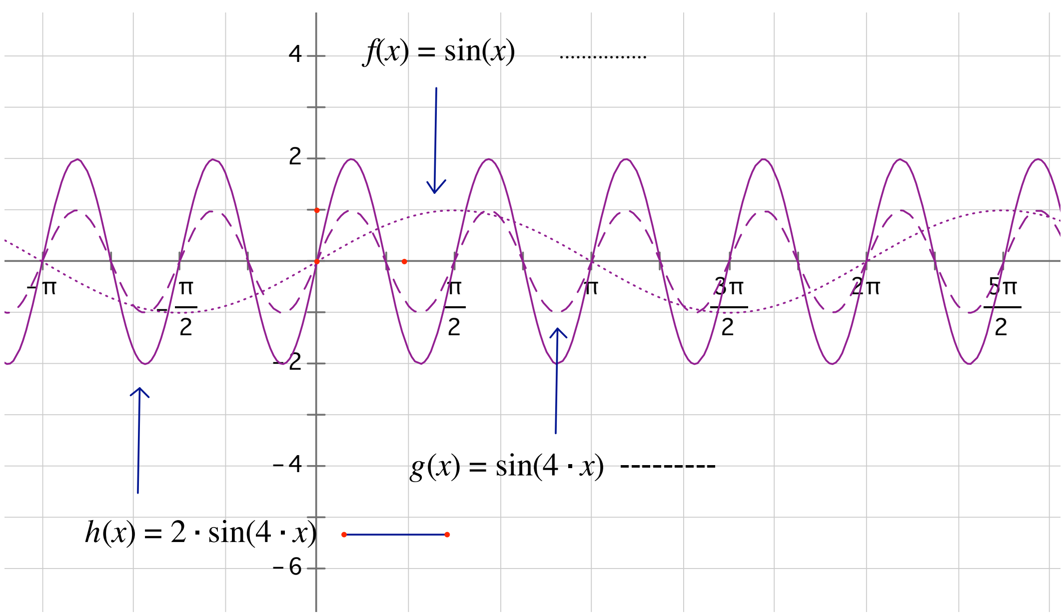


We see that the sine and cosine functions are wave functions with period(wavelength) 2π and amplitude 1.



We can apply our graphing techniques to graph variations of these functions. We see that multiplying a function by a constant (vertical stretch/shrink) changes the amplitude and multiplying x by a constant (horizontal stretch/shrink) changes the wavelength.

Example Sketch the graph of $y = 2\sin(4x)$. We see that this is a graph similar to that of $y = \sin(x)$, except with period $2\pi/4 = \pi/2$ and twice the amplitude:



Example Sketch the graph of $y = 4\cos(x/2) + 1$.

Example Sketch the graph of $y = |\cos x|$.

Note also since the point $(\cos\theta, \sin\theta)$ is always a point on the unit circle for any angle θ, we have **the important identity**

$$\boxed{\sin^2\theta + \cos^2\theta = 1.}$$