Optimal Mixed Strategy for Zero-Sum Games

In this section consider the existence of optimal mixed strategies for both players in zero sum games. We saw that if the reduced payoff matrix reduces to a matrix with a single strategy for both players, then optimal play by both players is given by a pure strategy for each player, namely the single strategy in the reduced pay-off matrix. However, we saw that not all reduced payoff matrices are 1×1 matrices.

Recall also that an **equilibrium point** in a pay-off matrix is a point such that once the point is reached neither player has any incentive to unilaterally change their (pure) strategy. However, we saw that an equilibrium point does not always exist and when they do, they may not be unique. So a study of equilibrium points in a matrix may not lead to a unique optimal pure strategy for either player. It is however useful to consider extending the concept of an equilibrium point to a situation where both players may use mixed strategies.

Definition An **equilibrium point** of a game where both players may use mixed strategies is a pair of mixed strategies such that neither player has any incentive to unilaterally change to another mixed strategy.

When searching for optimal mixed strategies for both players, we assume a number of things:

- The pay-off matrix is known to both players.
- Whatever mixed strategy is played by either player can be deduced by the opponent by observation.
- Both players strive to maximize their expected payoff (note that in a zero sum game, the expected payoffs of the players add to zero, therefore maximizing an expected payoff for one player is equivalent to minimizing the expected payoff of the other player).
- Whatever mixed strategy a player chooses, their opponent will choose the best counterstrategy.

In view of these assumptions, we see that if optimal mixed strategies exist for both players these strategies should occur at an equilibrium point, since neither player should be able to increase their payoff by unilaterally changing their strategy. The Minimax theorem below shows that optimal mixed strategies for both players always exist for a zero sum game.

Minimax Theorem: John Von Neumann For every zero sum game, there is a number ν for value and particular mixed strategies for both players such that

1. The expected payoff to the row player will be at least ν if the row player plays his or her particular mixed strategy, no matter what mixed strategy the column player plays.
2. The expected payoff to the row player will be at most ν if the column player plays his or her particular mixed strategy, no matter what strategy the row player chooses.

the number ν is called the **value of the game** and represents the expected advantage to the row player (a disadvantage if ν is negative).

Calculating Optimal mixed strategies and the value of the game for a 2×2 payoff matrix

We demonstrate the general principle with an example here.
Example Consider the situation where the payoff matrix for the row player is given by

$$\begin{pmatrix} 7 & 3 \\ 2 & 5 \end{pmatrix}$$

We denote the row player by R and the column player by C. We know from the last section that R's mixed strategy looks like $(p, 1-p)$ for some value of p with $0 \leq p \leq 1$ and C's mixed strategy looks like $\begin{pmatrix} q \\ 1-q \end{pmatrix}$ for some value of q with $0 \leq q \leq 1$. We wish to determine the values of p and q which give optimal strategies for both players.

We know that the expected vale of R's payoff is given by

$$\begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} 7p + 2(1-p) & 3p + 5(1-p) \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}$$

$$= \begin{pmatrix} 5p + 2 & 5 - 2p \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = (5p+2)q + (5-2p)(1-q) = 5pq + 2q + 5 - 2p - 5q + 2pq$$

$$= 5 + 7pq - 3q - 2p = 29/7 + 7(p - 3/7)(q - 2/7)$$

If Rsets p equal to $3/7$, then R is guaranteed an expected payoff of $29/7$. If R chooses any other value of p, C can choose a value of q that will make the expression $7(p-3/7)(q-2/7)$ negative giving a payoff for R which is less than $29/7$.
(If R sets $p < 3/7$ then the expression $7(p-3/7)(q-2/7)$ is negative as long as $q > 2/7$, if $p > 3/7$ then the expression $7(p-3/7)(q-2/7)$ is negative as long as $q < 2/7$.
Thus an expected payoff of $29/7$ is the best that R can do if C uses an optimal counterstrategy. Thus **R's optimal mixed strategy** is $(3/7, 4/7)$.
Similarly C's best counterstrategy is to set q equal to $2/7$ since otherwise R can choose a value of p making the expression $7(p-3/7)(q-2/7)$ positive to give an expected payoff for R that is greater than $29/7$ (thus a payoff for C which is less than -29/7).
Thus **C's optimal strategy** is given by $\begin{pmatrix} 2/7 \\ 5/7 \end{pmatrix}$

Since the expected pay-off is given by $29/7 + 7(p-3/7)(q-2/7)$, the value of the game is given by $\nu = 29/7$ (when both players play their optimal strategy).

Optimal Mixed Strategies and the Value of a Zero-Sum Game

A similar calculation to that given above with the general payoff matrix for the row player given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(where a, b, c and d are not all the same i.e. $a - b - c + d \neq 0$) yields an expected payoff for strategies $\begin{pmatrix} p & 1-p \end{pmatrix}$ and $\begin{pmatrix} q \\ 1-q \end{pmatrix}$ by the row and column players respectively of

$$\frac{ad - bc}{a - b - c + d} - (a - b - c + d)(p - \frac{d - c}{a - b - c + d})(q - \frac{d - b}{a - b - c + d}).$$

As above, we deduce that

- The optimal strategy for the row player is to set the probability of playing Row 1 equal to

$$p = \frac{d - c}{a - b - c + d}.$$

  The row player's probability of playing Row 2 is then determined as $1 - p$.

- The optimal strategy for the column player is to set the probability of playing Column 1 equal to
$$q = \frac{d-b}{a-b-c+d}$$
The column player's probability of playing Column 2 is then determined as $1-q$.

- The value of the game (expected payoff for the row player if both players play optimally) is given by
$$\nu = \frac{ad-bc}{a-b-c+d}.$$
The expected payoff for the column player is given by the negative of the expected payoff for the row player since it is a zero sum game.

This method can be used for any payoff matrix which is 2×2 or for and payoff matrix that has a reduced matrix which is 2×2. It can also be used for constant sum games.

Example Example: Football Run or Pass? [Winston] In football, the offense selects a play and the defense lines up in a defensive formation. We will consider a very simple model of play selection in which the offense and defense simultaneously select their play. The offense may choose to run or to pass and the defense may choose a run or a pass defense. One can use the average yardage gained or lost in this particular League as payoffs and construct a payoff matrix for this two player zero-sum game. In a previous lecture, we constructed a payoff matrix for this example based on the given statistics for the two teams involved.

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**	-5	5
	Pass	10	0

(a) What is the optimal strategy for both teams in this game?

(b) What is the value of the game given that both teams play optimally?

Example Our example of possible endgame strategies for basketball, had a payoff matrix where the payoff for each team was the probability of a win for each team under the given circumstances. This is a **Constant sum game** since the probabilities add to 1.

		Defending	Team
		Defend 2	**Defend 3**
Offense	**Shoot 2**	0.178	0.312
	Shoot 3	0.5	0.228

(a) What is the optimal strategy for both teams here?

(b) What is the value of the game?

(c) What is the expected payoff for the Offense?

(d) What is the expected payoff for the Defence?

(d) What do these numbers mean?

Example A baseball pitcher throws three pitches, a fastball, a slider and a change-up. As a measure of the payoff for this type of confrontation, we might use the expected number of runs the batter creates in each situation. (Note their are other possible measures that take into account the pitcher's abilities). We would expect that for any given pitch, the batter's performance is better if he anticipates the pitch. Lets assume that the batter has four possible strategies, To anticipate either a fastball, a slider or a change-up or not to anticipate any pitch.

		Pitcher		
		Fastball	**Change-up**	**Slider**
Batter	**Fastball**	0.38	0.37	0.39
	Change-up	0.25	0.4	0.41
	Slider	0.35	0.32	0.45
	None	0.38	0.3	0.42

(a) Find the reduced payoff matrix.

(b) Find the optimal mixed strategy for both players.

(c) Find the value of the game.

(d) Interpret the value of the game.

Example 4 Book In a simplified model of a tennis serve, the server must decide whether to serve to the receiver's forehand or to the backhand. The receiver must anticipate whether the serve will come to the forehand or the backhand. The payoff matrix shown below, shows the percentage of time the server (row player) wins the serve in each situation. Note this is a constant sum game with the payoff for the receiver equal to 100 minus the payoff for the server.

		Receiver	
		Guess Forehand	**Guess Backhand**
Server	**Forehand**	40	70
	Backhand	80	60

(a) Find each player's optional strategy given that both players play optimally.

(b) Find the value of the game.

References

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