Massey's Method.

Chapter 2 from "Who's # 1" [1], chapter available on Sakai.

Massey's method of ranking, which is also used in the BCS rankings makes use of the point differential in a game. The ratings produced can also be manipulated to produce an offensive and defensive rating for each team, which in turn can be used to produce an offensive and defensive ranking for each team.

For any given game between Team i and Team j, the point differential for team i is the score of team i minus the score for team j. At any given point in a tournament, the **point differential for team i** will be the sum of their point differentials for the games they have played.

Example In an inter-dorm basketball round robin, the final score data was as follows:

	Badin	Farley	Lyons	McGlinn	Pangborn	Record	Point Differential
Badin		37-82	51-54	37-68	30-75	0-4	-124
Farley	82-37		64-46	55-47	57-37	4-0	91
Lyons	54-51	46-64		37-35	33-60	2-2	-40
McGlinn	68-37	47-55	35-37		44-82	1-3	-17
Pangborn	75-30	37-57	60-33	82-44		3-1	90

Notice that the **point differentials** give us a rating of the teams which could be used to obtain a ranking of the teams:

Team	Point Differential
Badin	-124
Farley	91
Lyons	-40
McGlinn	-17
Pangborn	90

$\rightarrow$

Team	Rank
Badin	5
Farley	1
Lyons	4
McGlinn	3
Pangborn	2

Massey's ratings (which are used to derive his rankings), are based on the simple principle that for each game the difference in the ratings should be equal to the point differential i.e.

$$r_i - r_j = y_k$$

where r_i = rating for team i, r_j = rating for team j and y_k = score for team i - score for team j for game k.

This gives us a linear equation in the unknowns, r_i and r_j for every game played and thus gives us a (very large) system of linear equations.

[1] Who's # 1, *Amy N. Langville & Carl. D. Meyer*, Princeton University Press, 2012.

Example In our example above this would give us 10 equations in the 5 unknowns r_1, r_2, r_3, r_4 and r_5:

$$\begin{array}{cccccccc}
r_1- & r_2 & & & & = & \text{-45} \\
r_1- & & r_3 & & & = & \text{-3} \\
r_1- & & & r_4 & & = & \text{-31} \\
r_1- & & & & r_5 & = & \text{-45} \\
& r_2- & r_3 & & & = & 18 \\
& r_2- & & r_4 & & = & 8 \\
& r_2- & & & r_5 & = & 20 \\
& & r_3- & r_4 & & = & 2 \\
& & r_3- & & r_5 & = & \text{-27} \\
& & & r_4- & r_5 & = & \text{-38}
\end{array}$$

In general, if we have n teams, these equations translate to a matrix equation of the form

$$\mathbf{Xr} = \mathbf{y}$$

where the matrix $\mathbf{X}$ is very sparse and $\mathbf{r}$ is the column matrix $\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix}$. The matrix $\mathbf{X}$ has a row for each game played and n columns, with only two non-zero entries on each row. The matrix $\mathbf{y}$ is a column matrix, with a row for each game played and entry equal the point differential for that game (sign depends on the order in which the teams appear). Typically this turns out to be an inconsistent system of equations with no solutions for $r_1, r_2, r_3, ..., r_n$.

One can however use the statistical method of least squares to find a solution to the *normal equations* we get by multiplying by the transpose of $\mathbf{X}$, denoted by $\mathbf{X^T}$ (this is the matrix we get by switching the entries $\mathbf{X}_{i,j}$ and $\mathbf{X}_{j,i}$ in the matrix $\mathbf{X}$). The solution to the matrix equation

$$\mathbf{X^T X r} = \mathbf{X^T y}$$

is the best estimate (in a statistical sense of minimizing variance) for the ratings $\mathbf{r}$ in the original equation. The matrix $\mathbf{X^T X}$ is an $n \times n$ matrix because $\mathbf{X^T}$ has n rows and the matrix $\mathbf{X^T y}$ is a column matrix with n rows. Letting $\mathbf{M} = \mathbf{X^T X}$ and $\mathbf{P} = \mathbf{X^T y}$, we get n linear equations in the n unknowns, $r_1, r_2, \ldots, r_n$:

$$\mathbf{Mr} = \mathbf{P}.$$

Example In our example above, the equation $\mathbf{Xr} = \mathbf{y}$ translates to:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix}
=
\begin{pmatrix} -45 \\ -3 \\ -31 \\ -45 \\ 18 \\ 8 \\ 20 \\ 2 \\ -27 \\ -38 \end{pmatrix}$$

We have $\mathbf{X^T} =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

and $\mathbf{M} = \mathbf{X^T X} =$

$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

The matrix $\mathbf{X^T y}$ is

$$\begin{pmatrix} -124 \\ 91 \\ -40 \\ -17 \\ 90 \end{pmatrix}.$$

the equation $\mathbf{Mr} = \mathbf{P}$ looks like

$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -124 \\ 91 \\ -40 \\ -17 \\ 90 \end{pmatrix}$$

In general if we have n teams the Massey matrix, $\mathbf{M} = \mathbf{X^T X}$ is almost like the Colley matrix from the previous section. It is an $n \times n$ matrix with

$$M_{ii} = t_i = \text{total number of games played by team i}$$

$$M_{ij} = -\#\text{games played by team i and team j.}$$

The matrix $\mathbf{P} = \mathbf{X^T y}$ on the right hand side of the above equation is a column matrix with dimensions $1 \times n$ (where n is the number of teams) and the ith element is the sum of the point differentials for team i for every game played by team i that season.

Unfortunately the matrix $\mathbf{M}$ above is not invertible, so solving the system of equations $\mathbf{Mr} = \mathbf{P}$ using inverses is not an option. To get around this, Massey replaces the n-th row of the matrix $\mathbf{M}$ by a row of 1's and replacing the final row of $\mathbf{P}$ by a zero. This amounts to the requirement that the $r_i's$ add to 0. The new matrix is invertible and and the new system is solvable and **Massey's ratings** are the solutions for $r_1, r_2, \ldots, r_n$ for this system. We demote the adjusted Massey matrix by $\overline{\mathbf{M}}$ and the adjusted point differential column by $\overline{\mathbf{P}}$. The new equation looks like

$$\overline{\mathbf{M}}\mathbf{r} = \overline{\mathbf{P}}$$

and its solution $\mathbf{r}$ gives us the Massey ratings for the teams.

In Summary If there are n teams competing, the Massey matrix is given by $\mathbf{M}$ where

$$M_{ii} = t_i = \text{total number of games played by team i}$$

$$M_{ij} = -\#\text{games played by team i and team j.}$$

The adjusted Massey matrix is given by $\overline{\mathbf{M}}$ where

$$\overline{\mathbf{M}}_{ij} = M_{ii} \quad \text{if} \quad i < n$$

$$\overline{\mathbf{M}}_{nj} = 1 \quad \text{for all j}$$

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix}$$

where r_i represents the unknown Massey rating for team i.

$$\overline{\mathbf{P}} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_{n-1} \\ 0 \end{pmatrix}$$

where P_i denotes the point differential for Team i. **Massey's Ratings** are given by the solution to the system

$$\overline{\mathbf{M}}\mathbf{r} = \overline{\mathbf{P}}$$

Example In our running example, the adjusted equation looks like the following:

$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -124 \\ 91 \\ -40 \\ -17 \\ 0 \end{pmatrix}$$

The solution is given by

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -24.8 \\ 18.2 \\ -8. \\ -3.4 \\ 18. \end{pmatrix}$$

These ratings can be ordered to give us the **Massey Rankings**:

Team	Rank
Badin	5
Farley	1
Lyons	4
McGlinn	3
Pangborn	2

Example Suppose O'Neill, Carroll and Sorin have also been playing in a Round Robin with the following results:

	O'Neill	Carroll	Sorin	Record	Point Differential
O'Neill		47-64	52-54		
Carroll	64-47		64-96		
Sorin	54-52	96-64			

Set up the matrix equations, $\mathbf{Cr} = \mathbf{b}$ and $\overline{\mathbf{M}}\mathbf{r} = \overline{\mathbf{P}}$ for Colley's and Massey's method and find the rankings for the teams using both methods.

Team	Rank Colley	Rank Massey
O'Neill		
Carroll		
Sorin		

Example Sorin, Carroll, Pangborn and Farley will play in an end of season knockout tournament. You do not know yet which teams are drawn against each other in the first round (quarter-finals). You know that there have been three friendly matched between the dorms throughout the semester with results:
Sorin34 Pangborn 28
Carroll 56 Farley 92
Carroll 35 Pangborn 54

Use this information along with the information in bath examples above to set up the matrix equation $\overline{\mathbf{M}}\mathbf{r} = \overline{\mathbf{P}}$ for Massey's method for all of the teams in both tournaments. Derive a rating and a ranking for all teams which might be used to predict the outcome of the matches in the knockout tournament.