

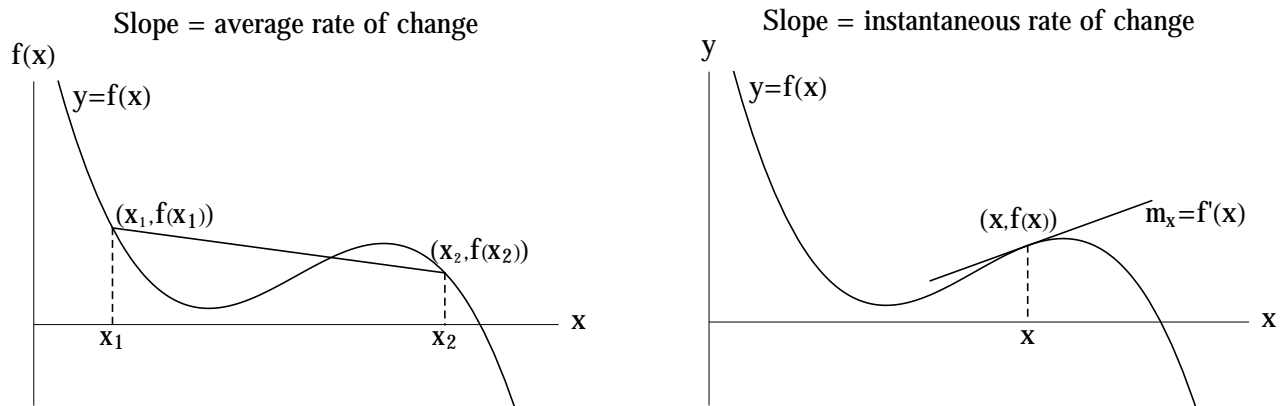
Derivatives, Instantaneous velocity.

Average and instantaneous rate of change of a function In the last section, we calculated the average velocity for a position function $s(t)$, which describes the position of an object (traveling in a straight line) at time t . We saw that the average velocity over the time interval $[t_1, t_2]$ is given by $\bar{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{\Delta s}{\Delta t}$. This may be interpreted as the average rate of change of the position function $s(t)$ over the interval $[t_1, t_2]$.

We can apply this general principle to any function given by an equation $y = f(x)$. Note the names of the independent and dependent variables have changed to x and y in place of t and s above. We can define **the average rate of change** of the function $f(x)$ over the interval $[x_1, x_2]$ as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

This can also be interpreted geometrically as the slope of the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of the function $y = f(x)$ as shown on the left below.

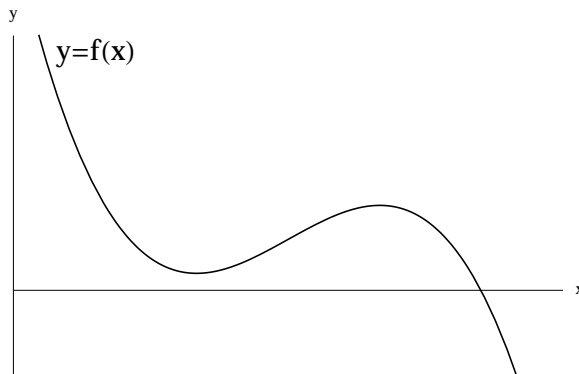


Now if the graph of $f(x)$ is smooth (no sharp points and continuous) at a point $(x, f(x))$, we can draw a tangent to the graph at the point $(x, f(x))$ as shown in the picture on the right above. The slope of this line, m_x is called the **derivative of f at x** and denoted by $f'(x)$ or $\frac{df(x)}{dx}$. Note that we can find a value for $m_x = f'(x)$ at any value of x in the domain of f where the graph of the function is smooth, therefore $f'(x)$ is a function of x and varies as x varies. The value of $f'(x)$ gives us the **instantaneous rate of change of f at x** .

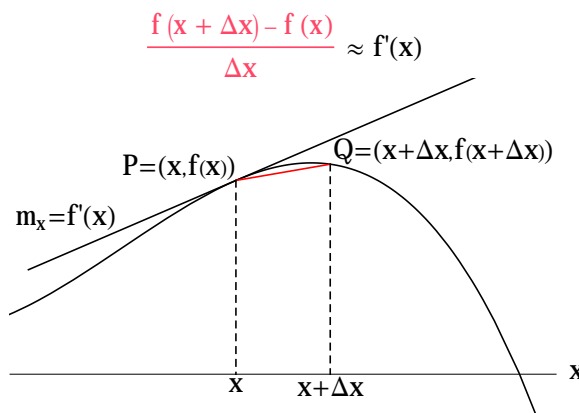
Although the definition of the derivative is relatively simple, the details of when it exists and how to calculate it are somewhat time consuming and require several lectures in a regular calculus course. Since we are mainly interested in its application to motion, we will focus on the essential features of the derivative below. From the tangent definition of the derivative, we can see the following relationship between the shape of the graph of $y = f(x)$ and the derivative function $f'(x)$:

- If the graph of $y = f(x)$ is smooth at x and increasing, then $f'(x)$ is positive.
- If the graph of $y = f(x)$ is smooth at x and decreasing, then $f'(x)$ is negative.
- If the graph of $y = f(x)$ is smooth at x and is at a turning point, then $f'(x)$ has value 0.
- If the graph of $y = f(x)$ has a sharp point or is not continuous at x , then the derivative is not defined.

Example Sketch the graph of the function $f'(x)$ using the graph of the function $y = f(x)$ shown below and the definition of the derivative as the slope of the tangent line.



Approximating and calculating Derivatives We see that the slope of the tangent line to the graph of $y = f(x)$ at a value of x where the curve is smooth can be approximated by the slope of a secant. The slope of the secant shown below, $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ will give us a reasonable **approximation to the slope of the tangent at $(x, f(x))$** , where Δx represent a relatively small change in x .



As the value of Δx approaches 0 in the diagram the point Q gets closer to the point P and the approximation to $f'(x)$ by the slope of the secant line shown gets more and more accurate and the value of $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ gets closer and closer to that of $f'(x)$ (the slope of the tangent line at P).

In mathematical language, we say the limit as Δx approaches 0 of $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is $f'(x)$ and we use the following notation to express this:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (0.1)$$

In a more in depth study of derivatives, one would use this formula to give a more rigorous definition of the derivative and to study existence and calculation of derivatives. In particular one would derive formulas and algebraic rules for the calculation of derivatives from the formula for a function. Since it takes a lot of time to develop the rules properly, we will restrict our algebraic exploration to a few simple rules derived from the above definition and concern ourselves mainly with a graphical exploration of

derivatives.

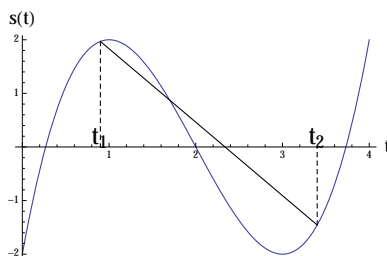
Some Rules of Differentiation We use the notation $\frac{df(x)}{dx}$ to denote $f'(x)$ below and we let $f(x), g(x)$ be functions of x and c a constant. The following rules of differentiation can be derived from Equation (0.1) above:

- $\frac{d(f(x) + g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$.
- $\frac{d c f(x)}{dx} = c \frac{df(x)}{dx}$.
- $\frac{dx^n}{dx} = nx^{n-1}$, where n is any real number.
- $\frac{d c}{dx} = 0$.

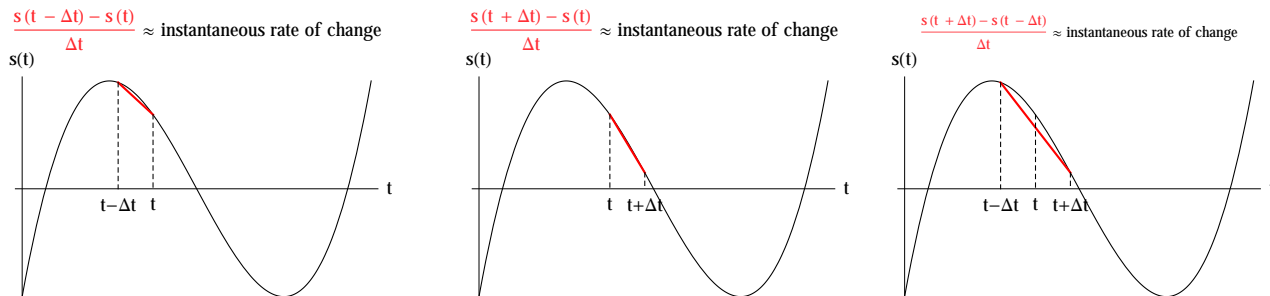
Example Let $f(x) = x^3 + 2x^2 + x + 1$. Find a formula for the function $f'(x)$.

Instantaneous Speed and Velocity We have already studied the concept of average speed and velocity and now we turn our attention to measuring instantaneous speed and velocity. The speedometer in a car gives us a measure of instantaneous speed. Our intuition tells us that a moving object has a speed (an instantaneous speed) at a particular instant in time. However at an given instant in time neither the position of the object nor the time changes, giving us something of a paradox to deal with. Such paradoxes were heavily discussed by the ancient Greeks and finally resolved by Isaac Newton and Gottfried Leibniz by introducing the concept of the **derivative** of a function.

In our discussion of the derivative above, we used the variables x and y . When dealing with position functions in one dimensional kinematics we use t for the independent variable, time, and s or y for the dependent variable, position. Let $s(t)$ be the position function of an object moving in a straight line. We saw that the average velocity over the time interval $[t_1, t_2]$ was given by $\bar{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ which corresponds to the slope of the secant shown on the graph below.



We can estimate the instantaneous speed at time t by taking the average speed in a small time interval containing t . Three different possibilities giving 3 different estimates are shown in the diagrams below, where Δt represents a small change in t which is positive in this case and the symbol \approx means “is approximately equal to”. The picture at the right corresponds to “the central distance method” discussed in your book.



The smaller the value of Δt in these estimates, the more accurate the estimate will be. It is natural to define the **instantaneous velocity** of the object at time t as the limiting value of these estimates as Δt tends to zero (which corresponds to the derivative, $s'(t)$, of the position function at time t and the slope of the tangent to the graph of $y = s(t)$ at t), that is

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t),$$

where $v(t)$ denotes the **instantaneous velocity of the object at time t** .

We see, as was the case for general derivatives, that instantaneous velocity changes as time changes and thus is a function of time. In biomechanics one needs to interpret graphical output and observational data in addition to motion which follows a formula as a result of the laws of physics. Therefore, we will discuss how to derive an estimate of velocity from graphical output and observational data as well as deriving the velocity function from a position function with a formula.

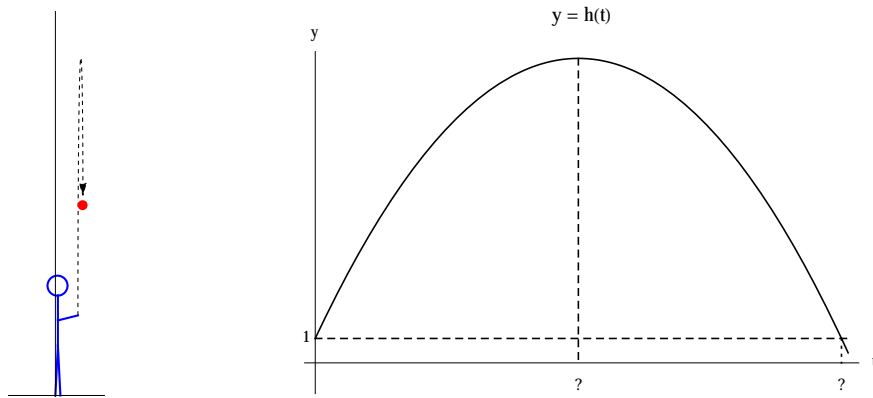
Instantaneous speed It is not hard to see that for movement of visible objects (where the position function is continuous and smooth), at any given point in time, t , we can choose a Δt so small that the distance travelled by the object on the time interval $[t, t + \Delta t]$ is equal to the absolute value of the displacement. Therefore when calculating instantaneous speed using the limiting process described above for velocity, we get that instantaneous speed at time t is equal to the absolute value of the instantaneous velocity:

$$\text{speed at time } t = \lim_{\Delta t \rightarrow 0} \frac{|s(t + \Delta t) - s(t)|}{\Delta t} = |s'(t)| = |v(t)|,$$

where $s(t)$ denotes the position function of an object moving in a straight line.

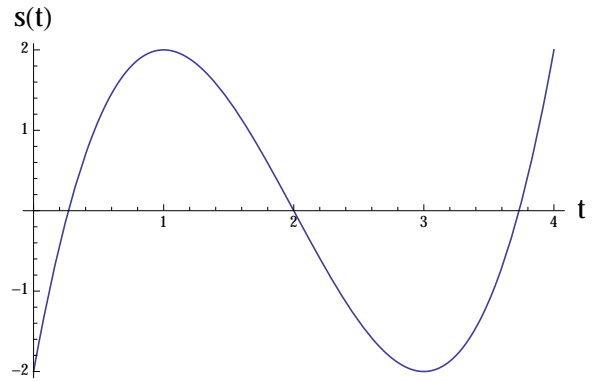
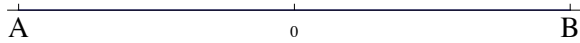
Note: Some books on biomechanics use the term velocity to denote speed. One can tell which they mean by how they define the function. Obviously a thorough understanding of the concepts helps you sort out exactly which function they are using independently of how it is labelled.

Example (Given a formula for the position function) A ball is thrown straight upwards with an initial velocity of $v_0 = 30$ m/s from a height of 1 meter above the ground. The height of the ball as a function of time measured in seconds after it is thrown is given (roughly) by $h(t) = 1 + 30t - 4.9t^2$.



- (a) What is the velocity function $v(t)$ showing the velocity of the ball at time t ?
- (b) What is the velocity of the ball after 1 second and after 2 seconds?
- (c) when does the ball reach its maximum height?
- (d) If the person who threw the ball catches it when it gets back to their hand height of 1 meter, how long does the ball stay in the air?
- (e) what is the speed of the ball when it gets back to the throwers hand?

Example (Velocity from the graph of the position function) An athlete doing agility training starts at point A and runs to point B and then turns and runs back to point A and turns again and runs back to point B. The position function for the athlete at time t is given by $s(t) = t^3 - 6t^2 + 9t - 2$. A graph of the position function is shown below for $0 \leq t \leq 4$.



(a) Identify the time intervals on which the velocity is positive. In which direction is the athlete running during these time intervals?

(b) Identify the time intervals on which the velocity is negative. In which direction is the athlete running during these time intervals?

(c) Identify the points where the velocity is zero. Give an interpretation of the athlete's movement at these points.

(d) Sketch the velocity function.

Example (Using Data to estimate velocity) (Central distance method) The following is a set of Position-Time data showing the vertical position of an object moving downwards in a straight line collected from 11 video frames collected at 60 frames per second.

Frame	Time (s)	Vertical Position (m)	Approx. (inst.) velocity v m/s
1	0.000	0.00	
2	0.0167	0.10	$\frac{0.21-0.00}{0.0334-0.000} = 6.29$ m/s
3	0.0334	0.21	$\frac{0.26-0.10}{0.0501-0.0167} = 4.79$ m/s
4	0.0501	0.26	
5	0.0668	0.32	
6	0.0835	0.10	
7	0.1002	0.00	
8	0.1169	0.20	
9	0.1336	0.26	
10	0.1503	0.34	
11	0.1670	0.04	

Note that the frames are collected at 60 frames per second, so the time between frames is $1/60 = 0.0167$ s. To estimate instantaneous velocity at a point using the **central distance method**, we take the average velocity over two consecutive time intervals to estimate the instantaneous velocity at its midpoint. For example if the positions for consecutive frames corresponding to times t_1 , t_2 and t_3 are given by $s(t_1)$, $s(t_2)$ and $s(t_3)$, then to estimate the instantaneous velocity at t_2 (the midpoint of the interval $[t_1, t_3]$), we use the average velocity on the interval $[t_1, t_3]$:

$$v(t_2) \approx \frac{s(t_3) - s(t_1)}{t_3 - t_1}.$$

Obviously we can only use this method for the times in the interior of our observational data.

We have filled in a few estimates for the instantaneous velocities in the table above, calculate the rest.

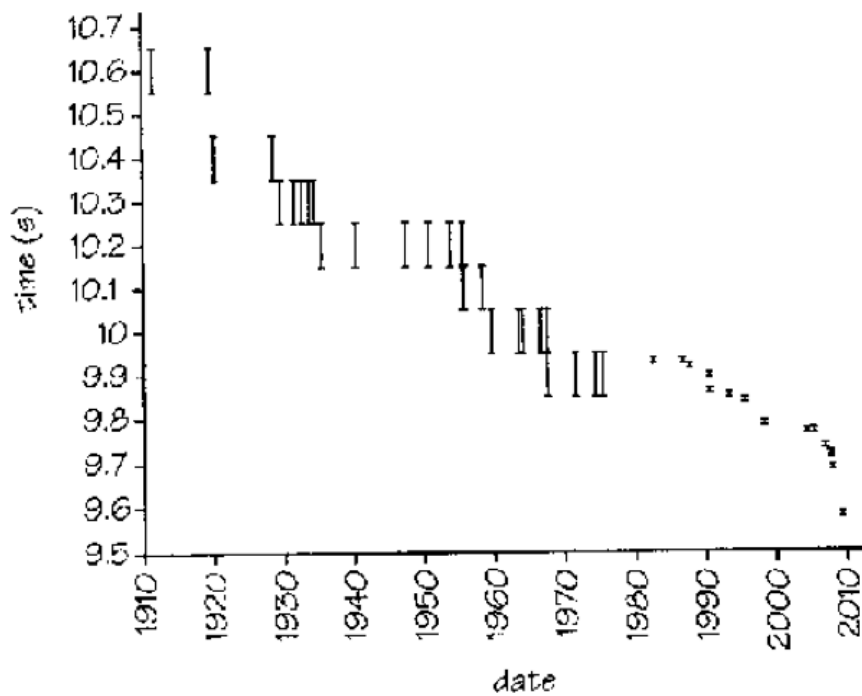
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How Usain Bolt Could Break His World Record with No Extra Effort

Usain Bolt is the best human sprinter there has ever been. Yet, few would have guessed that he would run so fast over 100 m after he started out running the 400m and 200m races when in his mid teens. His coach decided to shift him down to running the 100m one season so as to improve his basic sprinting speed. No one expected him to shine there. Surely he is too big to be a 100m sprinter? How wrong they were. Instead of shaving the occasional hundredth of a second off the world record, he took big chunks out of it, first reducing Asafa Powell's time of 9.74 s down to 9.72 in New York in May 2008, and then down to 9.69 (actually 9.683) at the Beijing Olympics later that year, before dramatically reducing it again to 9.58 (actually 9.578) at the 2009 Berlin World Championships. His progression in the 200m was even more astounding: reducing Michael Johnson's 1996 record of 19.32 s to 19.30 (actually 19.296) in Beijing and then to 19.19 in Berlin. These jumps are so big that people have started to calculate what Bolt's maximum possible speed might be. Unfortunately, all the commentators have missed the two key factors that would permit Bolt to run significantly faster without any extra effort or improvement in physical conditioning. "How could that be?" I hear you ask.

The recorded time of a 100m sprinter is the sum of two parts:

the reaction time to the starter's gun and the subsequent running time over the 100 m distance. An athlete is judged to have false-started if he reacts by applying foot pressure to the starting blocks within 0.10 s of the start gun firing. Remarkably, Bolt has one of the longest reaction times of leading sprinters—he was the second slowest of all the finalists to react in Beijing and third slowest in Berlin when he ran 9.58. Allowing for all this, Bolt's average running speed in Beijing was 10.50 m/s and in Berlin (where he reacted faster) it was 10.60 m/s. Bolt is already running faster than the ultimate maximum speed of 10.55 m/s that a team of Stanford human biologists recently predicted for him.¹



Bolt	$0.146 + 9.434 = 9.58$	Thompson	$0.119 + 9.811 = 9.93$
Gay	$0.144 + 9.566 = 9.71$	Chambers	$0.123 + 9.877 = 10.00$
Powell	$0.134 + 9.706 = 9.84$	Burns	$0.165 + 9.835 = 10.00$
Bailey	$0.129 + 9.801 = 9.93$	Patton	$0.149 + 10.191 = 10.34$

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In the Beijing Olympic final, where Bolt's reaction time was 0.165 s for his 9.69 run, the other seven finalists reacted in 0.133, 0.134, 0.142, 0.145, 0.147, 0.165 and 0.169 s.

From these stats it is clear what Bolt's weakest point is: he has a very slow reaction to the gun. This is not quite the same as having a slow start. A very tall athlete, with longer limbs and larger inertia, has got more moving to do in order to rise upright from the starting blocks.² If Bolt could get his reaction time down to 0.13, which is very good but not exceptional, then he would reduce his 9.58 record run to 9.56. If he could get it down to an outstanding 0.12 he is looking at 9.55 and if he responded as quickly as the rules allow, with 0.1, then 9.53 is the result. And he hasn't had to run any faster!

This is the first key factor that has been missed in assessing Bolt's future potential. What are the others? Sprinters are allowed to receive the assistance of a following wind that must not exceed 2 m/s in speed. Many world records have taken advantage of that, and the most suspicious set of world records in sprints and jumps were those set at the Mexico Olympics in 1968, where the wind gauge often seemed to record 2 m/s when a world record was broken. But this is certainly not the case in Bolt's record runs. In Berlin his 9.58 s time benefited from only a modest 0.9 m/s tailwind and in Beijing there was no wind, so he has a lot more still to gain from advantageous wind conditions. Many years ago, I worked out how the best 100m times are changed by wind.³ A 2 m/s tailwind is worth about 0.11 s compared to a no-wind performance, and a 0.9 m/s tailwind 0.06 s, at a low-altitude site. So, with the best possible legal wind assistance and reaction time, Bolt's Berlin time is down from 9.53 s to 9.47 s and his Beijing time becomes 9.51 s. And finally, if he were to run at a high-altitude site like Mexico City, then he could go faster still and effortlessly shave off another 0.07 s.⁴ So he could improve his 100m time to an amazing 9.4 s without needing to run any faster.⁵