Notes on Categorical Logic

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Introduction

The purpose of this course was to explore connections between contemporary model theory and category theory. By *model theory* we will mostly mean first order, finitary model theory. Categorical model theory (or, more generally, categorical logic) is a general category-theoretic approach to logic that includes infinitary, intuitionistic, and even multi-valued logics.

Say More Later.

Chapter I

A Brief Survey of Contemporary Model Theory

I.1 Some History

Up until to the seventies and early eighties, model theory was a very broad subject, including topics such as infinitary logics, generalized quantifiers, and probability logics (which are actually back in fashion today in the form of continuous model theory), and had a very set-theoretic flavour. In particular, the focus was usually on models and methods of constructing models. There was a general feeling of model theory as being a collection of techniques, such as compactness, which only really "came to life" in applications, such as in non-standard analysis or the Ax-Kochen theorem.

Starting in the mid-eighties, the focus of model theory tended towards the study of first-order finitary logic as well as the category of definable sets of models and not just the models themselves. On the pure side, the focus became the classification of theories and, in application, more sophisticated techniques were being used.

Elaborate/add more. This was all that was said in lecture.

I.2 Model Theory Basics

Model theory is a "set-based theory" in the sense that the objects being studied are sets. In recent times, model theory has adopted a more category-theoretic perspective, perhaps naïvely, in the form of the categories $\mathrm{Mod}(T)$ and $\mathrm{Def}(T)$, which we will introduce in this section. We also aim to introduce the basic concepts of model theory and briefly outline some important notions, such as hyperdefinability, and examples. adjust wording here. Very awkward.

The fundamental correspondence in model theory is the one between *syntax* and *semantics*. On the syntactic side, we have the notion of a vocabulary (we assume for convenience that everything is 1-sorted) or *language*, L, is a set consisting of:

- relation symbols R, each equipped with an airity $n_R \geq 0$;
- function symbols f, each equipped with an airity $n_f \geq 0$;
- constant symbols c (one may also consider constant symbols as 0-ary function symbols);
- logical symbols: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\forall$, $\exists$, $=$, $\top$, $\bot$, (,), and a countable list of variables $x, y, z, \ldots$.

In practice, we will omit the arity n_R of a relation symbol (similarly for function symbols) when the context is clear. We will also omit the logical symbols, and assume they are always in our language. For example, the language of graphs is $L_{graphs} = \{R\}$ where R is a binary relation symbol; the language of rings is $L_{rings} = \{+, \times, -, 0, 1\}$ where "+" and "×" are binary function symbols, "−" is a unary function symbol, and "0" and "1" are constant symbols.

Given a language L.... include inductive definition of formula.

Remark I.1. Say something about continuous model theory

We write "$\varphi(x) \in L$" to mean an L-formula with free-variable "x". That is, the variable "x" is not quantified over, and the truth of $\varphi(x)$ depends on our interpretation of "x". For example, in the language of rings,

$$P(x_1, \ldots, x_n) = 0,$$

where $P(x_1, \ldots, x_n)$ is a polynomial with integer coefficients, is a formula with free-variables $x_1, \ldots, x_n$. The formula

$$\exists z((x - y)^2 = z)$$

has free-variables x and y, and z is a bound variable. A formula φ with no free-variables is called a *sentence*.

On the semantic side, we have the notion of an L-structure, $\mathcal{M}$, which consists of a set M (the universe) and

- for each relation symbol R of arity n_R, we have an interpretation of R as a subset $R(M) \subseteq M^{n_R}$;
- for each function symbol f of arity n_f, we have an interpretation as a subset $f(M) \subseteq M^{n_f} \times M$ that is the graph of a total function $f : M^{n_f} \rightarrow M$;
- for each constant symbol c, we have an interpretation as an element $c^{\mathcal{M}} \in M$.

In practice, we will usually just identify $\mathcal{M}$ and M as well as each symbol with its interpretation.

The main definition is that of truth of a formula in a model. We write "$M \models \varphi(\bar{a})$" to mean that $\varphi(\bar{x})$ is true in M when $\bar{x}$ is interpreted as tuple $\bar{a} \in M$. If σ is a sentence, we say that "M models σ" if $M \models \sigma$. If Σ is a set of L-sentences, possibly infinite, we say M models Σ and write $M \models \Sigma$ if $M \models \sigma$ for every $\sigma \in \Sigma$. For a set of L-sentences Σ and another L-sentence σ, $\Sigma \models \sigma$ (Σ implies or entails σ) if, for any L-structure M, if $M \models \Sigma$, then $M \models \sigma$.

As mentioned earlier, contemporary model theory is concerned not only with models, but with the collection of *definable sets* of a structure. Given an L-formula $\varphi(\bar{x})$ and an L-structure M, we write

$$\varphi(M) := \{\bar{a} \in M^n : M \models \varphi(\bar{a})\}.$$

A set $X \subseteq M^n$ is said to be *definable* (0-definable or $\emptyset$-definable) if there is an L-formula $\varphi(\bar{x})$ such that $X = \varphi(M)$. If $A \subseteq M$, then a set X is called *A-definable* (or definable over A) if there is an L-formula $\psi(\bar{x}, \bar{y})$ and a tuple $\bar{b} \in A^m$ such that

$$X = \{\bar{a} \in M^n : M \models \psi(\bar{a}, \bar{b})\}.$$

Given to L-structures M and N, an embedding $f : M \hookrightarrow N$ is called an *elementary embedding* if it preserves all of the definable structure of M and N; that is, $f : M \hookrightarrow N$ is an elementary embedding if and only if, for every L-formula $\varphi(\bar{x})$ and every $\bar{a} \in M^n$,

$$M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})).$$

If $M \subseteq N$ and the inclusion map $\iota : M \hookrightarrow N$ is elementary, we say that "M is an elementary substructure of N" or, equivalently, "N is an elementary extension of M" and write $M \preccurlyeq N$. If $f : M \hookrightarrow N$ is elementary, we will often implicitly identify M with its image $f(M)$ and write $M \preccurlyeq N$ anyway.

Example I.2. Let $L = \{+, 0\}$ be the language of additive groups. The natural embedding

$$(\mathbb{Z}, +, 0) \hookrightarrow (\hat{\mathbb{Z}}, +, 0),$$

where $\hat{\mathbb{Z}}$ is the profinite completion of the integers, is an elementary embedding of additive abelian groups.

Given a language L, an *L-theory*, T, is a consistent set of L-sentences (often assumed to be closed under logical implication). By *consistent*, we mean that T has a model. We say that T is *complete* if for every L-sentence σ, either $\sigma \in T$ or $\neg\sigma \in T$. Given an L-structure M, we call the set

$$\mathrm{Th}(M) := \{\sigma \in L : M \models \sigma\}$$

the *theory of* M. $\mathrm{Th}(M)$ is always a complete L-theory. Observe that if $M \preccurlyeq N$ then $\mathrm{Th}(M) = \mathrm{Th}(N)$ (the converse is not true in general).

The fundamental theorem of model theory is the *compactness theorem*, which characterizes when a theory (or any set of sentences) is consistent in terms of its finite subsets:

Theorem I.3 (The Compactness Theorem)**.** *Let Σ be a set of L-sentences. Then Σ is consistent if and only if every finite subset Σ' of Σ is consistent.*

Remark I.4. It is arguable that model theory is interesting precisely because the compactness theorem holds.

The compactness theorem gives rise to "non-standard models" of a theory.

Example I.5.

1. Let $L = \{\in\}$. The axioms of Zermelo-Frankel set theory (ZF) give an incomplete L-theory.
2. ACF_0, the theory of algebraically closed fields of characteristic 0, is a complete L_{rings}-theory.

To a given L-theory T, we can naturally associate two categories: $\mathrm{Mod}(T)$, the category of models, and $\mathrm{Def}(T)$, the category of (0-) definable sets. $\mathrm{Mod}(T)$ is given by the following data:

- objects: models $M \models T$;
- morphisms: elementary embeddings $M \hookrightarrow N$.

The category $\mathrm{Def}(T)$ is given by

- objects: equivalence classes $[\varphi(\bar{x})]$ of formulas modulo T: two formulas $\varphi(\bar{x})$ and $\psi(\bar{x})$ are equivalent modulo T if $T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.
- morphisms: a morphism from $[\varphi(\bar{x})]$ to $[\psi(\bar{y})]$ is given by an equivalence class modulo T of L-formulas $\chi(\bar{x}, \bar{y})$ such that

  $$T \models \forall \bar{x} \left[\varphi(\bar{x}) \to \exists^{=1} \bar{y} \chi(\bar{x}, \bar{y})\right] \wedge \forall \bar{x}, \bar{y} \left[\varphi(\bar{x}) \wedge \chi(\bar{x}, \bar{y}) \to \psi(\bar{y})\right],$$

  i.e. in any model M of T, the formula $\chi(\bar{x}, \bar{y})$ defines a the graph of a function from $\varphi(M)$ to $\psi(M)$ (here, "$\exists^{=1}$" is an abbreviation for "there exists exactly one", which is expressable in a first-order way).

Remark I.6. Note that it is not totally necessary to take equivalence classes of formulas modulo T as the objects of $\mathrm{Def}(T)$; one could take formulas themselves and allow equivalent formulas modulo T to be isomorphic objects in $\mathrm{Def}(T)$. However, it is important to take morphisms as equivalence classes.

A reoccuring theme in categorical model theory (after Makkai) is the question of when $\mathrm{Def}(T)$ can be recovered completely from $\mathrm{Mod}(T)$. Lascar [3] showed that this is possible when T is $\aleph_0$-categorical and G-finite.

In many cases, $\mathrm{Def}(T)$ has real mathematical content.

Example I.7. 1. For ACF_0, the theory of algebraically closed fields of characteristic 0, Def(ACF_0) is essentially the category of of algebraic varieties over $\mathbb{Q}$ with morphisms regular maps.

2. If $T = RCF$, the theory of real closed fields, Def(T) is the category of semi-algebraic sets with semi-algebraic functions.

I.3 Morleyization and the T^{eq} Construction

Chapter II

Introduction to Category Theory and Toposes

II.1 Categories, functors, and natural transformations

Definition II.1. A *category* $\mathcal{C}$ is a collection of *objects* $X, Y, Z, A, B, \ldots, a, b, \ldots$ and a collection of *morphisms* $f, g, \ldots$ such that each morphism f has a *domain* $\mathrm{dom}(f)$ and a *codomain* $\mathrm{cod}(f)$ which are objects of $\mathcal{C}$. If $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) = Y$ we write $f : X \to Y$, but this does not mean that f is an actual function. In addition, for each object X there is a distinguished *identity* morphism $1_X : X \to X$ or $\mathrm{id}_X : X \to X$, and there is a *composition* operation: if $f : X \to Y$ and $g : Y \to Z$ then the composite is $g \circ f$ or $gf : X \to Z$. Moreover, we require that

- Composition is associative: $h(gf) = (hg)f$ whenever defined, and
- Composition is unital: $f1_X = f = 1_Y f$.

Remark II.2. It's an easy exercise to show that the identity 1_X is uniquely defined by condition of being unital.

Notation. Given $\mathcal{C}$ sometimes $\mathcal{C}_0$ denotes the set of objects, and $\mathcal{C}_1$ the set of morphisms. If $X, Y \in \mathcal{C}_0$, then $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ denotes the set of morphisms between X and Y.

Example II.3. Categories are everywhere. Some examples:

(a) Let $(P, \leq)$ be a partially ordered set. Then we define a category with object set P and such that there is a morphism betwen a and b iff $a \leq b$ in which case this morphism is unique. Formally, we may think of the morphism set as $\{(a, b) \mid a \leq b\}$ with $\mathrm{dom}(a, b) = a$ and $\mathrm{cod}(a, b) = b$. Unitality is given by reflexivity and associativity is given by transitivity.

(b) The category Set. Objects are sets, and morphisms are mappings between sets. Identities are the usual maps, and unitality and associativity are well-known.

(c) By a *monoid*, we mean a set X with a binary operation $(x, y) \mapsto x \cdot y$ which is associative and has a unit $e \in X$ such that $e \cdot x = x = x \cdot e$ for all $x \in X$. This can be thought of as a category with a single object $*$ and X as the set of morphisms, with composition given by $\cdot$. And conversely, any category with a single object can be thought of as a monoid: this is a 1-1 correspondence.

(d) The category Grp whose objects are groups and morphisms are homomorphisms.

(e) The category Top whose objects are topological spaces and morphisms are continuous maps. (Generally, if you're studying some class of mathematical object, you'll probably consider the category of those objects and structure-preserving maps at least implicitly. . .).

(f) Mod(T), where T is a theory. Objects are models of T, morphisms are elementary maps. (Or: you could take morphisms to be embeddings – cf. East-Coast Model Theory vs. West-Coast Model Theory.)

(g) Def(T), where T is a theory. Objects are definable sets and morphisms. This is analogous to algebraic geometry, where a morphism of affine varieties is a polynomial map – a sort of definable function rather than structure-preserving map (but it can be viewed that way by viewing it as a map of coordinate rings!).

(h) Def(M), the category of definable sets in a model M (over some fixed collection of parameters).

(i) Let $\mathcal{C}$ be a category. Then we say that $\mathcal{C}$ is *definable* in a structure M (over parameters A) if:

    (a) Each object and each morphism is an element of M, and the sets $\mathcal{C}_0$, $\mathcal{C}_1$ are A-definable sets in M.

    (b) The functions $dom(), codom() : C_1 \to C_0$ have graphs which are A-definable in M.

    (c) The graph of the morphism composition function $(- \circ -) : C_1^2 \to C_1$ is A-definable in M.

    Remark II.4. If $\mathcal{C}$ is an A-definable category in a structure M then the map assigning each object to its identity morphism has an A-definable graph.

(j) Likewise, a category $\mathcal{C}$ is *definable* in a theory T if for every model M of T, $\mathcal{C}$ is $\emptyset$-definable in T.

(k) The empty category 0 has no objects and no morphisms. The one-object category 1. More generally, if S is a set, there is a category with object set S and only identity morphisms. This sets up a bijection between sets and *discrete* categories – i.e. categories with all morphisms being identities.

Remark II.5. We will not take size issues very seriously in this course. But note that a category where the objects and morphisms both form sets is called *small* and if the collection of morphisms with any given domain and codomain is a set, the category is called *locally small*.

Definition II.6. Let $\mathcal{C}, \mathcal{D}$ be categories. By a *functor* $F : \mathcal{C} \to \mathcal{D}$, we mean a pair (F_0, F_1) with $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ begin mappings such that $F_0(\operatorname{dom} f) = \operatorname{dom}(F_1(f))$, $F_0(\operatorname{cod} f) = \operatorname{cod}(F_1(f))$ and composition and units are preserved: $F_1(gf) = F_1(g)F_1(f)$ and $F_1(1_X) = 1_{F_0(X)}$. Often we write F in place of F_0, F_1.

Example II.7. Functors are everywhere. Some examples:

(a) "Forgetful functors" (no formal definition). For example $\mathsf{Grp} \to \mathsf{Set}$ taking the underlying set. Or if you have a sub-theory, you an take a reduct, and this will be a forgetful functor.

(b) For any category $\mathcal{C}$, there is a unique functor $0 \to \mathcal{C}$ and a unique functor $\mathcal{C} \to 1$.

(c) There is a projection functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C}$. Here we introduce the *product* of two categories, with $(\mathcal{C} \times \mathcal{D})_0 = \mathcal{C}_0 \times \mathcal{D}_0$, and $\operatorname{Hom}_{\mathcal{C}\times\mathcal{D}}((CD), (C', D')) = \operatorname{Hom}_{\mathcal{C}}(C, C') \times \operatorname{Hom}_{\mathcal{D}}(D, D')$. Composition and units are likwise given by taking the product of the operations in $\mathcal{C}$ and $\mathcal{D}$. The projection functor sends $(C, D) \mapsto C$ and $(f, g) \mapsto f$. There is, of course, also a projection functor onto the second factor.

(d) "Free functors", for example if X is set, then let $F(X)$ be the free group on X, i.e. the group of words on the letters $\{x \mid x \in X\} \cup \{x^{-1} \mid x \in X\}$. This works for any variety in the sense of universal algebra.

(e) Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}_0$. Then there is a functor $y_X : \mathcal{C} \to \mathsf{Set}$ given by $y_X(Y) = \operatorname{Mor}_{\mathcal{C}}(X, Y)$ and if $f : Y \to Z$, then $y_X(f) : \operatorname{Mor}_{\mathcal{C}}(X, Y) \to \operatorname{Mor}_X(X, Z)$ is given by composition with f. Such a functor is called a *representable functor*.

(f) Let T be a theory and $\varphi(\vec{x})$ a formula. Then there is a functor $\underline{\varphi} : \operatorname{Mod}(T) \to \mathsf{Set}$, $\underline{\varphi}(M) = \varphi(M) = \{\vec{a} \in M^n \mid M \models \varphi(\vec{a})\}$. *Aside: Characterizing functors of the form* $\underline{\varphi}$ *is one of the themes we will explore as we go along.*

(g) Given T, $M \models T$, we have a category $\operatorname{Def}(M)$ of M-definable sets, and there is a functor $\operatorname{Mod}(T) \to \mathsf{Cat}$, $M \mapsto \operatorname{Def}(M)$.

Remark II.8. There is a category Cat of categories where an object is a category, a morphism is a functor.

Example II.9 (Slice category)**.** Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}_0$, the slice category $\mathcal{C}/X$ has objects morphisms in $\mathcal{C}$ with codomain X, like $Y \xrightarrow{f} X$. A morphism from $Y \xrightarrow{f} X$ to $Z \xrightarrow{g} X$ consists of a morphism $h : Y \to Z$ in $\mathcal{C}$ such that $gh = f$:

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & X \end{array}$$

There is a functor $\mathcal{C} \to \mathsf{Cat}$ sending $X \mapsto \mathcal{C}/X$. The action on morphisms is by postcomposition.

Definition II.10. (i) Let $\mathcal{C}$ be a category. There is a category $\mathcal{C}^{\mathrm{op}}$ the *opposite category of* $\mathcal{C}$ with the same objects and morphisms as $\mathcal{C}$, but with domain and codomain reversed.

(ii) A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is just a functor. A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Example II.11. Given $\mathcal{C}$, the map $\mathcal{C}_0 \times \mathcal{C}_0 \to \mathsf{Set}$ sending $(A, B) \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$ yields a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$. The action on morphisms is given by composition.

Definition II.12. Let $\mathcal{C}$ be a category.

(i) A morphism $f : X \to Y$ in $\mathcal{C}$ is *monic*, or a *monomorphism*, if for any $g, h : Z \to X$, if $fg = fh$, then $g = h$.

(ii) Dually, a morphism $f : X \to Y$ is called *epic*, or an *epimorphism* if it is monic in $\mathcal{C}^{\mathrm{op}}$, i.e. if for every $g, h : Y \to Z$, if $gf = hf$, then $g = h$.

(iii) A morphism $f : X \to Y$ is called a *split monomorphism* if there exists a $g : Y \to X$ such that $gf = \mathrm{id}_X$ (exercise: in this case f is indeed a monomorphism).

(iv) A morphism $f : X \to Y$ is called a *split epimorphism* if there exists a $g : Y \to X$ such that $fg = \mathrm{id}_Y$ (exercise: in this case, f is indeed an epimorphism).

(v) A morphism $f : X \to Y$ is called an *isomorphism* if there exists $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$ (exercise: in this case g is uniquely defined by these conditions), and we write $g = f^{-1}$. (exercise: a morphism which is split monic and epic is an isomorphism. Dually, a morphism which is epic and split monic is an isomorphism.)

Example II.13. (a) In Set, a map f is monic iff it is injective, and it is epic iff it is surjective.

(b) In Mon, the category of monoids, the inclusion map $f : \mathbb{N} \to \mathbb{Z}$ is both monic and epic, but not an isomorphism. Monicness is easy to see. For epicness, suppose that $g, h : \mathbb{Z} \to H$ with $gf = hf$. Then $g(n) = h(n)$ for all $n \in \mathbb{N}$. Then $g(-n) = g(n)^{-1} = h(n)^{-1} = h(-n)$ because inverses in a monoid are unique. So $g = h$.

(c) A group can be identified with a one-object category in which all morphisms are isomorphisms. The opposite group corresponds to the opposite category.

(d) A *groupoid* is a category where all morphisms are isomorphisms.

(e) Equivalence relations can be identitified with groupoids which are at the same time posets – that is, all morphisms are isomorphisms and there is at most one morphism $X \to Y$ for any X, Y. The correspondence works just as for posets in general.

Definition II.14. Let $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation* from F to G, written $\alpha : F \to G$, consists of a family of morphisms $\alpha_X : F(X) \to G(X)$ of morphisms in $\mathcal{D}$ for each $X \in \mathcal{C}_0$, which is *natural* in the sense that for any $f : X \to Y$, we have $\alpha_Y F(f) = G(f)\alpha_X$. That is, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

If each α_X is an isomorphism, then α is called a *natural isomorphism.*

Remark II.15. 1. Suppose that $F, G, H : \mathcal{C} \to \mathcal{D}$ are functors and $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are natural transformations, then there is a composite natural transformation $\beta\alpha : F \Rightarrow H$ with $(\beta\alpha)_X = \beta_X \alpha_X$ (check that this is natural!).

2. If $\mathcal{C}, \mathcal{D}$ are categories then the *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors $\mathcal{C} \to \mathcal{D}$, and morphisms are natural transformations. Composition is defined as in the previous item, and the identity on a functor $\mathrm{id}_F : F \Rightarrow F$ is the transformation with components $(\mathrm{id}_F)_X = \mathrm{id}_{FX}$. One can show natural isomorphisms are the isomorphisms of this category.

In Set, there is a bijection between the set of functions $X \times Y \to Z$ and the set of functions $X \to Z^Y$, where Z^Y is the set of functions $Y \to Z$.

Proposition II.16. *The same holds for the category of categories. That is, given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, there is a natural bijection* $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Cat}}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$. *That is, we have a natural isomorphism of functors* $\mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat} \to \mathsf{Set}$.

Remark II.17. This property is called being a *cartesian closed category*, which we will discuss more later. That is, we're observing that Cat and Set are both cartesian closed categories.

Proof. First we describe the map $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cat}}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$. Let $F : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$ be a functor. The image of F under the bijection will be a functor $\bar{F} : \mathcal{E} \to \mathcal{C}^{\mathcal{D}}$ defined as follows. First we define $\bar{F}_0 : \mathcal{E}_0 \to (\mathcal{C}^{\mathcal{D}})_0$. For each $E \in \mathcal{E}$, denote by $F_E : \mathcal{C} \to \mathcal{D}$ the functor which on objects is $F_E(C) = F(E, C)$ and on morphisms for $f : C \to C'$ in $\mathcal{C}$ we define $F_E(f) : F(E, C) \to F(E, C')$ to be $F_1(\mathrm{id}_E, f) : F(E, C) \to F(E, C')$. Check that this is a functor $F_E : \mathcal{C} \to \mathcal{D}$. We set $\bar{F}_0(E) = F_E$. Now we define the action on morphisms $\bar{F}_1 : \mathcal{E}_1 \to (\mathcal{D}^{\mathcal{C}})_1$. If $g : E \to E'$ is a morphism in $\mathcal{E}$, then $\bar{F}_1(g)$ should be a morphism $\bar{F}_1(g) : \bar{F}_0(E) \to \bar{F}_0(E')$ in $\mathcal{C}^{\mathcal{D}}$, i.e. a natural transformation $\bar{F}_1(g) : F_E \Rightarrow F_{E'}$. For $C \in \mathcal{C}_0$, we define the component $(\bar{F}_1(g))_C = F(g, \mathrm{id}_C) : F(E, C) \to F(E', C)$. Check that this defines a natural transformation $\bar{F}_1(g) : F_E$
$nattoF_{E'}$.

Now we describe the inverse map $\mathrm{Hom}_{\mathsf{Cat}}(\mathcal{E}, \mathcal{D}^{\mathcal{C}}) \to \mathrm{Hom}_{\mathsf{Cat}}(\mathcal{E} \times \mathcal{C}, \mathcal{D})$. Given $G : \mathcal{E} \to \mathcal{D}^{\mathcal{C}})$, we define a functor $\tilde{G} : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$ as follows. On objects, we define $\tilde{G}_0(E, C) = G(E)(C)$. On morphisms $(g, f) : (E, C) \to (E', C')$, we define $\tilde{G}_1(g, f) : G(E)(C) \to G(E')(C')$ to be the composite $G(g)_{C'} G(E)(f)$, or equivalently by the naturality of the natural transformation $G(g) : G(E) \Rightarrow G(E')$, the composite $G(E')(f) G(g)_{C'}$. Check that this defines a functor $\tilde{G} : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$.

Then we check that these two maps are inverse to one another. We can also check naturality in $\mathcal{C}, \mathcal{D}, \mathcal{E}$. □

II.2 Yoneda's Lemma

Often in mathematics, one defines some sort of abstract mathematical object with certain concerete examples in mind. It's important to ask to what extent the abstract objects can be represented concretely. For example, the Stone representation theorem allows one to represent an abstract Boolean algebra B concretely as an algebra of sets, i.e. to embed B in the powerset algebra of some set. Cayley's theorem in group theory allows one to represent an abstract group G concretely as a group of permutations, i.e. to embed G into the permutation group of some set (namely, the underlying set of the group itself). In this section, we will see how to represent an abstract category $\mathcal{C}$ concretely as a category of (multi-sorted, unary) algebras and homomorphisms between them, i.e. to embed $\mathcal{C}$ into a category of multisorted unary algebras. In fact, we will recover Cayley's theorem as a special case, by regarding a group as a 1-object category.

Exercise: Let $\mathcal{C}$ be a category. Define a language L as follows. The sorts of L are the objects of $\mathcal{C}$. There are no relation symbols, and the function symbols of L (which are all unary) are the morphisms of $\mathcal{C}$. The "input" sort of a morphism is its domain, and the "output" sort is its codomain. Define an L-theory T as follows. For every composable pair of function symbols f, g, there is an axiom $\forall x g(f(x)) = gf(x)$ (where gf is the composite in $\mathcal{C}$). Show that there is a bijection between models of T and functors $C \to \mathsf{Set}$, and that this extends to a bijection between homomorphisms of models of T and natural transformations between functors $C \to \mathsf{Set}$. The upshot is that categories of the form Set^C are certain categories of algebras.

Definition II.18. Fix a category $\mathcal{C}$. For $C \in \mathcal{C}$. In Example II.7.(e) we defined the *representable functor*

$$\begin{aligned} y_C : \mathcal{C}^{\mathrm{op}} &\to \mathsf{Set} \\ C' &\mapsto \mathrm{Mor}_{\mathcal{C}}(C', C) \\ f : C'' \to C' &\mapsto \mathrm{Mor}_{\mathcal{C}}(f, C) : \mathrm{Mor}_{\mathcal{C}}(C', C) \to \mathrm{Mor}_{\mathcal{C}}(C'', C) \\ &\quad \text{(i.e. precpompose by } f) \end{aligned}$$

Moreover, given $g : C_1 \to C_2$ we obtain a natural transformation

$$\begin{aligned} y_g : y_{C_1} &\Rightarrow y_{C_2} \\ &(y_g)_{C_3} = \mathrm{Mor}_{\mathcal{C}}(C_3, g) : \mathrm{Mor}_{\mathcal{C}}(C_3, C_1) \to \mathrm{Mor}_{\mathcal{C}}(C_3, C_2) \\ &\text{(i.e. postcompose by } g) \end{aligned}$$

Check that y_g is natural. Check that together we have defined a functor

$$Y = y_{(-)} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

This functor is called the *Yoneda embedding* of $\mathcal{C}$.

(Actually, earlier we defined a functor $\mathcal{C} \to \mathsf{Set}$ dual to this one: to translate between the two definitions, interchange $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$.)

Let us define the term "embedding" that we just used.

Definition II.19. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that

- F is *full* if for all C, C', the map $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(FC, FC')$ is surjective.
- F is *faithful* if for all C, C' the map $F : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(FC, FC')$ is injective.
- F is an embedding if it is injective on objects, full, and faithful.

Example II.20 (full,faitful,embedding)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that

- F is *full* if, for every $C, C' \in \mathcal{C}$, the map $F : \mathcal{C}(C, C') \to \mathcal{D}(FC, FC')$ is *surjective.*
- F is *faithful* if, for every $C, C' \in \mathcal{C}$, the map $F : \mathcal{C}(C, C') \to \mathcal{D}(FC, FC')$ is *injective.*
- F is an *embedding* if it is faithful and injective on objects.

Proposition II.21. *The functor Y is an embedding.*

To prove this, we will use:

Lemma II.22. *Given an object F of $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ and an object C of $\mathcal{C}$, there is a natural bijection*

$$\begin{aligned} f_{C,F} : \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, F) &\to F(C) \\ \kappa &\mapsto \kappa_C(\mathrm{id}_C) \end{aligned}$$

where naturality means that for all $g : C \to C' \in \mathcal{C}$ and $\varphi : F \Rightarrow F'$ in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$, the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, F) & \xrightarrow{f_{C,F}} & F(C) \\ \Big\downarrow{\scriptstyle \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(g,\mu)} & & \Big\downarrow{\scriptstyle \mu_{C'}F(g)} \\ \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(y_{C'}, F') & \xrightarrow{f_{C',F'}} & F(\mathcal{C}') \end{array}$$

Note on the righthand side of the diagram that by naturality of μ, $\mu_{C'}F(g)$ could equivalently be written as $F'(g)\mu_C$.

Proof. Let us show that $f_{C,F}$ is injective. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C', C) & \xrightarrow{\kappa_{C'}} & F(C') \\ {\scriptstyle y_C(f)}\Big\uparrow & & {\scriptstyle F(f)}\Big\uparrow \\ \mathrm{Hom}_{\mathcal{C}}(C, C) & \xrightarrow{\kappa_C} & F(C) \end{array}$$

The diagram commutes by naturality of κ. Consider $\mathrm{id}_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ in the bottom left corner. Comparing the two ways of getting to the top right, we have

$$\begin{aligned} F(f)(\kappa_C(\mathrm{id}_C)) &= \kappa_{C'}(y_C(f)(\mathrm{id}_C)) \\ &= \kappa_{C'}(f) \end{aligned} \tag{II.1}$$

That is, κ is entirely determined by where it sends $\kappa_C(\mathrm{id}_C)$. But recall that by definition, $f_{C,F}(\kappa) = \kappa_C(\mathrm{id}_C)$. So $f_{C,F}$ is injective.

For surjectivity, we note choose any $x \in F(C)$, and we define $\kappa_C(\mathrm{id}_C) = x$, and extend this definition by equation (II.1). That is, we define $\kappa_{C'}(f) = F(f)(x)$. We check that under this definition, κ is natural.

We check the naturality statement. On the one hand, $\mu_{C'}F(g)(f_{C,F}(\kappa)) = \mu_{C'}F(g)(\kappa_C(\mathrm{id}_C) = \mu_{C'}\kappa_{C'}(g)$. On the other hand, $f_{C',F'}(\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(g,\mu)(\kappa)) = \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}(g,\mu)(\kappa)_{C'}(1_{C'}) = \mu_{C'}\kappa_{C'}(g)$, so they agree. $\square$

We can now prove the proposition :

Proof. We first show that Y is injective on objects. Let $C \in \mathcal{C}$, then $\mathrm{id}_C \in Y(C)(C)$, as it is a morphism from C to itself. But for all $C' \in \mathcal{C}$ not equal to C and all $D \in \mathcal{C}$, the morphism id_C does not belong to $Y(C')(D)$, as this is the set $\mathrm{Hom}_{\mathcal{C}}(D, C')$.

We now show that it is bijective on Hom sets, which will complete the proof. Let $C, C' \in \mathcal{C}$. The previous lemma yields a bijection $f_{C,Y(C')}$ between $F(C)$

and $\mathsf{Set}^{\mathcal{C}^{\text{op}}}(Y(C), Y(C'))$. But $F(C) = \mathrm{Hom}_{\mathcal{C}}(C, C')$, so $f_{C,Y(C')}$ is a bijection between $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\mathsf{Set}^{\mathcal{C}^{\text{op}}}(Y(C), Y(C'))$. We check that this is induced by Y. □

Exercise II.23. Let $\mathcal{C}$ be a category, and A, B be objects of $\mathcal{C}$. Suppose that for all $X \in \mathcal{C}$, there is bijection $f_X : \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B)$. Moreover, suppose that for all $g \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, the following diagram commutes :

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, A) & \xrightarrow{f_X} & \mathrm{Hom}_{\mathcal{C}}(X, B) \\ \downarrow{\cdot \circ g} & & \downarrow{\cdot \circ g} \\ \mathrm{Hom}_{\mathcal{C}}(X', A) & \xrightarrow{f_{X'}} & \mathrm{Hom}_{\mathcal{C}}(X', B) \end{array}$$

Then there is an isomorphism between A and B in $\mathcal{C}$.

Remark II.24. Functors from $\mathcal{C}^{\text{op}}$ to Set are called presheaves on $\mathcal{C}$.

Example II.25 (of a natural transformation)**.** Let $U : \mathsf{Grp} \to \mathsf{Set}$ be the forgetful functor, and let $F : \mathsf{Set} \to \mathsf{Grp}$ be the free group functor. There are natural transformations :

$\epsilon : FU \Rightarrow \mathrm{id}_{\mathsf{Grp}}$
$\nu : \mathrm{id}_{\mathsf{Set}} \Rightarrow UF$

They are defined as follow : for $G \in \mathsf{Grp}$, the morphism $\epsilon_G : FU(G) \to G$ sends the word $g_1^{\pm 1} \cdots g_n^{\pm 1}$ to the product $g_1^{\pm 1} \cdots g_n^{\pm 1}$ in G. And for some $A \in \mathsf{Set}$, the morphism $\nu A \to UF(A)$ sends the element a to the word a. We can check that these are natural transformations.

Maybe this would fit better after the definition of a natural transformation

II.3 Equivalence of categories

An isomorphism between to categories $\mathcal{C}$ and $\mathcal{D}$ is defined as a functor from $\mathcal{C}$ to $\mathcal{D}$ with an inverse. That is, we consider this functor as a morphism in the category Cat of categories, it is an isomorphism if it has an inverse in this category.

Example II.26. Let T be a complete 1-sorted theory, and $M \models T$. Then the categories $\mathrm{Def}(T)$ and $\mathrm{Def}_{\emptyset}(M)$ are isomorphic.

This is often too strong, and categories we view as similar may fail to be isomorphic. This motivates the introduction of the following notion :

Definition II.27. A natural transformation α between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a natural isomorphism if for each $X \in \mathcal{C}$, the morphism α_X is an isomorphism.

Definition II.28. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mu : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\nu : \mathrm{id}_{\mathcal{D}} \Rightarrow FG$.

In that case, we say that F and G are equivalences of categories, pseudo inverse of each others.

Example II.29. Let $(P, \leq)$ be a preorder (i.e $\leq$ is reflexive and transitive), which we see as a category. Define an equivalence relation E on objects by $E(x, y)$ if and only if $x \leq y$ and $y \leq x$. Let $\pi : P \to Q$ be the quotient map. Then Q is a partial order, and π is an equivalence of categories.

Example II.30. A discrete category is a category in which the only morphisms are the identity morphisms.

A category $\mathcal{D}$ is equivalent to a discrete category if and only if it is given by an equivalence relation, that is $\mathcal{D}$ is a groupoid such that there is at most one morphism between any two objects. Note that this is equivalent to being both a groupoid and a preorder.

Remark II.31. Both of these examples are equivalent to the axiom of choice. Indeed, in both cases, to construct a pseudo inverse, we have to choose a representative for each equivalence class.

Exercise II.32. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then it is an equivalence of categories if and only if it is full, faithful, and essentially surjective, i.e : for any $D \in \mathcal{D}$, there is $C \in \mathcal{C}$ such that $F(C) \cong_{\mathcal{D}} D$

Definition II.33. A duality between two categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence of categories between $\mathcal{C}$ and $\mathcal{D}^{\text{op}}$.

We will illustrate this notion with an example that is relevant to logic. But first, we will need to state a few definitions. From now on, by a compact space, we mean a compact Hausdorff space.

Definition II.34. A topological space is said to be zero-dimensional if it has a basis of clopen sets.

Remark II.35. For a locally compact Hausdorff space, this is equivalent to being totally disconnected, meaning that each point is its own connected component.

We will call compact zero-dimensional space Stone spaces.

Definition II.36. A boolean algebra is a set B together with two distinguished elements 0 and 1, two binary operations $\vee$ (the join) and $\wedge$ (the meet), and an unary operation $\neg$ (the complement) such that :

- $\vee$ and $\wedge$ are associative
- $\vee$ and $\wedge$ are commutative
- for all a, b, we have $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ (absorption)
- for all a, we have $a \vee 0 = a$ and $a \wedge 1 = a$
- $\vee$ and $\wedge$ are distributive on each other
- for all a, we have $a \vee \neg a = 1$ and $a \wedge \neg a = 0$

We will denote $\mathcal{C}$ the category of boolean algebra with structure preserving maps, and $\mathcal{D}$ the category of Stone space with continuous maps.

Remark II.37. If B is a boolean algebra, then for all a, b one can define a partial order by $a \leq b$ if and only if $a \vee b = b$. It has greatest element 1 and smallest 0. Moreover, the meet and join operations correspond to the infimum and the supremum, respectively.

Example II.38. If X is any set, then its power set $\mathcal{P}(X)$ is a boolean algebra with the usual meet, join and complement. The partial order is in this case given by inclusion.

Definition II.39. Given a boolean algebra B, a filter on B is a subset $\mathcal{F}$ of B such that :

- $a, b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F}$
- $a \in \mathcal{F}$ and $a \leq b \Rightarrow b \in \mathcal{F}$
- $0 \notin \mathcal{F}$

Example II.40. Let $a \in B, a > 0$. The set $\mathcal{F}_a = \{x \in B, a \leq x\}$ is a filter. It is called the principal filter generated by $\{a\}$.

If B is infinite, non principal filter exist. If $B = \mathcal{P}(\mathbb{N})$, then the set of cofinite subset is a filter, called the Frechet filter.

Definition II.41. An ultrafilter is a maximal (for inclusion) filter. Equivalently, it is a filter $\mathcal{U}$ such that for all a, either $a \in \mathcal{U}$ or $\neg a \in \mathcal{U}$.

Fact II.42. *Using the axiom of choice, one can prove that every filter extends to an ultrafilter.*

We will now associate, to every boolean algebra B, a Stone space $S(B)$.

Construction II.43. Let B be a boolean algebra. Consider the set of ultrafilters on B, denoted $S(B)$. The collection of subsets $\{\{\mathcal{U} \in S(B), \mathcal{U} \supset \mathcal{F}\}, \mathcal{F}$ a filter $\} \cup \{\emptyset\}$ are the closed sets of a topology on $S(B)$.

To prove that this is a Stone space, we must find a basis of clopen sets, and prove the space is compact. The basis of clopen sets is given by $\{\{\mathcal{U}, a \in \mathcal{U}\}, a \in B\}$. These are closed because equal to an intersection of closed sets. Moreover, if X_a is the set associated to a, then $(X_a)^c = X_{\neg a}$, so they are open as well.

The space is Hausdorff because if $\mathcal{U} \neq \mathcal{V}$, there must be a such that $a \in \mathcal{U}$ and $\neg a \in \mathcal{V}$. So X_a and $X_{\neg a}$ separate them.

The reader is invited to check that to show compactness it is enough to prove that if $A \subset B$ is such that any finite part of A is contained in an ultrafilter, then A itself is contained in an ultrafilter. But this assumption on A is equivalent to every finite part of A having non-empty meet. Now consider the set $\mathcal{F}$ of elements $b \in B$ such that there is $a_1, \cdots, a_n \in A$ such that $a_1 \wedge \cdots \wedge a_n \leq b$. It is an ultrafilter, and contains A. So the space $S(B)$ is compact.

If $f : B \to C$ is a morphism of boolean algebra (that is, a structure preserving map), the reader can check that the map :

$$S(f) : S(C) \to S(B)$$

$$\mathcal{U} \to f^{-1}(\mathcal{U})$$

is well defined and continuous. This also preserves identities and composition, and therefore defines a functor $S : \mathcal{C} \to \mathcal{D}^{op}$.

We can define another functor $G : \mathcal{D}^{op} \to \mathcal{C}$. It maps a Stone space X to the boolean algebra of its clopen subsets. And if $f : X \to Y$ is a continuous map and $C \subset Y$ is clopen, then f^{-1} is clopen. Therefore we can define a map $G(B) : G(Y) \to G(X)$, which is easily checked to be a morphism of boolean algebras.

Theorem II.44. *The two functors S and G define a duality between $\mathcal{C}$ and $\mathcal{D}$.*

Remark II.45. This implies in particular that any boolean algebra is isomorphic to the boolean algebra of clopen sets of a Stone space.

The Stone duality applies to logic via boolean algebras of formulas.

Example II.46 (Propositional logic)**.** We consider the language given by propositional variables $P_1, P_2, \cdots$, the symbols $\vee$ and $\wedge$ for disjunction and conjunction, 0 and 1 for false and true, a symbol , for comma, and parenthesis (and).

We can then define formulas inductively, as was done in the introduction to model theory. Given variables P and Q, an example of a formula is $(\neg P) \vee G$. This particular formula is abbreviated as $P \to Q$, and the formula $(P \to Q) \wedge (Q \to P)$ is abbreviated as $P \leftrightarrow Q$.

A model of a collection of formulas is a truth assignment to each of the propositional variables, such that each of the formula is true. A theory is a consistent set of formulas (that is, it has a model).

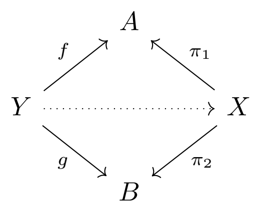
Let T be a theory, then we can define a boolean algebra $B(T)$ as the boolean algebra of formula with meet $\wedge$, join $\vee$, and complement $\neq$, up to equivalence modulo T. That is, to formulas φ and ψ are equivalent if and only if $\varphi \leftrightarrow \psi$ is a logical consequence of T.

Now consider $S(B(T))$, the reader can check that it is in one-one correspondance with models of T. Therefore in this case, Stone duality is a duality between the syntax $B(T)$ and semantics (models of T).

One of the objectives of categorical logic is to generalize this approach to predicate logic.

II.4 Product, Pullbacks, Equalizers

Definition II.47. Let $\mathcal{C}$ be a category, and A, B two objects of $\mathcal{C}$. A product of A and B is an object X of $\mathcal{C}$ and two morphisms $\pi_1 : X \to A$ and $\pi_2 : X \to B$, which are universal. That is, for any object Y and morphisms $f : Y \to A, g : Y \to B$, there exists a unique morphism from Y to X making the following commute :



this morphism is sometimes denoted (f, g).

Remark II.48. 1. Products, if they exist, are unique up to isomorphism, and denoted $A \times B$

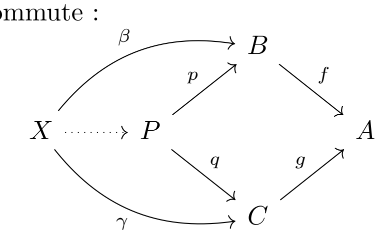
2. We can define in a similar way the product of an arbitrary family of objects. Again if it exists, it is unique up to isomorphism.

3. The product of the empy family, if it exists, is called the terminal object, and denoted 1. Any object has a unique morphism going to 1.

4. In Set, the categorical product is the cartesian product, and the terminal object is any singleton set.

5. If $(P, \leq)$ is a poset and $a, b \in P$, then $a \times b = \inf\{a, b\}$ whenever it exists.

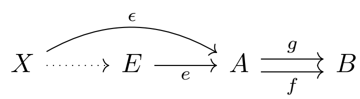
Definition II.49. Let $\mathcal{C}$ be a category. Let $f : B \to A$ and $g : C \to A$ be morphisms in $\mathcal{C}$. A pullback of f and g is an object P and morphisms $p : P \to A, q : P \to B$ such that for all X and morphisms $\beta : X \to B, \gamma : X \to C$ satisfying $f\beta = g\gamma$, there exists a unique morphism $X \to P$ making the following commute :



Remark II.50. 1. If it exists, a pullback is unique up to isomorphism.

2. In Set, pullbacks exist and are fibered products. If $f : B \to A$ and $g : C \to A$, then $B \times_A C = \{(b, c) \in B \times C, f(b) = g(c)\}$, and the morphisms to A are given by restriction of the projections.

Definition II.51. Let $\mathcal{C}$ be a category, and let $f, g : A \to B$ be morphisms in $\mathcal{C}$. An equalizer of these two morphisms is an object E along with a morphism $e : E \to A$ such that for any object X and morphism $\epsilon : X \to A$ satisfying $f\epsilon = g\epsilon$, there exists a unique morphism $X \to E$ making the following commutes :

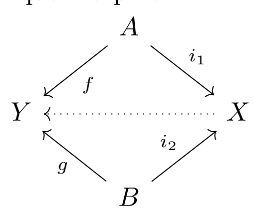


Remark II.52. 1. If it exists, an equalizer is unique up to isomorphism.

2. In Set, equalizers exist, and the equalizer of $A \underset{f}{\overset{g}{\rightrightarrows}} B$ is given by the subset $\{x \in A, f(x) = g(x)\}$ of A, with the inclusion in A.

If we consider products, pull backs and equalizers in the category $\mathcal{C}^{op}$, we obtain the same diagrams, but with every arrow reversed. These give the definition of coproduct, pushout and coequalizer. Once again, if these objects exist, they are unique up to isomorphism.

Example II.53. Given A and B objects in $\mathcal{C}$, a coproduct of A and B is given by an object X and two morphisms $i_1 : A \to X$ and $i_2 : B \to X$, such that for any object Y and any pair of morphisms $f : A \to Y$ and $g : B \to Y$, there exists a unique morphism $X \to Y$ making the following commute :



We can define in a similar way the product of any indexed family of objects. The product of the empty family, if it exists, is called the initial object, denoted 0. There is a unique morphism from zero to any object.

In Set, the coproduct correspond to the disjoint union, and the initial object is the empty set.

The reader in invited to work out the definitions of the two other notions, and construct them in Set.

Exercise II.54. Show that :

1. Equalizers are monic
2. Coequalizers are epic
3. If

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ X & & & & A \\ & \searrow & & \nearrow & \\ & & C & & \end{array}$$

is a pullback and $B \to A$ is monic, then $X \to C$ is monic too.

Definition II.55. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. Then F is said to be left adjoint to G (and G right adjoint to F) if there is a natural isomorphism between $\mathcal{C}(F(\cdot), \cdot) : \mathcal{D}^{op} \times \mathcal{C} \to \mathsf{Set}$ and $\mathcal{D}(\cdot, G(\cdot)) : \mathcal{D}^{op} \times \mathcal{C} \to \mathsf{Set}$. That is, there is a collection of bijections $\{M_{D,C} : \mathcal{C}(F(D), C) \to \mathcal{D}(D, G(C)), C \in \mathcal{C}, D \in \mathcal{D}\}$ such that for every $f : D \to D'$ and $g : C' \to C$, the following commutes :

$$\begin{array}{ccc} \mathcal{C}(F(D), C) & \xrightarrow{M_{D,C}} & \mathcal{D}(D, G(C)) \\ {\scriptstyle (Ff,g)}\uparrow & & {\scriptstyle (f,Gg)}\uparrow \\ \mathcal{C}(F(D'), C') & \xrightarrow{M_{D',C'}} & \mathcal{D}(D', G(C')) \end{array}$$

where the functions on the sides are obtained by precomposition on the left and composition on the right. For example, if $\alpha : F(D') \to C'$ then $(Ff, g)(\alpha) = g \circ \alpha \circ F(f)$. There might be a better notation for this

Adjunctions are ubiquitous in mathematics, and can often shed light on certain constructions. The case of forgetful functors is a good illustration of this.

Example II.56.

1. Consider the free group functor $F : \mathsf{Set} \to \mathsf{Grp}$ and the forgetful functor $G : \mathsf{Grp} \to \mathsf{Set}$. Then for $D \in \mathsf{Set}$ and $C \in \mathsf{Grp}$, can construct the bijections $M_{D,C} : \mathcal{C}(F(D), C) \to \mathcal{D}(D, G(C))$ like we did in II.25. It is easy to check that these make F left adjoint to G.

2. Let k be a field, and $\mathcal{C}$ be the category of k-vector spaces. Consider the forgetful functor $G : \mathcal{C} \to \mathsf{Set}$, and the functor $F : \mathsf{Set} \to \mathcal{C}$ which send a set X to the k vector space it generates. Then G is right adjoint to F. Essentially, this means that linear maps are determined by their restriction to a basis.

3. Consider the category $\mathcal{C}$ of compact topological groups, and the category $\mathcal{D}$ of topological group. The inclusion $I : \mathcal{C} \to \mathcal{D}$ is a functor, and it has a right adjoint B. For a group G, the group $B(G)$ is called the Borh compactification of G. Bohr compactification

Remark II.57. Given $F : \mathcal{D} \to \mathcal{C}$ left adjoint to $G : \mathcal{C} \to \mathcal{D}$, the family $\{M_{D,C}, C \in \mathcal{C}, D \in \mathcal{D}\}$ is determined by the family $\{M_{D,F(D)}(\mathrm{id}_{F(D)}) : D \to$

$GF(D), D \in \mathcal{D}\}$. Similarly, the family $\{M_{D,C}^{-1}, D \in \mathcal{D}, C \in \mathcal{C}\}$ is determined by the family $\{M_{G(C),C}(\mathrm{id}_{G(C)}) : FG(C) \to C, c \in \mathcal{C}\}$.

Proof. Consider $\alpha : F(D) \to C$, the adjunction yields the following commutative diagram :

$$\begin{array}{ccc}
\mathcal{C}(F(D), F(D)) & \xrightarrow{M_{D,F(D)}} & \mathcal{D}(D, G(F(D))) \\
\big\downarrow{\scriptstyle (\alpha, \mathrm{id}_{F(D)})} & & \big\downarrow{\scriptstyle (G(\alpha), \mathrm{id}_D)} \\
\mathcal{C}(F(D), C) & \xrightarrow{M_{D,C}} & \mathcal{D}(D, G(C))
\end{array}$$

Therefore, we obtain $M_{D,C}(\alpha) = G(\alpha) \circ M_{D,F(D)}(\mathrm{id}_{F(D)})$, which proves the first half of the remark. For the other half, observe in a similar way that $M_{D,C}^{-1}(\beta) = M_{G(C),C}(\mathrm{id}_G(C)) \circ F(\beta)$.

□

For $D \in \mathcal{D}$, we let $\eta_D = M_{D,F(D)}(\mathrm{id}_{F(D)}) : D \to GF(D)$ and for $C \in \mathcal{C}$, we let $\epsilon_C = M_{G(C),C}(\mathrm{id}_G(C)) : FG(C) \to C$. The family $\eta = (\eta_D)_D$ is a natural transformation from $\mathrm{id}_{\mathcal{D}}$ to GF, called the unit. Similarly, the family $\epsilon = (\epsilon_D)_D$ is a natural transformation from FG to $\mathrm{id}_{\mathcal{C}}$, called the co-unit.

Remark II.58. 1. Let $\beta : D \to G(C)$, then the following commutes :

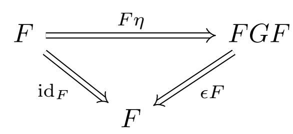
$$\begin{array}{ccc}
D & \xrightarrow{\beta} & G(C) \\
{\scriptstyle \eta_D}\big\downarrow & & \big\uparrow{\scriptstyle G(\epsilon_C)} \\
GF(D) & \xrightarrow[GF(\beta)]{} & GFG(C)
\end{array}$$

2. Similarly, for any $\alpha : F(D) \to C$, the following diagram commutes :

$$\begin{array}{ccc}
F(D) & \xrightarrow{\alpha} & C \\
{\scriptstyle F(\eta_D)}\big\downarrow & & \big\uparrow{\scriptstyle \epsilon_C} \\
FGF(D) & \xrightarrow[FG(\alpha)]{} & FG(C)
\end{array}$$

3. Let $\eta G = \{\eta_{G(C)}, C \in \mathcal{C}\}$, it defines natural transformation from G to GFG. We also let $G\epsilon = G(\epsilon_C)$, it defines a natural transformation from GFG to G. Define $F\eta$ and ϵF in a similar way. We then have two commutative diagrams of natural transformations :

$$\begin{array}{ccc}
G & \xRightarrow{\eta G} & GFG \\
& {\scriptstyle \mathrm{id}_G}\searrow \quad \swarrow{\scriptstyle G\epsilon} & \\
& G &
\end{array}$$



Proof. 1. There is $\alpha : F(D) \to C$ such that $\beta = M_{D,C}(\alpha)$, so $\alpha = M_{D,C}^{-1}(\beta)$. We also know that $M_{D,C}(\alpha) = G(\alpha) \circ \eta_D$ and $M_{D,C}^{-1}(\beta) = \epsilon_C \circ F(\beta)$. So we obtain :

$$\begin{aligned} M_{D,C}(MD, C^{-1}(\beta)) &= G(M_{D,C}^{-1}(\beta)) \circ \eta_D \\ \beta &= G(M_{D,C}^{-1}(\beta)) \circ \eta_D \\ &= G(\epsilon_C \circ F(\beta)) \circ \eta_D \\ &= G(\epsilon_C) \circ GF(\beta) \circ \eta_D \end{aligned}$$

2. Similar.

3. For the first diagram, apply 1. to $\beta = \mathrm{id}_{G(C)}$. For the second one, apply 2. to $\alpha = \mathrm{id}_{F(D)}$.

□

Lemma II.59. *Let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. Then F is a left adjoint to G if and only if there are natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ satisfying the identities $G\epsilon \circ \eta G = \mathrm{id}_G$ and $\epsilon F \circ F\eta = \mathrm{id}_F$ of the previous remark.*

Sketch of proof. To prove adjunction, we need to construct a natural isomorphism $\Phi : \mathcal{C}(F(\cdot), \cdot) \Rightarrow \mathcal{D}(\cdot, G(\cdot))$. Let $f \in \mathcal{C}(F(D), C)$. Then $G(f) \in \mathcal{D}(GF(D), G(C))$, so $G(F) \circ \eta_D \in \mathcal{D}(D, G(C))$. We let $\Phi_{D,C}(f) = G(f) \circ \eta_D$. This collection of maps defines a natural transformation because η is a natural transformation. We obtain, in a similar way, a natural transformation $\Psi : \mathcal{D}(\cdot, G(\cdot)) \Rightarrow \mathcal{C}(F(\cdot), \cdot)$. For $g : D \to G(C)$, let $\Psi_{D,C}(g) = \epsilon_C \circ F(g)$.

We obtain the following equalities, for $f \in \mathcal{C}(F(D), C)$:

$$\begin{aligned} \Psi\Phi(f) &= \epsilon_C \circ F(\Phi(f)) \\ &= \epsilon_C \circ F(G(f) \circ \eta_D) \\ &= \epsilon_C \circ FG(f) \circ F(\eta_D) \text{ by functoriality of } F \\ &= f \circ \epsilon_{F(D)} \circ F(\eta_D) \text{ by naturality of } \epsilon \\ &= f \circ \mathrm{id}_{F(D)} \text{ by assumption} \\ &= f \end{aligned}$$

We check that $\Phi \circ \Psi$ is the identity in the same way. So Φ is the natural isomorphism we are looking for. □

In practice, this lemma will allow us to prove adjunction between two functors by finding two natural transformations, and making sure the unit-co-unit equations hold.

Definition II.60. Let $\mathcal{C}$ be a category, and let A be an object in $\mathcal{C}$. Suppose that that there is a functor $A \times \cdot : \mathcal{C} \to \mathcal{C}$ such that for all B, the object $A \times B$ is a product of A and B.

If this functor has a right adjoint, we denote it $(\cdot)^A$, say that A is exponentiable, and call B^A the exponential of B with respect to A.

Note.
1. Unpacking this adjunction, we see that it means the existence of natural bijections between $\mathcal{C}(C, B^A)$ and $\mathcal{C}(A \times C, B)$.
2. Assuming the right adjoint exists, it has a co-unit ϵ, which is the collection of morphisms $\{\epsilon_B : A \times B^A \to B, B \in \mathcal{C}\}$, called evaluation maps.

Example II.61. In Set, any A is exponentiable, and for all B, the exponential is A^B, the set of functions from B to A. The co-unit is the evaluation map :

$$\begin{aligned} \epsilon_B : A \times B^A &\to B \\ (a, f) &\to f(a) \end{aligned}$$

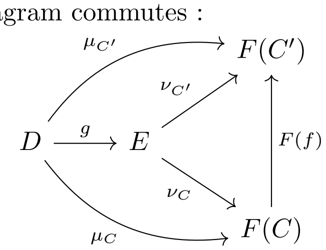
Definition II.62. The category $\mathcal{C}$ is cartesian closed if it has finite products, and each product functor has a right adjoint.

Exercise II.63. For any category $\mathcal{C}$, the category $\mathsf{Set}^{\mathcal{C}^{op}}$ is cartesian closed.

Let $\mathcal{C}, \mathcal{D}$ be categories, and D an object in $\mathcal{D}$. By Δ_D we mean the constant functor $\mathcal{C} \to D$ which sends every object to D and every arrow to id_D.

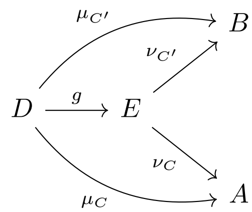
Construction II.64. Given a functor $F : \mathcal{C} \to \mathcal{D}$, by a cone for F we mean a natural transformation $\mu : \Delta_D \to F$, for some $D \in \mathcal{D}$. It is a family $\{\mu_C : D \to F(C), C \in \mathcal{C}\}$ such that for any $C, C' \in \mathcal{C}$ and $f : C \to C'$, we have $\mu_{C'} = F(f) \circ \mu_C$.

Given two cones (D, μ) and (E, ν) for F, a map from (D, μ) to (E, ν) is a morphism $g : D \to E$ such that for any $C, C' \in \mathcal{C}$ and $f : C \to C'$, the following diagram commutes :



Equipped with this notion of morphism, cones for F form a category. A limiting cone for F is a terminal objet in the category of cones for F.

Example II.65. Let $\mathcal{C}$ be the discrete category with two objects C and C', and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let A and B be the images of the two objects in $\mathcal{C}$. Then a morphism from a cone (D, μ) to (E, ν) is a commutative diagram :



Therefore, a limiting cone is simply a product of A and B. The reader is invited to work out similar conic definitions for pullbacks and equalizers.

Remark II.66. By reversing the direction of the arrows, we obtain the dual notion of a co-cone. It can be used to formalize co-products, push forwards and co-equalizers.

Definition II.67. We say the category $\mathcal{D}$ is complete, or has limits, if it has limiting cones for all functors $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is a small category.

We say that $\mathcal{D}$ has finite limits if it has limiting cones for all functors $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ is a finite category (meaning finitely many objects, and all Hom sets are finite).

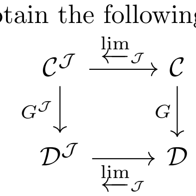
Example II.68. The category Set is complete.

For the remaining part of the section, we are going to use the following notations. We will use $\mathcal{J}$ to denote the index category and $\mathcal{C}$ to denote the category where we want to compute the limits. Hence the cateory of functors from $\mathcal{J}$ to $\mathcal{C}$ will be denoted by $\mathcal{C}^{\mathcal{J}}$. Also, there is a canonical functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$, which is defined to be such that Δ_c is the constant functor of c.

Lemma II.69. *Suppose every diagram $F : \mathcal{J} \to \mathcal{C}$ has a limiting cone, then Δ as above has a right adjoint, denoted by $\varprojlim_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$, where for each $F \in \mathcal{C}_0^{\mathcal{J}}$, its image under $\varprojlim_{\mathcal{J}}$ is the vertex of the limiting cone. Furthermore, the counit ε, which is a natural transformation $\Delta \varprojlim_{\mathcal{J}} \Rightarrow id_{\mathcal{C}^{\mathcal{J}}}$, where $\varepsilon_F : \Delta \varprojlim_{\mathcal{J}}(F) \to F$ is the limiting cone of F, or equivalently, the natural transformation $\Delta_c \Rightarrow F$, where c is the vertex of the limiting cone.*

Proof. From the definition. $\square$

Remark II.70. Suppose $G : \mathcal{C} \to \mathcal{D}$ is a functor and $\mathcal{J}$ is a small category. Furthermore, we assume that limits of type $\mathcal{J}$ exists in both $\mathcal{C}$ and $\mathcal{D}$, we can obtain the following diagram,



where $G^{\mathcal{J}}$ sends F to GF.

We have a canonical natural transformation $\alpha_{\mathcal{J}} : G \varprojlim_{\mathcal{J}} \Rightarrow \varprojlim_{\mathcal{J}} G^{\mathcal{J}}$, which is given by the universal property of limits. To be more precise, for each F a functor $\mathcal{J} \to \mathcal{C}$, $G \varprojlim_{\mathcal{J}}(F)$ is the veretex of a cone over GF, and $\varprojlim_{\mathcal{J}} G^{\mathcal{J}}$ is the limiting cone of GF, hence there is a unique map into it by the universal property.

Definition II.71. A functor G is said to preserve limits of type $\mathcal{J}$ is $\alpha_{\mathcal{J}}$ as above is an natural isomorphsim.

Need to check the following

Lemma II.72. *$G : \mathcal{C} \to \mathcal{D}$ preserves all limits iff G has a left adjoint.*

Before the end of the section, we define the dual notion of limits, which are called colimits, one can think of them as limits in ${\mathcal{C}^{op}}^{\mathcal{J}}$, and a cocone will be a natural transformation from F to some Δ_c for $c \in \mathcal{C}_0$. And if colimits exists for given $\mathcal{J}$, we have the functor $\varinjlim_{\mathcal{J}}$, which is a left adjoint to the functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$, given by the universal property.

Chapter III

More Advanced Category Theoy and Toposes

III.1 Subobject classifiers

Definition III.1. For a category $\mathcal{C}$ and $X \in \mathcal{C}_0$, a subobject of X is a monic $Y \rightarrowtail X$(acctually, it is an equivalence class of such monics up to isomorphsim). And $\mathrm{Sub}_{\mathcal{C}}(X)$ is the set of subobjects of X. Furthermore, we have a partial order on the set of subobjects. Let $g : Y_0 \rightarrowtail X$ and $h : Y_1 \rightarrowtail X$ be two subobjects of X, $Y_0 \leq Y_1$ if there is $f : Y_0 \to Y_1$ such that $hf = g$. Note that such f is automatically monic.

Remark III.2. In Set, we have as special set $2 = \{0, 1\}$.In particular $Y \subset X$ is characterized by its characteristic function $\chi_Y : X \to 2$ where $x \mapsto 0$ iff $x \in Y$. Working in categories other than Set, the special element 2 is replaced by an object Ω, which is called the subobject classifier provided it exists.

Definition III.3. Let $\mathcal{C}$ is a category with finite limits. In particular, it has a terminal object. A subobject classifier is a (monic) arrow, which we call it *true*, $1 \rightarrowtail \Omega$, where 1 is a terminal object, such that for every monic $S \rightarrowtail X$, there is unique $\varphi : X \to \Omega$ such that $\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ X & \xrightarrow[\varphi]{} & \Omega \end{array}$ is a pullback.

Example III.4. In Set, $1 = \{0\}$ and $\Omega = \{0, 1\}$ where $u : 1 \rightarrowtail \Omega$ is the inclusion map. If $S \subset X$, let i denote the inclusion of S into X. Then the following diagram is a pullback. $\begin{array}{ccc} S & \longrightarrow & 1 \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ X & \xrightarrow[\varphi=\chi_S]{} & \Omega = 2 \end{array}$

The following lemma characterizes the existence of subobject classifiers

Lemma III.5. *Suppose $\mathcal{C}$ has finite limits and small* Hom *sets, then $\mathcal{C}$ has a subobject classifier iff there is $\Omega \in \mathcal{C}_0$ such that for each X, there is a natural bijection $\theta_x : \mathrm{Sub}_{\mathcal{C}}(X) \to \mathrm{Hom}_{\mathcal{C}}(X, \Omega)$. The naturality condition means for each $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following diagram commutes, where the vertical maps are defined by pullback by g(Note that pullbacks of monics are monics).*

$$\begin{array}{ccc} \mathrm{Sub}_{\mathcal{C}}(Y) & \xrightarrow{\theta_Y} & \mathrm{Hom}_{\mathcal{C}}(Y, \Omega) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Sub}_{\mathcal{C}}(X) & \xrightarrow[\theta_X]{} & \mathrm{Hom}_{\mathcal{C}}(X, \Omega) \end{array}$$

Proof. Suppose we have subobject classifer $1 \rightarrowtail \Omega$, for each $S \to X$ a subobject, the unique $\varphi : X \to \Omega$ given by the definition of subobject classifiers gives us the natural bijection θ_X. It suffices to verify that it is sujective.

Let $\varphi : X \to \Omega$, by pullback, we can find $S \rightarrowtail X$ such that $\begin{array}{ccc} S & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[\varphi]{} & \Omega \end{array}$, and

this gives us the surjectivity.

Conversely, if the right hand side is satisfied, then there will be $1 \rightarrowtail \Omega$, a subobject of Ω that corresponds to $\mathrm{id}_\Omega : \Omega \to \Omega$. Now for each $S \rightarrowtail X$, there is $\varphi : X \to \Omega$ that corresponds to it. By naturality, we have the following diagram,

$\begin{array}{ccc} \mathrm{Sub}_{\mathcal{C}}(\Omega) & \xrightarrow{\theta_\Omega} & \mathrm{Hom}_{\mathcal{C}}(\Omega, \Omega) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Sub}_{\mathcal{C}}(X) & \xrightarrow[\theta_X]{} & \mathrm{Hom}_{\mathcal{C}}(X, \Omega) \end{array}$, where the vertical maps are induced by φ, and

in particular, S is the pullback of $1 \to \Omega$ along φ. Now, we have to show that 1 as above is a terminal object. But this is clear, since we consider $\varphi_1, \varphi_2 : X \to 1$

be two morphisms, then we would have $\begin{array}{ccc} X & \xrightarrow{\varphi_i} & 1 \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & \Omega \end{array}$ are trivially pullbacks.

And by the fact that $1 \rightarrowtail \Omega$ is monic, we have $\varphi_1 = \varphi_2$. □

Note that the right hand side condition in the above lemma is actually saying that the functor $\mathrm{Sub}_{\mathcal{C}} : \mathcal{C}^{op} \to \mathsf{Set}$ is representable and the representing object is Ω. Furthermore, Ω is unique up to isomorphism by Yoneda's lemma.

III.2 Elementary topos and Heyting algebra

Definition III.6. An *elementary topos* is a category $\mathcal{C}$ with all finite limits and exponentials and a subobject classifier.

The word elementary means that the above condition is expressible in the first order language of categories, where you have sorts for objects and morpisms and relation symbol for compositions and some additional data.

Definition III.7. (i) A *lattice* is a poset with sup for pairs, denoted by $\vee$ and inf for pairs, denoted by $\wedge$.

(ii) A lattice is *distriubtive* if it satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Note this implies the dual $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

(iii) Let L be a lattice with 0 as minimum and 1 as maximum, then a complement for $x \in L$ is a y such that $x \wedge y = 0$ and $x \vee y = 1$. Such y is unique if L is distributive.

(iv) A *Boolean algebra* is a distributive lattice with 0,1 and complement.

Remark III.8. The above definition can be viewed category theoretically, for example, a lattice with $0, 1$, is a poset with all finite products and coproducts(which implies finite limits and colimits).

Definition III.9. By a *Heyting algebra* H, we mean H is a poset with all finite products and finite coproducts and Cartesian closed, i.e. a lattice with $0, 1$ and for all $x, y \in H$, y^x exists.

Note that $-^x$ is the right adjoint to $x \times -$ and $- \times x$, so $\forall z z \leq y^x$ iff $z \wedge x \leq y$, usually we use $x \Rightarrow y$ to denote y^x. So, in notation, $z \leq (x \Rightarrow y)$ iff $z \wedge x \leq y$. Hence, $x \Rightarrow y$ is the sup of all z such that $z \wedge x \leq y$. In particular, in a lattice L where arbitrary sup exists, $x \Rightarrow y$ exists.

Exercise III.10. (i) A Boolean algebra is a Heyting algebra where $x \Rightarrow y$ is $\neg x \vee y$. (A Heyting algebra is distributive.)

(ii) Let X be a topological space, then the collection of open sets in X is a Heyting algebra where $U \Rightarrow V$ is the largest open set W such that $W \cap U \subseteq V$.

Remark III.11. We have the following easy facts.

(i) A Heyting algebra is distributive

(ii) In a Heyting algebra, we can define $\neg x$ as $x \Rightarrow 0$. For example, in a Heyting algebra of open sets of a topological space, $\neg U = (U^c)^\circ$, the interior of the complement of U. Note that $\neg x$ is the largest element u such that $u \wedge x = 0$.

(iii) H, a Heyting algebra is a Boolean algebra iff $\neg x \vee x = 1$ for all $x \in H$ iff $\neg\neg x = x$ for all $x \in H$.

An important example of Heyting algebra is $\mathrm{Sub}_{\mathcal{C}}(X)$ for $X \in \mathcal{C}_0$. Recall that a subobject of X is (an isomorphism class of) a monic $Y \rightarrowtail X$, wher an isomorphism is an isomorphism in this sense is an isomorphism that commutes with the monic arrows. Recall that in Definition III.1, we have a partial ordering on $\mathrm{Sub}_{\mathcal{C}}(X)$.
In Set, we have canonical representatives of subobjects of X, namely the images of the monic maps with inclusion. Via the above identification, $\mathrm{Sub}_{\mathsf{Set}}(X) \cong \mathcal{P}(X)$, the powerset of X, where the isomorphism above is actually an isomorphism of Boolean algebras. Namely, it preserves $\wedge, \vee, \neg, 0, 1$.
We can do similar constructions in $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. For the remaining part of the chapter, we use $\widehat{\mathcal{C}}$ to denote the category $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$. For each $F \in \widehat{\mathcal{C}}_0$, it is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. A subobject of F can be identified with a subfunctor G of F, where $G : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ and for each $x \in \mathcal{C}_0$, $G(x) \subseteq F(x)$ and for each $f : x \to y$, the following diagram commutes,

$$\begin{array}{ccc} F(y) & \xrightarrow{F(f)} & F(x) \\ \uparrow & & \uparrow \\ G(y) & \xrightarrow{G(f)} & G(x) \end{array}$$

, where the vertical arrows are inclusion maps.

Lemma III.12. *(i) Let $F \in \widehat{\mathcal{C}}_0$, then $\mathrm{Sub}_{\mathcal{C}}(F)$ with the canonical partial order is a Heyting algebra.*

(ii) Let $\varphi : F \Rightarrow G$ be a natural transformation. We can define $\varphi^{\sharp} : \mathrm{Sub}_{\widehat{\mathcal{C}}}(G) \to \mathrm{Sub}_{\widehat{\mathcal{C}}}(F)$ via pullback. Precisely, for each $G_1 \Rightarrow G$, a subobject, $\varphi^{\sharp}(G_1)(X) = \varphi_X^{-1}(G_1(X))$ for each $x \in \mathcal{C}_0$. Or we can see it through the following diagram.

$$\begin{array}{ccc} & & G_1(X) \\ & & \downarrow \\ F(X) & \xrightarrow[\varphi(X)]{} & G(X) \end{array}$$

$\varphi^{\sharp}(G_1)(X)$ is defined to be the pullback of it. Then we have that $\varphi^{\sharp}$ is a map between Heyting algebras that respescts $0, 1, \wedge, \vee, \leq, \Rightarrow$.

Proof of (i). Clearly the 0 element should be the empty functor and $1 = F$. We can defien $\leq$ pointwise, basically, say $G_1 \leq G_2$ iff $G_1(X) \leq G_2(X)$ for all $X \in \mathcal{C}$. Similarly, we defien $(G_1 \vee G_2)$ to be the functor such that $G_1 \vee G_2(X) = G_1(X) \cup G_2(X)$ and $(G_1 \vee G_2)(X) = G_1(X) \cap G_2(X)$ and arrows come from restricting arrows given by F.
However, the pointwise approach does not work for $\Rightarrow$ and $\neg$, they don't give subfunctors in general. We define, $(G_1 \Rightarrow G_2)(X) = \{x \in F(X) : \forall f : Y \to X \in \mathcal{C}_1, F(f) : F(X) \to F(Y), F(f)(x) \in G_1(Y) \text{ implies } F(f)(x) \in G_2(X)\}$. Likewise, for negation $(\neg G)(X) = \{x \in F(X) : \forall f : Y \to X, F(f)(x) \notin G(X)\}$. And the arrows come from restricting arrows given by F similarly. □

Remark III.13. Let us take a look at the above definition, the pointwise definition for $G_1 \wedge G_2$ works because whenever you have $f : X \to Y$, $G_i(f) : G_i(Y) \to G_i(X)$ is the restriction of $F(f)$ to $G_i(Y)$, and is defined. However, taking pointwise definition for negation does not work beacuse $F(f)$ maps $G(Y)$ into $G(X)$ does not necessarily guarantee that $G(Y)^c$ maps into $G(X)^c$.

Now, since we wish to develop logic, we need to define quantifiers categorically.
Naively, when working in Set. Let $f : Z \to Y$ be a function. Let $S \subset Z$. We can define $\exists_f(S) = \{y \in Y : \exists z \in S, f(z) = y\} = \{y \in Y : \exists z \in f^{-1}(y), z \in S\}$ and $\forall_f(S) = \{y \in Y : \forall z \in f^{-1}(y), z \in S\}$. Note, when f is the projection $p : X \times Y \to Y$, the above definitions agree with our usual notion of quantifiers.

Lemma III.14. *Work in* Set. *Let* $f : Z \to Y$ *be an arrow in* Set. *Let* $f^* : \mathcal{P}(Y) \cong \mathrm{Sub}_{\mathsf{Set}}(Y) \to \mathcal{P}(X) \cong \mathrm{Sub}_{\mathsf{Set}}(X)$, $Z \mapsto f^{-1}(Z)$. *Then* f^* *has a left adjoint* $\exists_f$ *and a right adjoint* $\forall_f$.

Proof. Note that f^* is induced by f^{-1} and the map $\exists_f : \mathcal{P}(Z) \to \mathcal{P}(Y)$ $U \mapsto f(U)$ is induced by f. And it can be easily checked that it is the left adjoint of f^*, namely, for $A \subseteq Z$ and $B \subseteq Y$, $\exists_f(A) = f(A) \subseteq B$ iff $A \subseteq f^{-1}(B)$. Likewise for $\forall_f$, for each $A \subseteq Z$, $\forall_f(A) = \{y \in Y : \forall z \in f^{-1}(y), z \in A\}$. Then for $A \subseteq Z$ and $B \subseteq Y$ $f^{-1}(B) \subseteq A$ iff $B \subseteq \forall_f(A)$. □

The above disgussion generalizes to the following.

Lemma III.15. *Let* $F, G \in \widehat{C}_0$, *let* $\varphi : F \Rightarrow G$ *be a natural transformation. Define* $\varphi^\sharp : \mathrm{Sub}_{\widehat{C}}(G) \to \mathrm{Sub}_{\widehat{C}}(F)$ *via pullback. Then* $\varphi^\sharp$ *has left and right adjoints. We call them* $\exists_\varphi$ *and* $\forall_\varphi$ *respectively.*

Proof. need to fill in the details □

Recall that $F \in \widehat{C} = \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is called a presheaf in $\mathcal{C}$. For every $X \in \mathcal{C}_0$, we have $F(X)$ as a set and $f : Y \to X$ gives $F(f) : F(X) \to F(Y)$. For each X, we can view $F(X)$ as $\{s : X \to E : \text{sections of } f\}$ for some total space E. $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is an elementary topos.

Lemma III.16. $\mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} = \widehat{C}$ *has all finite/small limits and colimits.*

Proof. □

III.3 More on limits

Recall that a presheaf $\mathcal{F}$ on $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{C}^{op} \to \mathsf{Set}$. The representable presheaves are those of the form $\mathrm{Hom}(-, A)$ for some $A \in \mathcal{C}_0$. The Yoneda lemma says that for any $\mathcal{F}$, $\mathcal{F}(X) \cong$ set of natural transformations from $\mathrm{Hom}(-, X)$ to $\mathcal{F}(-)$.
Now, assume that we have a functor $F : J \to \mathcal{C}$. For each $X \in \mathcal{C}_0$, to view X

as a cone, we need the following data $(\alpha_Y : X \to F(Y))_{Y \in J_0}$, which is a natural transformation between Δ_X and F. We use $\mathrm{cone}^F(X)$ to denote the set of all such natural formations and hence given a presheaf $\mathrm{cone}^F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. To say limit exists is the same as saying there is an object $A \in \mathcal{C}_0$ such that finding a cone on X is the same as finding a morphism $X \to A$, i.e. $\mathrm{Hom}(X, A) \cong \mathrm{cone}^F(X)$.

Proposition III.17. $\mathrm{cone}^F(X) \cong \lim \mathrm{Hom}(X, F(-))$.

Proof. $\lim \mathrm{Hom}(X, F(-))$, as X varies, we can view it as a presheaf on $\mathcal{C}$, and by Yoneda lemma, we have that it is in bijection with the set of all natural transformations $\alpha_Y : \mathrm{Hom}(Y, X) \to \lim \mathrm{Hom}(Y, F(-))$. Since giving a map to a limit is the same as giving each component maps, we have that the above is in bijection with the set of natural transformations $\alpha_{YZ} : \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y, F(Z))$. By Yoneda lemma again, when we vary Y, we can view the above as a presheaf and it will be in bijection with the set of natural transformations $\alpha_Z : X \to \mathrm{Hom}(X, F(Z))$, which by definition, is $\mathrm{cone}^F(X)$. ☐

As a corollary, we have the following.

Corollary III.18. $\mathrm{cone}^F(X) \cong \lim \mathrm{Hom}(X, F(-)) \cong \mathrm{Hom}(X, \lim F(-))$.

Theorem III.19. *Right adjoints preserve limiting cones.*

Proof. Let $L : \mathcal{C} \to \mathcal{D}$ be a functor and R is the right adjoint of L. Let $F : J \to C$ be a functor. Then we have $\mathrm{Hom}(X, R \lim F(-)) \cong \mathrm{Hom}(LX, \lim F(-)) \cong \lim \mathrm{Hom}(LX, F(-))$, where the latter bijection is by the above corollary. Now apply the property of adjoints again, $\lim \mathrm{Hom}(LX, F(-)) \cong \lim \mathrm{Hom}(X, RF(-)) \cong \mathrm{Hom}(X, \lim RF(-))$, where the latter bijection is by the above corollary again. But this is the same as saying, the limit of RF is the same as $R \lim F$. ☐

Next, we state a theorem that characterize the existence of small limits.

Theorem III.20. *Let $\mathcal{C}$ be a category, then $\mathcal{C}$ has small/finite limits iff $\mathcal{C}$ has equalizers and small/finite products.*

Proof. One direction is trivial since equalizers and products are limits. It suffices to prove the other one. The following proof works for both finite and small case by restricting J to be a small/finite category respectively.
Let $F : J \to \mathcal{C}$ be a functor. Then $\mathrm{cone}^F(X)$ is the set of natural transformations $\alpha_A : X \to F(A)$. Since Set has products, we have the above set is the same as $\{\alpha \in \prod_{A \in J_0} \mathrm{Hom}(X, F(A)) : \text{for all } f \in J_1,\ F(\alpha)\alpha_{\mathrm{dom}(f)} = \alpha_{\mathrm{cod}(f)}\}$. The above set is in bijection with $\{\alpha \in \prod_{A \in J_0} \mathrm{Hom}(X, F(A)) : (F(f)\alpha_{\mathrm{dom}(f)})_{f \in J_1} = (\alpha_{\mathrm{cod}(f)})_{f \in J_1}\}$. By the universal property of products(in $\mathcal{C}$), we have the above set is in bijection with $\{\alpha \in \mathrm{Hom}(X, \prod_{A \in J_0}(F(A)) : (F(f)\mathrm{proj}_{\mathrm{dom}(f)})\alpha = (\mathrm{proj}_{\mathrm{cod}(f)})\alpha\}$. Note that the last condition is a equalizer diagram.

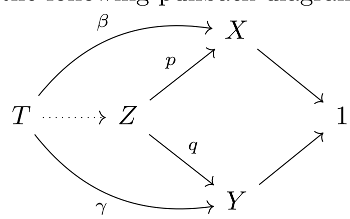
$$X \xrightarrow{\alpha} \prod_{A \in J_0}(F(A)) \underset{\mathrm{proj}_{\mathrm{cod}(f)}}{\overset{F(f)\mathrm{proj}_{\mathrm{dom}(f)}}{\rightrightarrows}} F(\mathrm{cod}(f))$$

Hence it is in bijection with $\mathrm{Hom}(X, \mathrm{Eq}((F(f)\mathrm{proj}_{\mathrm{dom}(f)})_{f\in J}, (\mathrm{proj}_{\mathrm{cod}(f)})_{f\in J_1}))$ by the universal property of equalizers. And by our discussion preceeding the proposition, it is the same as saying that $\mathrm{Eq}((F(f)\mathrm{proj}_{\mathrm{dom}(f)})_{f\in J}, (\mathrm{proj}_{\mathrm{cod}(f)})_{f\in J_1})$ is the limit of F. $\square$

Similarly, we have the following statement.

Theorem III.21. *Let $\mathcal{C}$ be a category, then $\mathcal{C}$ has finite limits iff $\mathcal{C}$ has pullbacks and a terminal object.*

Proof. Since pullbacks and terminal objects are finite limits, we have one direction is trivial. It suffices to show the reverse direction.
First,given $X, Y \in \mathcal{C}_0$, let 1 denote the terminal object in the category. Consider the following pullback diagram.



Clearly, Z satisfies the universal property of $X \times Y$. Hence finite products exists in $\mathcal{C}$. It remains to show that equalizers exists. let $f, g : X \to Y$, then the equalizer of f, g is the pullback of the following.

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow[f\times g]{} & Y \times Y \end{array}$$

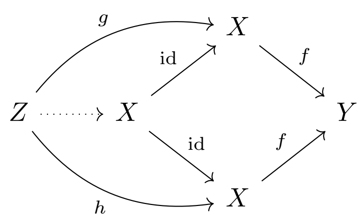
Hence we can conclude the theorem from the previous theorem. $\square$

Theorem III.22. *$f : X \to Y$ is monic iff*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback.

Proof. f is monic iff for all $g, h : Z \to X$, $fg = fh$ implies $g = h$. Hence the diagram is a pullback.
Conversely, if the above diagram is a pullback, then for $g, h : Z \to X$ such that $fg = fh$, the following diagram (without the dotted arrow) commutes.



no matter I interpret the dotted arrow as g or h, the above diagram commutes, and hence by the universal property of pullback (uniqueness of the dotted arrow), we have that $g = h$. Hence f is monic.

□

III.4 Elementary Topos

In this section, we wish to show that $\hat{\mathcal{C}}$ is an elementary topos. In particular, we show that $\hat{\mathcal{C}}$ has exponentials and a subobject classifier.Then, we will give some more definitions regarding toposes. Recall that $y : \mathcal{C} \to \hat{\mathcal{C}}$ is the Yoneda where $y(C) = y_C$ and $y_C(C') = Hom_{\mathcal{C}}(C', C)$.

Lemma III.23. *Every $X \in \hat{\mathcal{C}}$ is a colimit of y_C's.*

Proof. Recall that the Yoneda lemma gives a natural bijection between elements of $X(C)$ (for $X \in \hat{\mathcal{C}}$, $C \in \mathcal{C}$) and arrows $y_C \Rightarrow X$. If $x \in X(C)$, we have some $\mu_x : y_C \to X$ such that $(\mu_x)_C(id_C) = X$ (i.e. $(\mu_x)_C : y_C \to X(C)$). Let $y \downarrow X$ be the category whose objects are pairs (C, μ) where $C \in \mathcal{C}$ and $\mu : y_C \Rightarrow x$ and arrows between $(C, \mu) \to (C', \nu)$ are given by $f : C' \to C$ in $\mathcal{C}$ such that:

$$
\begin{array}{ccc}
y_C & \rightarrow_{y_f} & y_{C'} \\
{}_{\mu}\searrow & & \swarrow_{\nu} \\
& X &
\end{array}
$$

Let $U_X : y \downarrow X \Rightarrow \mathcal{C}$ be the forgetful functor where $(C, \mu) \mapsto C$ and $f \mapsto f$. So, $y \circ U_X : y \downarrow X \to \hat{\mathcal{C}}$. Notice that $y \circ U_X$ is a diagram in $\hat{\mathcal{C}}$ with $y \downarrow X$ as its indexing category. Let ρ be the natural transformation from $y \circ U_X$ to Δ_X (where Δ_X is the constant functor at X) from $y \downarrow X$ to $\hat{\mathcal{C}}$.

$\rho_{(C,\mu)}$ is a map from $(y \circ U_X)(C, \mu)$ to X. I.e. $y_c \Rightarrow X$ and is precisely μ. Notice that ρ is a natural transformation from $(y \circ U_X)$ to the constant functor at X (otherwise known as a cocone from $y \circ U_X$ to X). We claim that this cocone is colimiting. □

Proposition III.24. *ρ is colimiting.*

Proof. Given $Z \in \hat{\mathcal{C}}$ and given $\gamma : y \circ U_X \Rightarrow \Delta_Z$ we want to find a $g : X \to Z$ such that $\Delta_g \circ \rho = \gamma$. Claim: $g_C(\mu) = \gamma_{(C,\mu)}(id_C)$ is the unique solution. Notice that the domain of g_C is $X(C)$ and we have used the Yoneda lemma to identify

$X(C)$ with the set of natural transformation from y_C to X. The verification that g is natural, $\Delta_g \circ \rho = \gamma$, and the uniqueness are routine. □

In fact, the Yoneda embedding is the free colimit completion of $\mathcal{C}$.Whenever $F : \mathcal{C} \to \mathcal{D}$ where D is cocomplete, there is a unique (up to isomorphism) colimit preserving functor $\tilde{F} : \hat{\mathcal{C}} \to D$ such that $\tilde{F} \circ y \cong F$. Concretely $\tilde{F}(X)$ is the colimit in $\mathcal{D}$ of the diagram $y \downarrow X \to_{U_X} \mathcal{C} \to_F D$. Also note that F is the left Kan extension of F along y.

When $\mathcal{C}$ is the 1 point category, then $\hat{\mathcal{C}} \cong \mathsf{Set} \cong \mathsf{Set}^1$.

Proposition III.25. *$\hat{\mathcal{C}}$ has exponentials*

Proof. Let $X, Y \in \hat{\mathcal{C}}$. Then, we define Y^X as follows: $Y^X : \mathcal{C}^{\text{op}} \to \mathsf{Set}$ where $Y^X(C) = \hat{\mathcal{C}}(y_C \times X, Y) = Hom_{\hat{\mathcal{C}}}(y_C \times X, Y)$ and if $f : C' \to C$, then $Y^X(f) : \hat{\mathcal{C}}(y_{C'} \times X, Y) \to \hat{\mathcal{C}}(y_C \times X, Y)$ which is precisely the composition with $y_f \times id_X : y_{C'} \times X \to y_C \times X$.

Now we need to show that Y^X is the required exponential. It suffices to show that for any Z in $\hat{\mathcal{C}}$ there is a natural bijection between $\hat{\mathcal{C}}(Z, Y^X)$ and $\hat{\mathcal{C}}(Z \times X, Y)$. Notice that when Z is representable by y_C, the Yoneda Lemma gives a natural bijection between $\hat{\mathcal{C}}(y_C, Y^X)$ and $Y^X(C) = \hat{\mathcal{C}}(y_C \times X, Y)$. This extends to arbitrary Z in $\hat{\mathcal{C}}$ by III.23. □

Proposition III.26. *$\hat{\mathcal{C}}$ has a subobject classifier.*

Proof. We want to show that there exists 1 and Ω in $\hat{\mathcal{C}}$ and a morphism $true : 1 \to \Omega$ such that for every $Y \rightarrowtail X$ there exists a map from X to Ω such that the following diagram is a pull-back:

$$\begin{array}{ccc} Y & \to & 1 \\ \downarrow & & \downarrow \\ X & \to & \Omega \end{array}$$

Notice that if Ω exists, then Ω has the property that $\forall F \in \hat{\mathcal{C}}$ there exists a bijection between $Sub_{\hat{\mathcal{C}}} \leftrightarrow Hom_{\hat{\mathcal{C}}}(F, \Omega)$ natural in F. Define Ω as follows: we let $\Omega(C) =$ the set of all subfunctors of y_C. If $f : C' \to C$, let $\Omega(f) : \Omega(C') \to \Omega(C)$ be the pull back along y_f. By this, we mean the $f^*(A)$ in the following pullback diagram:

$$\begin{array}{cccc} C & \leftarrowtail & A & \in \Omega(C) \\ \uparrow_f & & \uparrow & \\ C' & \leftarrow & f^*(A) & \in \Omega(C') \end{array}$$

We define $1 : \mathcal{C} \to Set$ as $1(C) = \{\emptyset\}$. So, $true : 1 \to \Omega$ is a natural transformation which is a map which takes the unique element $1(C)$ to $y_C \in \Omega(C)$.

So, if $X = y_C$ for some C, then $Hom(y_C, \Omega) \cong \Omega(C) \cong Sub(y_C)$ □

As a remark, all of this can be expressed in the language of Sieves. So, $y_C : \mathcal{C}^{op} \to \mathsf{Set}$ takes C' to $Hom_{\mathcal{C}}(C', C)$ (which is a functor). A subfunctor R of y_C takes $C' \in \mathcal{C}$ to a subset of $Hom_{\mathcal{C}}(C', C)$. So the subfunctor R can be viewed as a collection of arrows in the category $\mathcal{C}$ with codomain C, i.e. $\bigcup_{C' \in \mathcal{C}} R(C')$.

Now, the naturalness means if $f : C' \to C$ is in R ($R(C')$) and $g : C'' \to C'$, then $f \circ g : C'' \to C$ is in R. Such an R is called a sieve on C. Therefore, $\Omega(C)$ is the collection of associated sieves on C. More explicityl, we have that if $f : C' \to C$ is in $\mathcal{C}$ and R is a sieve on C then $\Omega(f)(R) = f^*(R) = \{g : D \to C' | fg \in R, D \in \mathcal{C}\}$. Finally, $true : 1 \to \Omega$ takes $1(C)$ to the maximum sieve on $\Omega(C)$ (which is y_C itself).

Therefore, we can conclude that $\hat{\mathcal{C}}$ is an elementary topos.

Definition III.27. Let C be a category and let $X \in C$. Then A is a power object of X if there is a natural one-to-one correspondence between $C(Y, A) \cong Sub_C(Y \times X)$ for any $Y \in C$.

Proposition III.28. *An elementary topos also has power objects. The power object for X is Ω^X.*

Definition III.29. A category is called cartesian if it has all finite limits. A functor between cartesian categories is called cartesian if it preserves all finite limits (also called left exact).

Definition III.30. Regular Categories.

- $\mathcal{C}$ has images; Let $A, B \in \mathcal{C}$ and if $f : A \to B$ there exists a smallest subjobject $C \rightarrowtail B$ through which f factors.
- A regular epimorphism $f : B \to C$ is an epimorphism which is a coequalizer, i.e. $\exists A$ such that
$$A \rightrightarrows^{g_1}_{g_2} \to B \to_f C$$
- A category $\mathcal{C}$ is regular if $\mathcal{C}$ is cartesian, has images, and regular epimorphisms are stable under pull-back, i.e. if
$$\begin{array}{ccc} X_1 & \to & X_2 \\ a \downarrow & & \downarrow f \\ X_3 & \to & X_4 \end{array}$$
is a pull-back and f is a regular epimorphism, then so is a.

Proposition III.31. *$\mathcal{C}$ has images iff for any morphism $f : A \to B$ $f^* : Sub_{\mathcal{C}}(B) \to Sub_{\mathcal{C}}(A)$ has a left adjoint $\exists_f$.*

Definition III.32. Assume $\mathcal{C}$ has images. Let $f : A \to B$ and assume that C is the image of f. Then, $g : A \to C$ is the cover of f.

Proposition III.33. *In a regular category, the covers are regular maps precisely when the epimorphisms are regular.*

Proof. ☐

Fact III.34. *In a regular category, coeualizers need not exist but we always have coequalizers of "Kernal Pairs".*

Proof. Assume that $f : X \to Y$ is a kernal pair. Then, the following diagram in a pull-back:

$$\begin{array}{ccc} Z & \to^{p_1} & X \\ \downarrow_{p_2} & & \downarrow_f \\ X & \to_f & Y \end{array}$$

Then, the image of f will be an equalizer for $Z \rightrightarrows X$. ☐

Definition III.35. A regular functor between regular categories is one which preserves finite limits.

Example III.36.

- Set and Groups are regular categories and covers are surjective maps.
- The category of monoids is regular.
- Top is **not** a regular category since covers (surjective continuous maps) are not stable under pullbacks. Consider the map $f : [0, 1) \to \mathbb{R}/\mathbb{Z}$.
- CAT is not regular.

Definition III.37. A category $\mathcal{C}$ is coherent if it is regular and for each $A \in \mathcal{C}_0$, $Sub_{\mathcal{C}}(A)$ has finite coproducts.

Definition III.38. $\mathcal{C}$ is positive if it has disjoint unions. If $A_1, A_2 \in \mathcal{C}$ then there exists $f_1 : A_1 \rightarrowtail A$ and $f_2 : A_2 \rightarrowtail A$ such that

$$\begin{array}{ccc} 0 & \to & A_1 \\ \downarrow & & \downarrow_{f_1} \\ A_2 & \to_{f_1} & A \end{array}$$

is a pullback.

In a positive category, coproducts have a special property. Namely, if $A_1 + A_2$ is the coproduct of A_1, A_2, there there exists $f_1 : A_1 \to A_1 + A_2$ and $f_2 : A_1 + A_2$ such that both f_1, f_2 are monic.

What is an equivalence relation? Well, in the set theory case, R is a relation on $A \times A$ which is reflexive, symmetric, and transitive. They bring this into the category theory context. Let $A \in \mathcal{C}$ and let R be a subobject of A. Now we consider the following diagram:

$$\begin{array}{ccccc} & & & & A \\ & & & \pi_1 \nearrow & \\ R & \rightarrowtail & A & & \\ & & & \pi_2 \searrow & \\ & & & & A \end{array}$$

Notice that R is determined by its cordinate maps.

Definition III.39. (Equivalence Relation) Let $A \in \mathcal{C}$. We say that $R \in \mathcal{C}$ is an equivalence relation on A if R is a subobject such that:

- $\Delta_A : A \to A \times A$ factors through R.
- (reflexive) $\exists r : A \to R$ such that $\pi_1 r = \pi_2 r = id_A$
- (symmetry) $\exists s : R \to R$ such that $\pi_1 s = \pi_2$ and $\pi_2 s = \pi_1$
- (transitivity) Consider the pullback below. Now, $\exists t : P \to R$ such that $\pi_1 t = \pi_1 p$ and $\pi_2 t = \pi_2 q$

$$\begin{array}{ccc} P & \to_q & R \\ \downarrow_p & & \downarrow_{\pi_1} \\ R & \to_{\pi_2} & A \end{array}$$

Example: A kernal pair is an equivalence relation.

Definition III.40. (Effective, Pre-topos, Boolean)

- A Coherent category, $\mathcal{C}$ is effective if every equivalence relation in $\mathcal{C}$ is given by a kernal pair.
- A pre-topos is a coherent category that is positive and effective.
- A coherent category is Boolean if $Sub_{\mathcal{C}}(X)$ is a boolean algebra for all $X \in \mathcal{C}$

For example, For any first order theory T, $Def(T^{eq})$ is a Boolean Pre-topos.

III.5 Grothendieck Topologies and Sheaves

First, Let X be a topological space and let $O(X)$ be the category of open sets viewed as a poset where if $V \subset U$ we let i_{VU} be the inclusion morphism. Therefore, if F is a presheaf on $O(X)$, then $F : O(X) \to \mathsf{Set}$ where:

- if $U \in O(X)$, then $F(U) \in \mathsf{Set}$.
- if $V \subset U$, we can view this relation as the canonical inclusion map of $i_{VU} : V \to U$. Then $F(i_{VU}) : F(U) \to F(V)$.

For example, we let $X = \mathbb{R}$, the $O(X)$ is the collection of open subsets of the reals. Let $F(U)$ be the collection of continuous real valued functions on U. Assume that $V \subset U$. Then, we let $F(i_{VU}) : F(U) \to F(V)$ be the restriction map. What makes a presheaf into a sheaf? Well, we want to be able to glue things together.

Definition III.41. (Gluing Axiom) Assume that $U \subset_{open} X$ and let $\{U_i\}_{i \in I}$ be an open cover of U, i.e. $\bigcup_{i \in I} U_i = U$. Let $x_i \in F(U_i)$. Then, we say that the collection $\{x_i\}_i$ are compatible if $F(i_{(U_i)(U_i \cap U_j)})(x_i) = F(i_{(U_j)(U_i \cap U_j)})(x_j)$ for each pair x_i, x_j. If this is the case, then $\exists! x \in F(U)$ such that $F(i_{(U)(U_i)})(x) = x_i$.

Definition III.42. A sheaf is a presheaf which satisfies the gluing axiom.

Definition III.43. A presheaf is called seperated if for any $U \subset_o penX$ and $x, y \in F(U)$ and convering $\{U_i\}_i$ of U, we have that if $F(i_{U_i U})(x) = F(i_{U_i U})(y)$ for each i then $x = y$.

Example III.44. Let $X = \mathbb{R}$. $F(U)$ be the collection of bounded continuous functions on U and $F(i_{VU})$ is the corresponding restriction map. Then F is a separeated presheaf, but not a sheaf.

Recall that a *sieve* on C is a subfunctor of the Yoneda embedding y_C, which is also the functor $Hom(-, C)$.)

It is easy to see that a monic arrow in the functor category is a precisely a natural transformation η where each component η_C is a monomorphism in SET, such that the relevant diagrams commute. Thus if R is a subfunctor of y_C, each $R(D)$ can be identified with a set of maps from D to C. We will make this identification from here forward.

Definition III.45. A **Grothendieck topology** on a category $\mathcal{C}$ is an assignment J of each object $C \in \mathcal{C}$ to a family of sieves over C, $J(C)$, called covering sieves of C. Each family $J(C)$ has the following properties:

- The maximal sieve y_C is in $J(C)$ for each $C \in \mathcal{C}$.
- For every subfunctor R of y_C and arrow $f : C' \to C$ define the collection $f^*(C)$ of arrows $g : C \to C'$ with $f \circ g \in R(\text{dom}(g))$. If $R \in J(C)$ and $f : C' \to C$ then $f^*(R) \in J(C)$.
- *Transitivity:* Whenever R is some sieve over C and $S \in J(C)$ such that $f \in S$ implies $f^*(R) \in J(dom(f))$, then $R \in J(C)$.

Definition III.46. A **site** is a category $\mathcal{C}$ equipped with a Grothendieck topology.

Definition III.47. A **basis** for a Grothendieck topology (also known as a *pretopology*) on a category $\mathcal{C}$ is a family $\{K(C) : C \in \mathcal{C}\}$ of morphisms (sometimes denoted *Cov(C)*) with codomain C with the following properties:

- *Every set covers itself:* The singleton $\{id_C : C \to C\}$ is in $K(C)$ for each C.
- *Stability under pullbacks, or, a cover of a set leads to a cover of a subset:* If $\{f_i : i \in \Delta\} \in K(C)$ and $g : D \to C$ then each pullback $D \times_C dom(f_i)$ exists; namely we have the following pullback diagram:

$$\begin{array}{ccc} D \times_C dom(f_i) & \xrightarrow{\varphi_i} & dom(f_i) \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle f_i} \\ D & \xrightarrow{\quad g \quad} & C \end{array}$$

  Moreover, the family $\{g_i : i \in \Delta\}$, is in $K(C)$.

- *Refinements of covers lead to covers:* Suppose $\{f_i : C_i \to C\} \in K(C)$ and for all i, $\{g_{ij} : D_{ij} \to C_i\} \in K(C_i)$. Then $\{f_i \circ g_{ij}\} \in K(C)$.

Exercise III.48. If a category $\mathcal{C}$ has a basis for a Grothendieck topology, then the family $\{J(C) : C \in \mathcal{C}\}$, where each $J(C)$ is the set of sieves on C containing some $f \in K(C)$, is a Grothendieck topology on $\mathcal{C}$.

Example III.49.

1. TOP is a site, where for each open set U, $K(U)$ is the collection of open covers of U. More precisely, $K(U)$ is the collection of sets of inclusions $\{f_i : U_i \to U\}$ such that $\cup_i dom(f_i) = U$.
2. The coarsest Grothendieck topology $\{J(C) = \{y_C\}\}$ is a Grothendieck topology.
3. The finest Grothendieck topology $\{J(C) : C \in \mathcal{C}\}$, where each $J(C)$ is the collection of all subfunctors of y_C, is a Grothendieck topology. In this context, we denote $J(C)$ by $\Omega(C)$.

Definition III.50. Let $\mathcal{C}$ be a site with Grothendieck topology $\{J(C) : C \in \mathcal{C}\}$. A **compatible family** of a sieve $R \in J(C)$ with a presheaf $F \in SET^{\mathcal{C}^{OP}}$ is a family of *elements* $\{x_f : f \in R(dom(f))\}$ such that:

- If $f : C' \to C$ is in R then $X_f \in F(C')$ and if $g : C'' \to C$ is any arrow then $x_{fg} \in F(C'')$.

Recall that F is a functor to SET, so talking about elements of $F(D), F(E)$, etc. makes sense.

Exercise III.51. Such a compatible family is "precisely" an arrow $R \to F$ in the category of functors. Recall that an arrow is a natural transformation in this category.

Solution. (⇒) Let $\eta : R \Rightarrow F$ be a natural transformation. Define, for each f such that $f \in R(dom(f))$, $x_f := \eta(f)$. Recall $\eta_{dom(f)} : R(domf) \to F(dom(f)$ so this notation makes sense. We claim $\{x_f\}$ is a compatible family.
Pick some $f : D \to C$ such that $f \in R(dom(f)$ and let $g : E \to D$ be any morphism. Since R is a subfunctor of y_C, there is a monic natural transformation $\epsilon : R \to y_C$ such that $\epsilon_E(R(g)[f]) = y_C(g)[\epsilon_D(f)]$ [To self: insert the relevant diagram]. Since ϵ is monic, each component is an inclusion in SET, so we may write $R(g)[f] = y_C(g)[f]$. But by definition $y_C(g)[f] = Hom(g, C)[f] = f \circ g$ so $R(g)[f] = f \circ g$.
Now since η is a natural transformation, we have the following commutative diagram: [Insert] Thus $\eta_E(R(g)[f]) = F(g)[\eta_d(f)]$ which implies $x_{f\circ g} = F(g)[x_f]$ as was desired.
(⇐) Conversely, let $\{x_f\}$ be a compatible family. Define, for each object D, η_D element-wise by letting $\eta_D(f) = x_f$. In order to show that $\eta = \{\eta_D : D \in \mathcal{C}\}$ is a natural transformation, it remains only to show that the relevant diagrams commute. But for each $g : D \to E$, the relevant diagram commutes precisely by definition of compatible family and since $R(g)[f] = f \circ g$ for each $f \in R(dom(f))$. □

Definition III.52. An **amalgamation** of such a compatible family is some $x \in F(C)$ such that $x_f = F(f)[x]$ for all $f \in R$.

Definition III.53. A preseheaf F is a **sheaf** if every compatible family has a unique amalgamation. More precisely, let $\mathcal{C}$ be site with Grothendieck topology J. A presheaf is a sheaf with respect to J if for every $J(C)$ and every $R \in J(C)$, every family $\{x_f\}$ of R with F has a unique amalgamation.

Definition III.54. A presheaf is called **separated** if every compatible family has at most one amalgamation.

Definition III.55. A **Grothendieck topos** is a category of sheaves on a site; namely, let $\mathcal{C}$ be a site with topology J. Consider the collection of all presheaves which are sheaves with respect to J. This collection forms a category which is called the Grothedieck topos over $\mathcal{C}$.

Exercise III.56.

1. If $\mathcal{C}$ is site with respect to the coarsest Grothendieck topology J then every presheaf is a sheaf.
2. Every Grothendieck topos is an elementary topos.

We now proceed to discuss the correspondence between sheaves and so-called ètale bundles. In fact, ètale bundles are often called sheaves.

Definition III.57. A map $p : E \to X$ between topological spaces is a local homeomorphism if for every point $e \in E$ there is an open set U of E containing e and an open set $V \subset X$ such that the restriction map $p \restriction_U$ is a homeomorphism.

Remark III.58. Every local homeomorphism is continuous.

Definition III.59. An **ètale bundle** is a map $p : E \to X$ between topological spaces which is a local homeomorphism.

Remark III.60. Given the above definition, one may ask why one would call a local homeomorphism an ètale bundle if the two notions are precisely the same. The answer is that the local homeomorphism contains all of the *data* of an ètale bundle, but the bundle properly speaking is a tuple $(p, E, (E_x)_{x \in X}, X)$. For each $x \in X$ define $E_x = p^{-1}(\{x\})$; we call this set the **stalk over x** or the fiber over x. However, this latter definition conflicts with another notion we shall call a *fiber* later, so we will not use it. The elements of each stalk E_x are called germs at x. We call $E = \cup E_x$ the **stalk space** and X the **base space**, and the whole bundle is sometimes called a *bundle of stalks over* X. This terminology is motivated by the following picture: [To self: add picture]

Example III.61. The map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a local homeomorphism and thus an ètale bundle. [To self: add diagram]

Definition III.62. A function $s : X \to E$ is a **section** of a bundle $p : E \to X$ if $s(x) \in E_x$ for each $x \in X$. In other words, s is precisely any continuous function which "picks" a germ from the stalk above x; one can think of "slicing" the stalks horizontally, motivating the use of the word 'section'. Equivalently, s is a function so that for all x, $p \circ s = id_X$. The section of an open set $U \subseteq X$ is a continuous map s such that $p \circ s = id_U$.

Lemma III.63. *Let X be a topological space whose Grothendieck topology is given by basis with $K(U)$ being the collection of open coverings of U, for each open U. Then every sheaf over X corresponds to an ètale bundle and vice verse.*

Proof. ($\Leftarrow$) Let $p : E \to X$ be an ètale bundle. We define a sheaf F over X as follows:

- On objects, define $F(U)$ to be the set of sections of U.
- On morphisms, for each inclusion $i : U \to V$, define $F(i)$ to be the "restriction" function which, on input $s \in F(V)$, outputs the function $s \restriction_U$. Note that $s \restriction_U$ is clearly a section on U.

It remains to check that F is a sheaf.

($\Rightarrow$) Let F be a sheaf in on a topological space X. We define an ètale bundle over X. Let U, V be open subsets of X containing $x \in X$ let $s \in F(U)$ and $t \in F(V)$. We say that $s \sim_x t$ or that *s,t have the same germ at x* if there is an open neighborhood $W \subseteq U \cap V$ containing x such that $s \restriction_W = t \restriction_W$, where $s \restriction_W$ is defined as $F(i)[s]$, where i is the inclusion $i : U \Rightarrow V$.
Note briefly this is a generalization of the case when F is the functor taking U to itself and sending inclusions to literal restrictions (To self: Is this correct???), we can identify each $s \in F(U)$ with U and define, for each x, an equivalence relation $U \sim_x V$ on open sets containing x, when "U, V look the same locally at x," or when there is a $W \subseteq U$ such that $W \cap U = W \cap V$. This motivates why we should say that $s \sim_x t$ if "s, t have the same germ at x."

One must check $s \sim_x t$ is an equivalence relation. Let E_x be the set of equivalence classes with x fixed but U, V, s, t vary. Note that each E_x is disjoint. Define $E = \cup_{xX} E_x$. Define $p : E \to X$ by sending each $e \in E_x$ to x. This is the ètale bundle we want. In order for this map to be a local homeomorphism, we must define a topology on E.

For each open set U in X and each $s \in F(U)$, define $\tilde{s}(U)$ to be the collection of germs of s at x for each $x \in U$. We let the $\tilde{s}(U)$'s be a basis for the desired topology. It only remains to be shown that p is a local homeomorphism. □

Chapter IV

Categorical Logic

IV.1 Categorical Semantics

Let L be a many-sorted, finitary language with propositional symbols $\top$, $\bot$, and let $\mathcal{E}$ be an elementary topos. In this section, we define the notion of an $\mathcal{E}$-valued L structure, and explain the semantic interpretation of the L-terms and L-formulas in such a structure.

In the following definition, it is useful to note that if $\mathcal{E}$ is Set, we recover the notion of L-structure familiar to model theory.

Definition IV.1 ($\mathcal{E}$-valued L-structures)**.** Suppose L is a many-sorted, finitary language and $\mathcal{E}$ is an elementary topos. An $\mathcal{E}$-valued L-structure M consists of the following:

1. For each sort X of L, an object $X(M)$ of $\mathcal{E}$,
2. For each relation symbol R in L of type $X_1 \times \cdots \times X_n$, a subobject $R(M)$ of the product $X_1(M) \times \cdots \times X_n(M)$ in $\mathcal{E}$,
3. For each function symbol f in L of type $X_1 \times \cdots \times X_n \to X$, a morphism $f(M) : X_1(M) \times \cdots \times X_n(M) \to X(M)$ in $\mathcal{E}$ and
4. For each constant symbol c in L of sort X, a morphism $c(M) : 1 \to X(M)$ in $\mathcal{E}$, where 1 is the terminal object of $\mathcal{E}$.

For the rest of the section, we let M be an $\mathcal{E}$-valued L structure. Before we can define the semantic value of L-formulas and L-sentences in M, we must assign interpretations to the L-terms.

Definition IV.2 (Interpretations of terms)**.** Suppose $t(x_1, \ldots, x_n)$ is an L-term of type $X_1 \times \cdots \times X_n \to X$. We assign to t a morphism $t(M) : X_1(M) \times \cdots \times X_n(M) \to X(M)$ in $\mathcal{E}$ inductively, as follows:

1. If t is a constant symbol c, where c has sort X, then $t(M)$ is $c(M)$.

2. If t is the variable x, where x has sort X, then $t(M)$ is $id_{X(M)}$.

3. If t is $f(t_1, \dots t_n)$ where $t_i : X_1 \times \cdots \times X_n \to Y_i$ and $f : Y_1 \times \cdots \times Y_n \to Z$, then $f(t_1, \dots, t_n)(M) : X_1(M) \times \cdots \times X_n(M) \to Z$ is the composition $f(M) \circ (t_1(M), \dots, t_n(M))$.

We now describe how to interpret an L-formula $\varphi(x_1, \dots, x_n)$, where x_i has sort X_i, as a subobject $\varphi(M)$ of $X_1(M) \times \cdots \times X_n(M)$ in $\mathcal{E}$. If φ is a sentence, then $\varphi(M)$ is a subobject of the terminal object 1.

We showed in Chapter 3 that if $\mathcal{E}$ is $Set^{\mathcal{C}^{op}}$, then for any $E \in \mathcal{E}$, $Sub_{\mathcal{E}}(E)$ is a Heyting algebra. Furthermore, if $f : C \to D$ is a morphism in $\mathcal{E}$, then $f^\sharp : Sub_{\mathcal{E}}(D) \to Sub_{\mathcal{E}}(C)$ has left and right adjoints $\exists_f$, $\forall_f : Sub_{\mathcal{E}}(C) \to Sub_{\mathcal{E}}(D)$. These facts are true in a general elementary topos $\mathcal{E}$, and we use them freely in the definition below.

Definition IV.3 (Interpretation $\varphi(M)$ of an L-formula φ)**.** Suppose $\varphi(x_1, \dots, x_n)$ is an L-formula where x_i is of sort X_i.

1. If φ is $t_1 = t_2(\bar{x})$ where t_1 and t_2 are L-terms of sort $X_1 \times \cdots \times X_n \to Y$, then $\varphi(M)$ is the equalizer of the following diagram:

$$X_1(M) \times \cdots \times X_n(M) \underset{t_2(M)}{\overset{t_1(M)}{\rightrightarrows}} Y(M) \quad .$$

2. If φ is $R(t_1(\bar{x}), \dots, t_n(\bar{x}))$, where each $t_i : X_1 \times \cdots \times X_m \to Y_i$ is an L-term and R is a relation of type $Y_1 \times \cdots \times Y_n$, then $\varphi(M)$ is the pullback

$$\begin{array}{ccc} \varphi(M) & \longrightarrow & R(M) \\ \downarrow & & \downarrow \\ \Pi X_i(M) & \xrightarrow{(t_1(M) \dots t_n(M))} & \Pi Y_i(M) \end{array} \quad .$$

3. If φ is $\top(x_1, \dots, x_n)$, then $\varphi(M)$ is the top subobject of $X_1(M) \times \cdots \times X_n(M)$; similarly, if φ is $\bot(x_1, \dots, x_n)$, then $\varphi(M)$ is the bottom subobject of $X_1(M) \times \cdots \times X_n(M)$.

4. If φ is $\varphi_1 \wedge \varphi_2(\bar{x})$, $\varphi_\vee \varphi_2(\bar{x})$, $\neg\varphi_1(\bar{x})$, or $\varphi_1 \Rightarrow \varphi_2(\bar{x})$, then $\varphi(M)$ is interpreted according to the Heyting algebra $Sub_{X_1(M) \times \cdots \times X_n(M)}$, i.e., $\varphi_1 \wedge \varphi_2(M)$ is $\varphi_1(M) \wedge \varphi_2(M)$, etc.

5. Finally, suppose φ is $(\exists y)(\varphi_1(\bar{x}, y))$ or $(\forall y)(\varphi_1(\bar{x}, y))$, where y has sort Y. Let $\pi : X_1(M) \times \cdots \times X_n(M) \times Y(M) \to X_1(M) \times \cdots \times X_n(M)$ be the projection. Then $(\exists y)(\varphi_1(\bar{x}, y))(M)$ is $\exists_\pi(\varphi_1(M))$, and $(\forall y)(\varphi_1(\bar{x}, y))(M)$ is $\forall_\pi(\varphi_1(M))$.

A theory T is a collection of formulas φ of L. $\varphi(\bar{x})$ is *valid* (true) in M if $\varphi(M)$ is the maximal subobject of $X_1(M) \times \cdots \times X_n(M)$. M is a model of T if every formula φ in T is valid in M. If φ is a sentence, then validity of φ means that $\varphi(M)$ is the maximal subobject of 1.

IV.2 Geometric Theories

Throughout this section, all topoi we consider are elementary topoi. In this section, we consider which functors preserve theories and which maps are the "good" maps between topoi.

Definition IV.4. Let $\mathcal{F}, \mathcal{E}$ be topoi. A **geometric morphism** $f : \mathcal{F} \to \mathcal{E}$ is a pair of functors (f^*, f_*) where $f_* : \mathcal{F} \to \mathcal{E}$, $f^* : \mathcal{E} \to F$, f^* is left adjoint to f_* and f^* is a left exact functor. We call f_* the **direct image** part of f and f^* the **inverse image** part of f.

Example IV.5. Let X, Y be Hausdorff topological spaces. Let $Sh(X)$ be the Grothendieck topos of sheaves over X considered as a site, and $Sh(Y)$ the same for Y. Then a geometric morphism $Sh(X) \to Sh(Y)$ is precisely a continuous map $f : X \to Y$. More precisely, to every such geometric morphism there corresponds such a continuous map and vice verse.

Proof. ($\Leftarrow$) Suppose $f : X \to Y$ is a continuous map. We first define f_*, the direct image functor. On objects, for each sheaf functor F in $Sh(X)$ we define the functor $f_*[F]$ in $Sh(Y)$ element-wise, as follows. For each open set V in Y, let

$$f_*[F](V) = F(f^{-1}(Y)).$$

Since f is continuous, $f^{-1}(Y)$ is open in X and the expression on the right hand side is therefore well-defined.
Defining f_* on arrows is left to the reader.

We now define f^*, the inverse image functor. On objects, for each sheaf functor G in $Sh(Y)$, we must define a sheaf functor $f^*(G)$ in $Sh(X)$. We will take advantage of the correspondence between sheaves and étale bundles. Let $p : E \to Y$ be the étale bundle over Y associated with G. We will construct, from p, an étale bundle over X, and then define $f^*(G)$ to be the sheaf in $Sh(X)$ associated with this bundle.
In the category of topological spaces, we let E' be the topological space that makes the following a pullback diagram:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

The map p' is the desired étale bundle. It remains for the reader to check that this will be a local homeomorphism.
Finally, one must show that (f^*, f_*) is an adjoint pair and that f^* is left exact.
($\Rightarrow$) Let (f^*, f_*) be a geometric morphism from $Sh(X)$ to $Sh(Y)$. We want to construct a continuous function $\bar{f} : X \to Y$. Note that f^* preserves finite limits, arbitrary colimits, and the terminal object. Therefore f^* takes subobjects of the terminal object in $Sh(Y)$ to subobjects of the terminal object in $Sh(X)$. Note that the terminal object in $Sh(Y)$ is the functor F that takes every open V to $F(V) = \{a\} = 1$ or equivalently, the identity bundle. The subobjects of F as defined above are the open subsets of V considered as subfunctors. So f^* takes open subsets of Y to open subsets of X. [To self: elaborate.]. In particular, $f^*(Y) = X$. Now f^* preserves finite intersections and arbitrary unions so we can define the map $\bar{f} : X \to Y$ such that $\bar{f}(x) = y$ if $x \in f^*(V)$ for all neighborhoods V of y in Y, by the Hausdorff condition. [To self: elaborate] We now check that $\bar{f}$ is well-defined. Now there is at most one such point y by the Hausdorff condition, since f^* preserves intersection, and since $f^*(\emptyset) = \emptyset$ [Proceeds by contradiction.] There is at least one such y since otherwise, for all $y \in Y$ there is a neighborhood V_y of y in Y such that $x \notin f^*(V_y)$. So

$$x \notin \cup_{y \in Y} f^*(V_y) = f^*\left(\cup_{y \in Y} V_y\right) = f^*(Y) = X.$$

But this is a contradiction.
It remains to be checked that $\bar{f}$ is continuous.
Note briefly that $f^*(V) = \bar{f}^{-1}(V)$, motivating the name *inverse image functor*
It remains to be shown that $\bar{f}_*$ is naturally isomorphic to f_*; that is, this map we have defined really does correspond to the geometric morphism in a strong way. Recall that we defined, for each $F \in Sh(X), V$ open in Y, $\bar{f}_*(F)[V] = F(\bar{f}^{-1}(V))$.
So by the Yoneda lemma,

$$\begin{aligned} \bar{f}_*(F)[V] &= F(\bar{f}^{-1}(V)) \\ &\cong HOM_{Sh(X)}(\bar{f}^{-1}(V), F) \\ &\cong HOM_{Sh(X)}(f^*(V), F) \\ &\cong HOM_{Sh(Y)}(V, f_*(F)) \cong f_*(F)[V] \end{aligned}$$

where the second to last natural isomorphism holds by definition of geometric morphism; in particular, by the adjunction. □

Lemma IV.6. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism between elementary topoi.*

...[There is a lacuna in the text]...

The following makes reference to Anand's numbering and needs to be adjusted once the previous contents are filled in.

Definition IV.7 (Geometric theory)**.** Let $\mathcal{L}$ be a first order finitary possibly many-sorted language. Recall that an $\mathcal{L}$-formula is geometric if it is built up from atomic formulas using $\wedge$, $\vee$ and $\exists$.

A *geometric theory* in $\mathcal{L}$ is a collection of $\mathcal{L}$-formulas, which are called *axioms*, of the form

$$\forall \bar{x}(\varphi(\bar{x}) \Rightarrow \psi(\bar{x}))$$

where φ and ψ are geometric formulas with variables among $\bar{x} = (x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$.

Note that in ordinary first-order model theory, first-order theories are usually assumed to be closed under logical consequence. However, we make no assumption on geometric theories.

Furthermore, note that geometric theories may include axioms of the form $\forall \bar{x}\varphi(\bar{x})$ and $\forall \bar{x}\neg\varphi(\bar{x})$, where $\varphi(\bar{x})$ is a geometric $\mathcal{L}$-formula with free variables among $\bar{x}$ since $\forall \bar{x}\varphi(\bar{x})$ is equivalent to the formula $\forall \bar{x}(\top \Rightarrow \varphi(\bar{x}))$, and similarly, $\forall \bar{x}\neg\varphi(\bar{x})$ is equivalent to $\forall \bar{x}(\varphi(\bar{x}) \Rightarrow \bot)$.

However, one should be careful to note that $\forall \bar{x}(\varphi(\bar{x}) \Rightarrow \psi(\bar{x}))$ is not a geometric formula even when φ and ψ are geometric formulas. In other words, one should not confuse geometric formulas with axioms of a geometric theory.

To make one last minor point, note that our logical (syntactical) symbol for implication is '$\Rightarrow$' rather than the usual '$\rightarrow$' seen in ordinary first-order model theory. We use the former to be consistent with Mac Lane and Mordeijk's notation.

Corollary IV.8. *Let T be a geometric theory in a fixed first-order and possibly many-sorted language $\mathcal{L}$. Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism (between topoi $\mathcal{F}$ and $\mathcal{E}$) where $f = (f^*, f_*)$. Let M be a model of T in the sense of $\mathcal{E}$.*

Then the inverse image $f^*(M)$ of M is a model of T in the topos $\mathcal{F}$.

Moreover, f^* induces a functor from the category of models of T in $\mathcal{E}$ to the category of models of T in $\mathcal{F}$.

Proof. Let T be a geometric theory and let $\forall \bar{x}(\varphi(\bar{x}) \Rightarrow \psi(\bar{x}) \in T$. For this axiom to be valid in M just means, by definition, that $\varphi(M) \leq \psi(M)$ as subobjects of the relevant product $X_1(M) \times \ldots X_n(M)$. By Theorem 4.11 and by the fact that f^* preserves the inclusion of subobjects (Lemma 4.8), it follows that $\varphi(f^*(M)) \leq \psi(f^*(M))$ as subobjects of $X_1(f^*(M)) \times \ldots \times X_n(f^*(M))$, which means that the axiom is valid in f^* as well.

Check the moreover part. □

Remark IV.9. Note that the previous theorem uses three claims that we will discuss in more detail, i.e. (i) Lemma 4.8 according to Anand's numbering, (ii) preservation of inclusion of subobjects by f^*, and (iii) the equivalence of the validity of $\forall \bar{x}(\varphi(\bar{x}))$ and $\varphi(\bar{x})$.

Remark IV.10. Note that Mac Lane and Mordeijk define the notion of an 'open' [4]. Mac Lane and Mordeijk prove that Theorems 4.11 and IV.8 hold for *all* formulas and *all* theories when f is an open geometric morphism.

Example IV.11. *(i)* Rings: Let $\mathcal{L}_{rings}$ be the (one-sorted) first-order language of rings, i.e. $\mathcal{L}_{rings} = \{+, \times, 0, 1\}$. We will often suppress the multiplication symbol when there is no ambiguity. The theory of commutative rings is a geometric theory consisting of the following axioms:

- $\forall x(1x = x)$
- $\forall x(0 + x = x)$
- $\forall x, y(xy = yx)$
- $\forall x, y(x + y = y + x)$
- $\forall x, y, z((xy)z = x(yz))$
- $\forall x, y, z((x + y) + z = x + (y + x))$
- $\forall x \exists y(x + y = 0)$
- $\forall x, y, z(x(y + z) = xy + xz)$

(ii) Local rings: Let $\mathcal{L}$ be $\mathcal{L}_{rings}$. Then the theory of commutative local rings is geometric and given by the following axioms:

- The axioms for commutative rings
- $\forall x((\exists y(xy = 1) \vee \exists y((1 - x)y = 1)))$.

Intuitively, local rings are rings that have a unique (proper) maximal ideal.

(iii) Linear (not strict) orderings with endpoints: Let $\mathcal{L} = \{\leq, b, t\}$ where b and t are the least and greatest elements. Then the theory of linear orderings with endpoints is geometric and given by the following axioms:

- $\forall x, y(x \leq y \vee y \leq x)$
- $\forall x(x \leq x)$.
- $\forall x, y((x \leq y \wedge y \leq x) \Rightarrow x = y)$
- $\forall x, y, z((x \leq y \wedge y \leq z) \Rightarrow x \leq z)$
- $\forall x(b \leq x \leq t)$
- $(b = t) \Rightarrow \bot$

(iv) Fields: Let $\mathcal{L} = \{+, \times, -, 0, 1\}$. The theory of fields is geometric and given by the ring axioms and the following axiom:

$$\forall x(x = 0 \vee \exists y(xy = 1)). \tag{IV.1}$$

Note that in the category **Set**, the last axiom is equivalent to

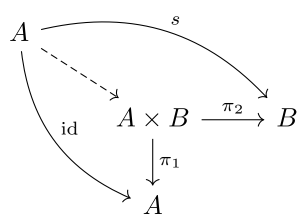
$$\forall x(\neg y(xy = 1) \Rightarrow x = 0) \tag{IV.2}$$

However, this equivalence does not hold in every topoi. For example, let X be a Hausdorff topological space, and let $\mathcal{E}$ be the category of sheaves on X. Then the sheaf of real-valued functions on X is a model of the ring axioms in the sense of $\mathcal{E}$ satisfying (IV.2) but not (IV.1).

Now, for M an $\mathcal{L}$-structure in $\mathcal{E}$, we want to define a category $\mathrm{Def}(\mathbf{M})$.

Remark IV.12. First, one should note that there is a correspondence between "functions" and their "graphs." in suitable categories where the composition of functions correspond to pullbacks of "graphs." In the case of **Set**, there is a trivial relationship between functions and graphs.

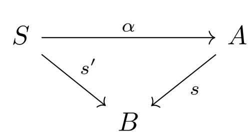
Definition IV.13. Let $\mathcal{E}$ be a topos, $s : A \to B$ a morphism in $\mathcal{E}$. The *graph* of s the subobject of $A \times B$ corresponding to the induced map $A \to A \times B$:



Note that this map is monic because $\mathrm{id} : A \to A$ is monic; one can verify monicity directly from the definition.

In Definition IV.13, we described the graph of a morphism $s : A \to B$ as a subobject of $A \times B$ by finding a particular monomorphism into $A \times B$. It is useful to remember that subobjects are defined only up to isomorphism. Lemma IV.14 characterizes when a subobject of $A \times B$ is equivalent (as subobjects) to the graph of s.

Lemma IV.14. *Let $s : A \to B$ be a morphism in a topos, and let S be an object together with a morphism $S \to A \times B$. By the universal property of $A \times B$, we may view this morphism as a pair (α, s'), with $\alpha : S \to A$ and $s' : S \to B$. Then (α, s') is monic and presents S as an equivalent subobject to the graph of s if and only if α is an isomorphism over B. That is, if α is an isomorphism and the following diagram commutes:*

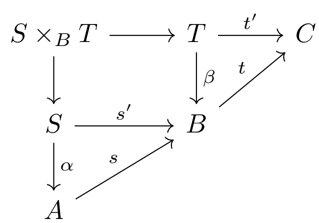


Proof. ($\Rightarrow$) Assume $(\alpha, s') : S \to A \times B$ is monic and equivalent to the graph of s as a subobject of $A \times B$. That is, recalling that (id, s) is the graph of s, there is an isomorphism $\beta : S \to A$ such that $(\mathrm{id}, s) \circ \beta = (\alpha, s')$. Consequently, $\beta = \alpha$ and $s \circ \alpha = s'$, as desired.

($\Leftarrow$) Assume α is an isomorphism over B. We want to show that α is moreover an isomorphism over $A \times B$ (monicity of the map into is immediate from this isomorphism and the monicity of $(\mathrm{id}, s) : A \to A \times B$). That is, we want to show that $(\mathrm{id}, s) \circ \alpha = (\alpha, s)$. Since it is sufficient by the universal property of products to check each coordinate separately, the result is immediate. □

Via Lemma IV.14, we can observe that the graph of the composition of two morphisms is the pullback of their individual graphs over the common domain/codomain.

Lemma IV.15. *Let $s : A \to B$ and $t : B \to C$ be morphisms in a topos, and let $(\alpha, s') : S \to A \times B$ and $(\beta, t') : T \to A \times B$ be their respective graphs. Then $S \times_B T$ is the graph of ts, were the relevant morphism into $A \times C$ is given by the following diagram:*



Proof. The result is immediate from Lemma IV.14 and the fact that both pullbacks of isomorphisms and compositions of isomorphisms are isomorphisms. □

We can now define the category $\mathrm{Def}(M)$, where M is a structure in a topos.

Definition IV.16. Let L be a language, $\mathcal{E}$ a topos, and M an L-structure in $\mathcal{E}$. By $\mathrm{Def}(M)$, we mean the following category:

- Objects are pairs (X, A), where $X = (X_1, \ldots, X_n)$ is a tuple of sorts and A is a subobject of $X(M)$ that arises as the interpretation in M of some geometric formula $\varphi(x_1, \ldots, x_n)$.
- Morphisms are $(X, A) \to (Y, B)$ are morphisms $s : A \to B$ in $\mathcal{E}$ whose graph, as a subobject of $X(M) \times Y(M)$, arises as the interpretation in M of some geometric formula $\sigma(\overline{x}, \overline{y})$.

Composition and identity morphisms are induced by the category $\mathcal{E}$.

Lemma IV.17. *The category* $\mathrm{Def}(M)$ *is well defined by Definition IV.16.*

Proof. We defined identity and composition morphisms by deferring to identity and composition in $\mathcal{E}$; we must check that the graphs of these morphisms arise as the interpretations of geometric formulas. (We must also check that identity and composition obey the necessary algebraic laws, but this follows immediately from the fact that $\mathcal{E}$ is a category obeying these laws.)

(Identity.) The graph of the identity morphism is given by coordinate-wise equality, which can be expressed as a finite conjunction of atomic equality assertions, and is therefore given by a geometric formula.

(Composition.) By Lemma IV.15, the graph of a composition is a pullback of the graphs of the morphisms being composed. Pullbacks of definable functions are expressible by geometric formulas: let $\sigma(x, z), \tau(y, z)$ be the graphs of $s : A \to C$ and $t : B \to C$, respectively. Then their pullback is defined by the formula $\exists z \in C : \sigma(x, z) \wedge \tau(y, z)$, where "$z \in C$" is shorthand for the formula defining C as a subobject of $Z(M)$. □

Note that Definition IV.16 of $\mathrm{Def}(M)$ gives rise to a canonical "forgetful" functor $\mathrm{Def}\, M \to \mathcal{E}$ by sending the object $(X, A) \in \mathrm{Def}(M)$ to $A \in \mathcal{E}$ and $f : (X, A) \to (Y, B)$ to $f : A \to B$. (To be needlessly technical, since the A in (X, A) is a subobject of $X(M)$, it only determines an equivalence class of objects in $\mathcal{E}$, rather that a specific object. Consequently, the forgetful functor is "weak" in that it is only defined up to unique isomorphism in the functor category, rather than as a specific functor in particular. In practice, since it is defined up to unique isomorphism, this distinction is irrelevant.)

Proposition IV.18. *The category* $\mathrm{Def}(M)$ *has a finite limits, and the forgetful functor* $\mathrm{Def}(M) \to \mathcal{E}$ *is left exact.*

Proof. Let $F : \mathrm{Def}(M) \to \mathcal{E}$ denote the forgetful functor. Note that F is by definition full and faithful, meaning that it induces bijections on hom sets:

$$\mathrm{Hom}_{\mathrm{Def}(M)}(X, Y) \cong \mathrm{Hom}_{\mathcal{E}}(F(X), F(Y))$$

Also, since $\mathcal{E}$ is a topos, it has finite limits. Letting J be a finite diagram in $\mathrm{Def}(M)$, we must check that $\lim_{X \in J} \mathrm{Hom}_{\mathrm{Def}(M)}(-, X)$ is representable in $\mathrm{Def}(M)$. Given the above observations, it suffices to check that the finite limits present in $\mathcal{E}$ arise as images of objects in $\mathrm{Def}(M)$ itself. Let $L \in \mathrm{Def}(M)$ be the assumed object such that $F(L) = \lim_{X \in J}(F(X))$. Then

$$\begin{aligned}
\lim_{X \in J} \mathrm{Hom}_{\mathrm{Def}(M)}(-, X) &\cong \lim_{X \in J} \mathrm{Hom}_{\mathcal{E}}(F(-), F(X)) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(F(-), \lim_{X \in J}(F(X))) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(F(-), F(L)) \\
&\cong \mathrm{Hom}_{\mathrm{Def}(M)}(-, L)
\end{aligned}$$

so $\mathrm{Def}(M)$ has finite limits which are preserved by F.

Now we show that the finite limiting cones (of diagrams that are themselves images under F) that exist in $\mathcal{E}$ are images of diagrams in $\mathrm{Def}(M)$. By Theorem III.20, it suffices to check finite products and equalizers. The result is then immediate, since the diagrams corresponding to finite products and equalizers are directly definable by geometric formulas. □

Given a topos $\mathcal{E}$, language $\mathcal{L}$, and $\mathcal{L}$-structure M in the sense of $\mathcal{E}$, we aim to define a *Grothendieck Topology* on $\mathrm{Def}(\mathbf{M})$.

Recall that a **Grothendieck topology** on a category $\mathcal{C}$ is an assignment J of each object $C \in \mathcal{C}$ to a family of sieves over C, $J(C)$, called covering sieves of C. Moreover, each family $J(C)$ has the following properties:

- The maximal sieve y_C is in $J(C)$ for each $C \in \mathcal{C}$ where y_C is the presheaf (functor) $\text{Hom}_{\mathcal{C}}(-, C)$.
- For every subfunctor R of y_C and arrow $f : C' \to C$ define the collection $f^*(C)$ of arrows $g : C \to C'$ with $f \circ g \in R(\text{dom}(g))$. If $R \in J(C)$ and $f : C' \to C$ then $f^*(R) \in J(C)$.
- *Transitivity:* Whenever R is some sieve over C and $S \in J(C)$ such that $f \in S$ implies $f^*(R) \in J(\text{dom}(f))$, then $R \in J(C)$.

Furthermore, recall that a **basis** for a Grothendieck topology (also known as a *pretopology*) on a category $\mathcal{C}$ is a family $\{K(C) : C \in \mathcal{C}\}$ of morphisms (sometimes denoted *Cov(C)*) with codomain C with the following properties:

- *Every set covers itself:* The singleton $\{id_C : C \to C\}$ is in $K(C)$ for each C.
- *Stability under pullbacks, or, a cover of a set leads to a cover of a subset:* If $\{f_i : i \in \Delta\} \in K(C)$ and $g : D \to C$ then each pullback $D \times_C \text{dom}(f_i)$ exists; namely we have the following pullback diagram:

$$\begin{array}{ccc} D \times_C \text{dom}(f_i) & \xrightarrow{\varphi_i} & \text{dom}(f_i) \\ \Big\downarrow g_i & & \Big\downarrow f_i \\ D & \xrightarrow{\quad g \quad} & C \end{array}$$

  Moreover, the family $\{g_i : i \in \Delta\}$, is in $K(C)$.
- *Refinements of covers lead to covers:* Suppose $\{f_i : C_i \to C\} \in K(C)$ and for all i, $\{g_{ij} : D_{ij} \to C_i\} \in K(C_i)$. Then $\{f_i \circ g_{ij}\} \in K(C)$.

It was an exercise to show that if a category $\mathcal{C}$ has pullbacks and a basis for a Grothendieck topology, then $\mathcal{C}$ has a Grothendieck topology. That is, suppose $\mathcal{C}$ has pullbacks and a basis. Then one can generate a Grothendieck topology by defining, for for each $C \in \mathcal{C}$, the set $J(C)$ of sieves S on C that contains some $R \in K(C)$.

Since we have shown that $\text{Def}(\mathbf{M})$ has finite limits, it follows that $\text{Def}(\mathbf{M})$ has pullbacks, and thus, we can show that $\text{Def}(\mathbf{M})$ has a Grothendieck Topology by defining a basis.

Definition IV.19. We define the following basis on $\text{Def}(\mathbf{M})$. For each object (B, Y) in $\text{Def}(\mathbf{M})$ where $B = \varphi(M)$ for some φ a geometric $\mathcal{L}$-formula and Y is a list of sorts $Y_1, \ldots Y_n$, a member of $K(B, Y)$, i.e. a cover for (B, Y), is a finite family of arrows in $\mathbf{Def}(\mathbf{M})$,

$$s_i : (A_i, X^{(i)}) \to (B, Y) \text{ for } i = 1, \dots, m.$$

such that the induced map $\amalg_i^m A_i \to B$ is an epimorphism in $\mathcal{E}$.

Lemma IV.20. *Let $\mathcal{E}$ be an elementary topos, $\mathcal{L}$ be a first order language, and M an $\mathcal{L}$-structure in the sense of $\mathcal{E}$. Moreover let (B, Y) be an object of* $\mathbf{Def}(\mathbf{M})$ *and let*

$$s_i : (A_i, X^{(i)}) \to (B, Y), \text{ where } i = 1, \dots, m$$

be a family of maps. Then the induced map $\amalg_i^m A_i \to B$ is an epimorphism in $\mathcal{E}$ if and only if the formula

$$\forall y(\psi(y) \Rightarrow (\exists x^1 \sigma_1(x^1, y) \vee \dots \vee \exists x^n(\sigma_n(x^n, y))) \tag{IV.3}$$

is valid in M (in $\mathcal{E}$) where $B = \psi(M)$ for ψ a geometric formula, and where $\sigma_i(M)$ defines the graph of s_i.

Proof. Note that (IV.3) is valid in M (in $\mathcal{E}$) if and only if

$$B = \{y : \psi(y)\}(M) \subset Y(M)$$

is contained in the subobject

$$S = \exists x^1(\sigma_1(x^1, y)) \vee \dots \vee \exists x^n(\sigma_n(x^n, y))(M).$$

By definition of our logical symbols, this implies that $S = Im(s_1) \vee \dots \vee Im(S_n) \subseteq B$ where $\vee$ is the supremum and where $s_i : A_i \to$ in $\mathcal{E}$. This supremum S can be described as the image of the map $\amalg_i^m A_i \to B$ induced by $\{s_i\}_i^m$. This map is an epimorphism if and only if its image contains all of B, which just means that (IV.3) is valid in M (in $\mathcal{E}$) □

Definition IV.21. A Grothendieck topology on a category $\mathcal{C}$ is *subcanonical* if, for every $C \in \mathcal{C}$, the representable presheaf $\mathrm{Hom}_{\mathcal{C}}(-, C)$ is a sheaf for this topology.

Fact IV.22. *The Grothendieck topology on* Def($\mathbf{M}$) *is generated by the basis defined in IV.19 is* canonical.

Given a geometric theory T, we define the category Def($\mathbf{T}$) and a Grothendieck topology on Def($\mathbf{T}$).

Definition IV.23. Let T be a geometric theory. We define the category Def($\mathbf{T}$) as follows:

- *Objects:* The objects of Def($\mathbf{T}$) are given by a finite list of sorts $X = (X_1, \dots, X_n)$ and an equivalence class $[\varphi(x_1, \dots, x_n)]$ of geometric formulas $\varphi(x_1, \dots, x_n)$ with variables x_i of sort X_i, and where the equivalence relation is as follows:

$$\varphi(\bar{x}) \sim \psi(\bar{x}) \text{ if } \varphi(M) = \psi(M) \text{ as subobjects of } X_1(M) \times \ldots \times X_n(M)$$

in every model model M of T in every topos $\mathcal{E}$. We denote such an object by $[\varphi, X]$

- *Morphisms:* A morphism in Def($\mathbf{T}$) between $[\varphi, Y]$ and $[\psi, Y]$ is an equivalence class of certain geometric formulas $\sigma(\bar{x}, \bar{y}) \subset X \times Y$ where $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$ with x_i (resp. y_i) of sort X_i (resp. y_i). Moreover, we require that morphisms in $\mathbf{Def}(\mathbf{T})$ have the property that for every $\mathcal{E}$ and every model M of T in the sense of $\mathcal{E}$, $\sigma(\bar{x}, \bar{y})(M) \subset X(M) \times Y(M)$ is the graph of the arrow in $\mathbf{Def}(\mathbf{M})$ from (A, X) to (B, Y) where $A = \varphi(M)$ and $B = \psi(M)$. In particular, $\sigma(\bar{x}, \bar{y})$ is a subobject of $\varphi(M) \times \psi(M)$ and $\sigma(\bar{x}, \bar{y}) \sim \sigma'(\bar{x}, \bar{y})$ if $\sigma(\bar{x}, \bar{y})(M) = \sigma'(\bar{x}, \bar{y})(M)$ for every M in every $\mathcal{E}$; or equivalently, if $\sigma(\bar{x}, \bar{y})(M)$ and $\sigma'(\bar{x}, \bar{y})$ define graphs of the same arrow in $\mathbf{Def}(\mathbf{M})$.

Lemma IV.24. *Let T be a geometric theory. Then*

(i) Def($\mathbf{T}$) *is a well-defined category.*

(ii) Def($\mathbf{T}$) *has all finite limits.*

(iii) For each model M of T in $\mathcal{E}$, the following functor is left exact, i.e preserves limits,

$$F_M : \mathrm{Def}(\mathbf{T}) \to \mathrm{Def}(\mathbf{M})$$

where $F_M([\varphi, X]) = (\varphi(M), X)$.

Proof. In the proof of Lemma IV.17, we described how identity morphisms and composition are witnessed by geometric formulas. We now note that the specific choice of formula did not depend on M itself, but only on the defining formulas for the objects and morphisms involved. Define identity and composition morphisms in Def(T) according to the scheme described in that proof. Given that objects and morphisms in Def(T) are defined as formulas up to having equivalent behavior in all models, the fact that Def(M) is itself a category for all M implies that are necessary algebraic laws are satisfied.

Similarly we noted in the proof for Proposition IV.18 that the limit cones for finite products and equalizers are definable by geometric formulas. Again, these formulas did not depend on M. These cones are vacuously preserved by the functors Def(T) $\to$ Def(M), so we must only check that they are limiting cones in Def(T). Note that the functors Def(T) $\to$ Def(M) need not be full or faithful, so we can't use the same "trick" we used in the proof of Proposition IV.18. However, directly verifying the universal property for finite products and equalizers is routine. □

Now we define the Grothendieck topology on Def(T)

Definition IV.25. Let $s_i : A_i \to B$ be a finite family of morphisms in $\mathrm{Def}(T)$. Say that these s_i *cover* B if, for every model M of T in every topos, the induced functor $\mathrm{Def}(T) \to \mathrm{Def}(M)$ sends this family to a cover with respect to the topology on $\mathrm{Def}(M)$.

Lemma IV.26. *Definition IV.25 defines a basis for a Grothendieck topology on* $\mathrm{Def}(T)$ *(see Definition III.47). Moreover, the induced functors* $\mathrm{Def}(T) \to \mathrm{Def}(M)$ *send covers to covers.*

Proof. Per Definition III.47, we must show that every set covers itself, that covers are stable under pullback, and that refinements of covers are covers. These properties follow immediately from the definition of the topology on $\mathrm{Def}(T)$, the fact that the topologies on $\mathrm{Def}(M)$ follow the required laws, and that the functors $\mathrm{Def}(T) \to \mathrm{Def}(M)$ preserve identities, pullbacks, and compositions, respectively.

That the functors $\mathrm{Def}(T) \to \mathrm{Def}(M)$ send covers to covers is vacuous. □

Lemma IV.27. *Suppose a finite family of morphisms* $s_i : A_i \to B$ *in* $\mathrm{Def}(T)$ *are given by geometric formulas* $\sigma_i(x^i, y)$*. Then the family covers* B *if and only if, in every model* M *of* T *in every topos,* M *models*

$$\forall y \in B, \bigvee_i \exists x^i \in A_i : \sigma_i(x^i, y)$$

Proof. Immediate from Lemma IV.20. □

At this point, Anand defined what it means for a functor $C \to \mathcal{E}$ to be continuous by requiring it to send covering sieves to epimorphic families. That definition requires that $\mathcal{E}$ have infinitary coproducts, which isn't true in an elementary topos in general. I don't actually think there is a good definition for when a functor from a site to an elementary topos is continuous: the definition "ought" to be equivalent to requiring the induced adjunction between $\mathsf{Set}^{C^{\mathrm{op}}}$ and $\mathcal{E}$ to be a geometric morphism (i.e., the left adjoint is left exact) that moreover factors through the sub-topos (of $\mathsf{Set}^{C^{\mathrm{op}}}$) of sheaves on C. Unless $\mathcal{E}$ is actually an elementary topos, or at least has satisfies assumptions for some sufficient Adjoint Functor Theorem, the induced adjunction need not exist, however. The only way to fix it, as far as I can tell, is to outright require the existence of the adjunction as part of the definition of "continuous".

Looking ahead, the problem is actually even worse. We've been using "topos" to mean "elementary topos", and marching on toward building $\mathbb{B}(T)$ as the universal model for T in any topos. But that universal property only actually works for models in Grothendieck toposes, not elementary ones.

Before formally defining the classifying topos $\mathbb{B}(T)$ and its properties, we need more background on geometric morphisms, which form the correct notion of morphisms between toposes.

Remark IV.28. Given toposes $\mathcal{F}$ and $\mathcal{E}$, the collection of geometric morphisms $\mathcal{F} \to \mathcal{E}$ forms a category. Given $f, g : \mathcal{F} \to \mathcal{E}$ geometric morphisms, the arrows $f \Rightarrow g$ are given by natural transformations $f^\star \Rightarrow g^\star$, or equivalently by transformations $g_\star \Rightarrow f_\star$.

Remark IV.29. Given a geometric morphism $g : \mathcal{G} \to \mathcal{F}$ (between toposes $\mathcal{G}$ and $\mathcal{F}$) where $g = (g^*, g_*)$, we can define the following functor:

$$\text{Hom}(g, \mathcal{E}) : \text{Hom}(\mathcal{F}, \mathcal{E}) \to \text{Hom}(\mathcal{G}, \mathcal{E})$$

Such that we have the following maps on objects and arrows, respectively:

Objects: For $f \in Hom(\mathcal{F}, \mathcal{E})$, $\text{Hom}(g, \mathcal{E})(f) = f \circ g \in \text{Hom}(\mathcal{G}, \mathcal{E})$;

Arrows: Let $\alpha : f_1^* \to f_2^*$ where $f_1, f_2 \in \text{Hom}(\mathcal{F}, \mathcal{E})$ and $f_1 = (f_1^*, f_{1*})$ and analogously for f_2. Then $\text{Hom}(g, \mathcal{E})(\alpha) = g^*\alpha$, which is an arrow in $\text{Hom}(\mathcal{G}, \mathcal{E})$ between $(f_1 g)^* = g^* f_1^*$ and $(f_2 g)^* = g^* f_2^*$ such that for $E \in \mathcal{E}$, $(g^*\alpha)_E = g^*(\alpha_E) : g^* f_1^* E \to g^* f_2^* E$.

Definition IV.30. Let T be a geometric theory in a language $\mathcal{L}$. Let $\text{Def}(T)$ be equipped with its Grothendieck topology $J(T)$ (see Definition IV.25). We denote by $\mathbb{B}(T)$ the topos of sheaves on the site $\text{Def}(T)$ (with respect to its Grothendieck topology $J(T)$).

Theorem IV.31. *The topos $\mathbb{B}(T)$ is the classifying topos for T, i.e., for any cocomplete topos $\mathcal{E}$ (i.e. $\mathcal{E}$ has all small colimits), there is an equivalence of categories*

$$\text{Hom}(\mathcal{E}, \mathbb{B}(T)) \cong \text{Mod}(T, \mathcal{E}) \tag{IV.4}$$

that is natural in $\mathcal{E}$ in the following sense:

Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism and $\text{Hom}(\mathcal{F}, \mathbb{B}(T)) \cong Mod(T, \mathcal{F})$, *then the following diagram commutes:*

$$\begin{array}{ccc} \text{Hom}(\mathcal{F}, \mathbb{B}(T)) & \xrightarrow{\cong} & \text{Mod}(T, \mathcal{F}) \\ \downarrow{\scriptstyle \text{Hom}(f,\ \mathbb{B}(T))} & & \downarrow{\scriptstyle f^*} \\ \text{Hom}(\mathcal{E}, \mathbb{B}(T)) & \xrightarrow{\cong} & \text{Mod}(T, \mathcal{E}) \end{array} \tag{IV.5}$$

where $f^* : \mathcal{F} \to \mathcal{E}$ is left exact, and so it takes models of T in $\mathcal{F}$ to models of T in $\mathcal{E}$ by Corollary IV.8.

Proof. First, by [4] Chapter VII, Corollary 9.4, there is an equivalence of categories between $\text{Hom}(\mathcal{E}, \mathbb{B}(T))$ and the category of left exact continuous functors from $\text{Def}(T)$ to $\mathcal{E}$. One direciton of this equivalence is given by a geometric morphism $f : \mathcal{E} \to \mathbb{B}(T)$ where $f = (f^*, f_*)$. Take $f^* : \mathbb{B}(T) \to \mathcal{E}$ and compose with Yoneda embedding $y : \text{Def}(T) \to \mathbb{B}(T)$. One checks that $f^* \circ y$ is left exact.

Now given a model M of T in a topos $\mathcal{E}$, we construct a left exact continuous functor $A_M : \text{Def}\, T \to \mathcal{E}$, which is the composition of $F_M : \text{Def}(T) \to \text{Def}(M)$ (i.e. evaluating objects of $\text{Def}(T)$ at M) and the 'forgetful' functor $\text{Def}(M) \to \mathcal{E}$. We have seen that A_M is left exact and continuous. Note that the objects of $\text{Def}(T)$ are of the form $[\varphi(x), X]$ and $A_M([\varphi(x), X]) = \varphi(M)$ as an object of $\mathcal{E}$ and similarly for arrows.

There are a few things to check: First, that $M \to A_M$ is a functor, i.e. given a homomorphism of models $M \to M'$, we get a natural transformation $A_M \to A_{M'}$.

For the other direction of the proof, let $A : \mathrm{Def}(T) \to \mathcal{E}$ be a left exact continuous functor $A : \mathrm{Def}(T) \to \mathcal{E}$. We want to construct a model M_A of T in the topos $\mathcal{E}$.

Let X_i be a sort in the language $\mathcal{L}$. We will use the formula $x_i = x_i$ for a variable x_i of the sort X_i to define the object

$$X_i(M_A) = A([x_i = x_i, X_i]) \tag{IV.6}$$

We assume that $\mathcal{L}$ has just relation symbols and equality, although the proving where $\mathcal{L}$ has function symbols is not difficult. Let $R \subset X_1 \times \ldots \times X_n$ be a relation symbol of $\mathcal{L}$. Then we define

$$R(M_A) = A([R(x), x]) \tag{IV.7}$$

where $x = (x_1, \ldots, x_n)$. Since A preserves monomorphisms, we have that A yields the monomorphism

$$R(M_A) \rightarrowtail X_1(M_A) \times \ldots \times X_N(M_A) \tag{IV.8}$$

This gives us M_A from the continuous left-exact functor $A : \mathbb{B}(T) \to \mathcal{E}$. To complete the proof, we need the following two lemmas.

Lemma IV.32. *For any geometric formula $\varphi(x_1, \ldots, x_n)$ where x_i is of the sort X_i, there is a natural isomorphism between the subobjects $\varphi(M_A) \subset X_1(M_A) \times \ldots \times X_n(M_A)$ and $A([\varphi, X])$ as subobjects of $X_1(M_A) \times \ldots \times X_n(M_A)$.*

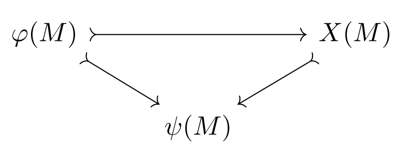
Proof. The proof follows by induction on φ, which uses the existence of covers and pullbacks, and the fact that A is left exact and continuous. □

Lemma IV.33. *M_A is a model of T in $\mathcal{E}$. That is, every axiom $\forall \bar{x}(\varphi(\bar{x}) \to \psi(\bar{x})$ of T is valid in M_A in $\mathcal{E}$.*

Proof. By assumption, for every model M of T in any topos $\mathcal{E}$, we have $\varphi(M) \subset \psi(M)$ as subobjects of $X(M)$.

Claim. *There is a corresponding inclusion $[\varphi, X] \rightarrowtail [\psi, X]$ in* Def(T).

Proof of claim. In every model M, we have that the formula $x = x$ yields the arrows:



(IV.9)

So we obtain the same diagram in Def(T):

$$\begin{array}{ccc} [\varphi, X] & \rightarrowtail & [x = x, X] \\ & \searrow \quad \swarrow & \\ & [\psi, X] & \end{array} \tag{IV.10}$$

Now we know that A preserves inclusion of subobjects, so $A[\varphi, X] \subset A[\psi, X]$ as subobjects of $A[x = x, X]$. By Lemma **??**, we have that $\varphi(M_A) \subset \psi(M_A)$ as subobjects of $X_1(M_A) \times \ldots \times X_n(M_A)$. □

To complete the proof of Theorem IV.31 , one checks that the functors taking $M \to A_M$ and $A \to M_A$ are inverses of each other up to natural isomorphism, natural in $\mathcal{E}$. □

□

Definition IV.34. The *universal (topos-valued) model* $\mathcal{U}_T$ is the model of T in $\mathbb{B}(T)$ corresponding to the identity geometric morphism $\mathbb{B}(T) \to \mathbb{B}(T)$.

Proposition IV.35. *Let M be a model of T in a complete topos $\mathcal{E}$. Let c_M : $\mathcal{E} \to \mathbb{B}(T)$ be the corresponding geometric morphism. Then M is the image of $\mathcal{U}_T$ under $c_M^* : \mathbb{B}(T) \to \mathcal{E}$, which is a left-exact continuous functor.*

Proof. By the naturality of (IV.4), we have the following commutative diagram as a special case of (IV.5)

$$\begin{array}{ccc} \mathcal{U}_T \in \mathrm{Mod}(T, \mathbb{B}(T)) & \xrightarrow{\cong} & \mathrm{Hom}(\mathbb{B}(T), \mathbb{B}(T)) \ni \mathrm{id} \\ \downarrow{\scriptstyle c_M^*} & & \downarrow{\scriptstyle \mathrm{Hom}(c_M,\, \mathbb{B}(T))} \\ M \in \mathrm{Mod}(T, \mathcal{E}) & \xrightarrow{\cong} & \mathrm{Hom}(\mathcal{E}, \mathbb{B}(T)) \ni c_m \end{array} \tag{IV.11}$$

□

Remark IV.36. One can provide the following description of $\mathcal{U}_T$. Given a geometric theory T, and the topos $\mathbb{B}(T)$ of sheaves on $\mathrm{Def}(T)$. Note that $id : \mathbb{B}(T) \to \mathbb{B}(T)$ corresponds to the Yoneda embedding $y : \mathrm{Def}(T) \to \mathbb{B}(T)$. So by the constructions of Theorem IV.31, we have that $\mathcal{U}_T$ is precisely the model M_y.

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