# STABLE GROUPS AND EXPANSIONS OF $(\mathbb{Z}, +, 0)$

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Abstract. We show that if $G$ is a stable group of finite weight with no infinite, infinite-index, chains of definable subgroups, then $G$ is superstable of finite $U$-rank. Combined with recent work of Palacín and Sklinos, we conclude that $(\mathbb{Z}, +, 0)$ has no proper stable expansions of finite weight. A corollary of this result is that if $P \subseteq \mathbb{Z}^n$ is definable in a finite dp-rank expansion of $(\mathbb{Z}, +, 0)$, and $(\mathbb{Z}, +, 0, P)$ is stable, then $P$ is definable in $(\mathbb{Z}, +, 0)$. In particular, this answers a question of Marker on stable expansions of the group of integers by sets definable in Presburger arithmetic.

## 1. Introduction and Summary of Main Results

The work in this paper is motivated by questions surrounding first-order expansions of the group $(\mathbb{Z}, +, 0)$, which are well-behaved with respect to some notion of model theoretic tameness (e.g. stability or NIP). The group $(\mathbb{Z}, +, 0)$ is a well-known example of a stable group, and so this program is a natural analog of the very fruitful study of "tame" (e.g. o-minimal or NIP) expansions of the real ordered field $(\mathbb{R}, +, \cdot, <, 0)$. Expansions of $(\mathbb{Z}, +, 0)$ have emerged in the context of definable subgroups of finitely generated free groups, as well as the general growing industry of research on ordered abelian groups satisfying notions of tameness coming from dp-rank in NIP first-order theories (e.g. [7], [9], [24]). We will provide more detail on these contexts toward the end of the introduction. For now, we state an explicit question, originally asked by Marker in 2011.

**Question 1.1** (Marker)**.** Is there a set $P \subseteq \mathbb{Z}^n$, definable in Presburger arithmetic $(\mathbb{Z}, +, <, 0)$, such that $(\mathbb{Z}, +, 0, P)$ is a proper stable expansion of $(\mathbb{Z}, +, 0)$?

The focus on Presburger arithmetic in the previous question is not unnatural. Indeed, $(\mathbb{Z}, +, <, 0)$ is an ordered structure, and thus unstable, but is still well understood and very well behaved model theoretically (to be precise, its theory is NIP of dp-rank 1 [8]). Our first main result will show that, in fact, these model theoretic notions completely control the answer to Marker's question.

**Theorem 1.2.** *If $P \subseteq \mathbb{Z}^n$ is definable in a finite dp-rank expansion of $(\mathbb{Z}, +, 0)$, and $(\mathbb{Z}, +, 0, P)$ is stable, then $P$ is definable in $(\mathbb{Z}, +, 0)$.*

The notion of dp-rank in NIP theories has been an important tool in extending the work of stability theory to the unstable stable setting (see, e.g., [23]), and so Theorem 1.2 establishes a fundamental fact about the behavior of NIP expansions of $(\mathbb{Z}, +, 0)$. The proof of this theorem will be obtained from a more general result

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on stable groups (Theorem 1.4 below), combined with the following recent result of Palacín and Sklinos [15].

**Fact 1.3.** [15] $(\mathbb{Z}, +, 0)$ *has no proper stable expansions of finite* $U$*-rank.*

Precise definitions of these notions will be given in Section 2. For now, we emphasize that, *a priori*, Fact 1.3 alone is not sufficient to answer Marker's question, or obtain Theorem 1.2. In particular, while the dp-rank of complete stable theory is *bounded above* by its $U$-rank, there is no further general relationship between these two ranks. Indeed, there are stable groups of dp-rank 1 and infinite, even undefined, $U$-rank (see Example 2.9). Therefore, the work involved in proving Theorem 1.2 consists of showing that if a stable expansion of $(\mathbb{Z}, +, 0)$ has finite dp-rank, then it must have finite $U$-rank (we also show that these ranks are the same). We will in fact obtain this conclusion from a general characterization of superstable groups of finite $U$-rank, which exploits the notion of *weight* in stable theories. Before stating this result, we clarify the following terminology.

Given a group $G$, definable in (a model of) a complete theory $T$, we say $G$ *has finite dp-rank* if there is a uniform finite bound on the dp-rank of any type containing the formula defining $G$. If $G$ is a *stable* group (i.e. if $T$ is stable), then the *$U$-rank of $G$*, denoted $U(G)$, is the supremum of the $U$-ranks of types containing a formula defining $G$. Replacing $U$-rank with *weight*, we similarly define the *weight of* $G$, denoted $\mathrm{wt}(G)$. We let $<_\infty$ denote the partial order on groups given by: $H <_\infty K$ if $H \leq K$ and $[K : H] = \infty$. If $G$ is superstable of finite $U$-rank then, by well-known facts, $G$ necessarily has finite weight and no infinite $<_\infty$-chains of definable subgroups. Our main result is that these conditions are also sufficient.

**Theorem 1.4.** *Let $G$ be a stable group. The following are equivalent.*

($i$) *$G$ is superstable of finite $U$-rank.*
($ii$) *$G$ has finite weight and no infinite $<_\infty$-chains of definable subgroups.*

The proof is given in Section 3, and involves a new application of Zilber indecomposability in the setting of weight. Theorem 1.4 quickly yields Theorem 1.2, modulo Fact 1.3 and general results on weight and dp-rank in stable theories.

*Proof of Theorem 1.2.* Let $G = (\mathbb{Z}, +, 0, P)$, where $P \subseteq \mathbb{Z}^n$ is definable in a finite dp-rank expansion of $(\mathbb{Z}, +, 0)$. Since dp-rank cannot increase after taking a reduct, $G$ has finite dp-rank $n < \omega$. Assume $G$ is stable. Then $\mathrm{wt}(G) = n$ (see [1], [14]), and so $U(G)$ is finite by Theorem 1.4 (in fact, $U(G) = \mathrm{wt}(G) = n$ by Corollary 1.5 below). By Fact 1.3, $P$ is definable in $(\mathbb{Z}, +, 0)$. □

Altogether, the real content of this paper is the proof of Theorem 1.4. From the proof of this result, we will also obtain the following corollary.

**Corollary 1.5.** *Suppose $G$ is a stable group of finite weight, with no infinite $<_\infty$-chains of definable subgroups. Then there is a uniform finite bound on the length of a $<_\infty$-chain of definable subgroups of $G$. Moreover, if $n < \omega$ is the length of the longest $<_\infty$-chain of definable subgroups of $G$, then $U(G) \leq n\,\mathrm{wt}(G)$.*

In Section 2, we will also recall some classical examples showing that the upper bound in this result cannot be improved in general.

We end this section with a discussion of related work and open questions. The motivation for Question 1.1 partly arose from interest in the induced structure on proper definable subgroups of finitely-generated free groups, which are examples of stable groups [21]. In particular, the maximal proper definable subgroups of such groups are exactly the centralizers of some nontrivial element (see [16]), and thus isomorphic *as groups* to $(\mathbb{Z},+,0)$. Therefore, studying stable expansions of $(\mathbb{Z},+,0)$ was seen as an alternate approach toward the unpublished result of Perin that the induced structure on centralizers in the free group is always a pure group. Another proof of this has been recently given by Byron and Sklinos [4].

Beyond this connection to the free group, there has been a recent flurry of interest in expansions of $(\mathbb{Z},+,0)$. On the stable side, we have the following ambitious question (which is similar to a question of Goodrick quoted in [15]).

**Question 1.6.** Characterize the sets $P \subseteq \mathbb{Z}^n$, for which $(\mathbb{Z},+,0,P)$ is stable.

On the unstable side, Dolich and Goodrick [7] have shown that $(\mathbb{Z},+,<,0)$ has no proper *strong* expansions (which includes expansions of finite dp-rank). Concerning *reducts* of Presburger arithmetic, a recent result of the second author [6] is that there are no intermediate structures strictly between $(\mathbb{Z},+,0)$ and $(\mathbb{Z},+,<,0)$. In a different direction, Kaplan and Shelah [11] show that if $P = \{z \in \mathbb{Z} : |z| \text{ is prime}\}$ then $(\mathbb{Z},+,0,P)$ is unstable and, assuming a fairly strong conjecture in number theory, $(\mathbb{Z},+,0,P)$ is supersimple of $SU$-rank 1 (see also Remark 1.8(3) below).

The investigation of stable expansions of $(\mathbb{Z},+,0)$ also fits naturally into the general question of when good properties of a structure are preserved after adding a new predicate. For example Pillay and Steinhorn [17] proved that there are no proper o-minimal expansions of $(\mathbb{N},<)$, while Marker [12] exhibited proper strongly minimal expansions of $(\mathbb{N}, x \mapsto x+1)$. Zilber [26] showed that there are proper $\omega$-stable expansions of the complex field $(\mathbb{C},+,\cdot,0,1)$ (in particular, adding a predicate for the roots of unity), while Marker [13] proved that there are no proper stable expansions of $(\mathbb{C},+,\cdot,0,1)$ by a semialgebraic set.

Even more generally, Theorem 1.4 fits into the investigation of when stronger forms of stability can be proved for stable groups satisfying various assumptions on definable subgroups. For example, in [2], Baldwin and Pillay prove that if $G$ is superstable of finite $U$-rank, and $G$ has no proper connected type-definable normal subgroups, then $G$ is $\omega$-stable. In [10], Gagelman proves that if $G$ is superstable of finite $U$-rank and satisfies the descending chain condition on definable subgroups, then $G$ is $\omega$-stable. It would be interesting to know if the finiteness conditions on weight and $U$-rank in Theorem 1.4 can be relaxed to obtain a characterization of superstable groups of a similar flavor. In particular, it is well known that if $G$ is a superstable group, then every type in $G$ has finite weight (i.e. $G$ is *strongly stable*) and $G$ has no infinite descending $<_\infty$-chains of definable subgroups (i.e. $G$ satisfies the *superstable descending chain condition*). Therefore, we ask the following question, which is an analog of Theorem 1.4 for superstable groups.

**Question 1.7.** Suppose $G$ is a strongly stable group satisfying the superstable descending chain condition. Is $G$ is superstable?

We end with some important remarks.

**Remark 1.8.**

(1) Many of the results above on $(\mathbb{Z},+,0)$ do not hold if one considers expansions of structures elementarily equivalent to $(\mathbb{Z},+,0)$. For example, there are models $(M,+,0)$ of $\mathrm{Th}(\mathbb{Z},+,0)$ with proper stable expansions of finite $U$-rank.

(2) Theorem 1.2 also holds with inp-rank in place of dp-rank, since these ranks coincide in the stable case (see [1]). Therefore the theorem can be applied in the more general class of $\mathrm{NTP}_2$ theories.
(3) Fact 1.3 does not hold if stable is replaced by simple. For example, using results of Chatzidakis and Pillay [5] on "generic" predicates, one can find a set $P \subseteq \mathbb{Z}$ such that $(\mathbb{Z}, +, 0, P)$ is unstable, but supersimple of $U$-rank 1.

## 2. Preliminaries

The purpose of this section is to collect the preliminary tools and facts that we will need in the proof of the main result (Theorem 1.4). Our intent is to include sufficient detail so as to make this paper accessible to a wider audience beyond those researchers well-versed in stability theory. For example, Lemma 2.3 and Proposition 2.11 are folkloric facts, which seem to be primarily used in the superstable context, and to not appear in the literature in more general settings. Therefore we have included proofs suitable for the general stable case.

Throughout this section, $T$ is a stable first-order theory, and we assume $T = T^{\mathrm{eq}}$. We work in a sufficiently saturated monster model $\mathbb{M}$ of $T$, and use letters $A, B, \ldots$ for small parameter sets in $\mathbb{M}$, where a parameter set $A$ is *small* (written $A \subset \mathbb{M}$) if $\mathbb{M}$ is $|T(A)|^+$-saturated. In general, a cardinal $\kappa$ is *small* or *bounded* if $\mathbb{M}$ is $\kappa^+$-saturated. We use letters $X, Y, \ldots$ for definable or type-definable sets, and we always identify such a set $X$ with its set of realizations $X(\mathbb{M})$ in the monster model. As usual, by a *type-definable* set we mean an intersection of a small collection of definable sets. Given a type $p$, and a type-definable set $X$, we write $p \models X$ if $p$ extends a type defining $X$. We use $\mathop{\smile\!\!\!\!\!\mid}$ for nonforking independence in $T$. We assume familiarity with stability and $U$-rank.

**Definition 2.1.**

(1) Given a sequence $(\bar{b}_i)_{i \in I}$ of tuples and $C \subset \mathbb{M}$, we say $(\bar{b}_i)_{i\in I}$ is **$C$-independent** if $\bar{b}_i \mathop{\smile\!\!\!\!\!\mid}_C \{\bar{b}_j : j \neq i\}$ for all $i \in I$.
(2) Given $C \subset \mathbb{M}$ and $p \in S(C)$, define the **weight of** $p$, denoted $\mathrm{wt}(p)$, to be the supremum over cardinals $\kappa$ for which there is $B \supseteq C$, a realization $\bar{a} \models p$, and a $B$-independent sequence $(\bar{b}_i)_{i<\kappa}$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C B$ and $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_B \bar{b}_i$ for all $i < \kappa$.
(3) Let rk denote either $U$-rank or weight.
 (*i*) If $\bar{a} \in \mathbb{M}$ and $C \subset \mathbb{M}$ then $\mathrm{rk}(\bar{a}/C)$ denotes $\mathrm{rk}(\mathrm{tp}(\bar{a}/C))$.
 (*ii*) If $X$ is type-definable, then $\mathrm{rk}(X) = \sup\{\mathrm{rk}(p) : p \models X\}$.

We will use the following basic properties of $U$-rank and weight.

**Proposition 2.2.** *Let* rk *denote either $U$-rank or weight.*

(*a*) *Given $\bar{a} \in \mathbb{M}$ and $C \subset \mathbb{M}$,* $\mathrm{rk}(\bar{a}/C) = 0$ *if and only if* $\bar{a} \in \mathrm{acl}(C)$.
(*b*) *Fix $\bar{a}, \bar{b} \in \mathbb{M}$ and $C \subset \mathbb{M}$. If $\bar{a} \in \mathrm{acl}(\bar{b}, C)$ and $\bar{b} \in \mathrm{acl}(\bar{a}, C)$ then* $\mathrm{rk}(\bar{a}/C) = \mathrm{rk}(\bar{b}/C)$.
(*c*) *Suppose $X$ is type-definable and $f$ is a definable function with domain containing $X$. Then* $\mathrm{rk}(f(X)) \leq \mathrm{rk}(X)$.

*Proof.* These are straightforward exercises. Parts (*b*) and (*c*) follow easily from part (*a*) together with Lascar's inequality for $U$-rank (see [18, Theorem 19.4]), and a sufficiently similar inequality for weight (see [22, Lemma V.3.11(2)]). □

In a superstable theory, the weight of a type $p$ is bounded by the sum of the integer coefficients in the Cantor normal form of $U(p)$ (see [18, Theorem 19.9]). In particular, one has $\mathrm{wt}(p) \leq U(p)$, which still holds for stable theories in general.

**Lemma 2.3.** *If $C \subset \mathbb{M}$ and $p \in S(C)$, then $\mathrm{wt}(p) \leq U(p)$.*

*Proof.* Fix $p \in S(C)$. Suppose we have a set $B \supseteq C$, a realization $\bar{a} \models p$, and a $B$-independent sequence $(\bar{b}_i)_{i<\kappa}$, for some cardinal $\kappa$, such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$ and $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_B \bar{b}_i$ for all $i < \kappa$. We prove $U(\bar{a}/B) \geq \kappa$, which implies $U(p) \geq \kappa$. Given $i \leq \kappa$, define $B_i = B \cup \{\bar{b}_j : i \leq j\}$ (so $B_\kappa = B$). We prove, by induction on $i \leq \kappa$, that $U(\bar{a}/B_i) \geq i$. Given this, we will then have $U(\bar{a}/B) = U(\bar{a}/B_\kappa) \geq \kappa$.

The base case is trivial; so suppose $\lambda \leq \kappa$ is a limit ordinal and $U(\bar{a}/B_i) \geq i$ for all $i < \lambda$. For any $i < \lambda$, we have $B_\lambda \subseteq B_i$, and so $U(\bar{a}/B_\lambda) \geq U(\bar{a}/B_i) \geq i$. Therefore $U(\bar{a}/B_\lambda) \geq \lambda$. Finally, fix $i < \kappa$ and suppose $U(\bar{a}/B_i) \geq i$. Since $B_{i+1} \mathop{\smile\!\!\!\!\!\!|}_B \bar{b}_i$ and $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_B \bar{b}_i$, we have $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_{B_{i+1}} \bar{b}_i$ by transitivity. Therefore $U(\bar{a}/B_{i+1}) \geq i+1$. □

For general stable theories, Lemma 2.3 is the most one can say concerning the relationship between weight and $U$-rank for arbitrary types (see Example 2.9). However, when working "close" to types of $U$-rank 1, weight and $U$-rank coincide. This will be a key tool in the proof of our main result.

**Proposition 2.4.** *Fix $C \subset \mathbb{M}$, and suppose $X \subseteq \mathbb{M}$ is such that $U(a/C) \leq 1$ for all $a \in X$. If $\bar{b}$ is a finite tuple in $\mathrm{acl}(XC)$ then $U(\bar{b}/C) = \mathrm{wt}(\bar{b}/C)$.*

*Proof.* We first consider the case that $\bar{b}$ is a tuple of elements of $X$. The essential observation is the following:

(†) If $a \in X$ and $C \subseteq B \subseteq D \subset \mathbb{M}$ then, since $U(a/C) \leq 1$, we have $a \not\mathop{\smile\!\!\!\!\!\!|}_B D$ if and only if $a \in \mathrm{acl}(D) \backslash \mathrm{acl}(B)$.

In particular, by (†) and symmetry of forking, we have the following exchange property: given $a, a' \in X$ and $C \subseteq B \subset \mathbb{M}$, if $a \in \mathrm{acl}(B, a') \backslash \mathrm{acl}(B)$ then $a' \in \mathrm{acl}(B, a)$. Therefore, a tuple $\bar{a} \in X$ has a well-defined basis over $C$, and we show $U(\bar{a}/C) = \dim_{\mathrm{acl}}(\bar{a}/C) = \mathrm{wt}(\bar{a}/C)$. The first equality follows from Proposition 2.2($a$) and repeated application of Lascar's inequality. For the second equality, it follows from (†) that a basis for $\bar{a}$ over $C$ is $C$-independent (with respect to nonforking) and, moreover, $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_C a_i$ for any $a_i \in \bar{a} \backslash \mathrm{acl}(C)$. Therefore $\mathrm{wt}(\bar{a}/C) \geq \dim_{\mathrm{acl}}(\bar{a}/C)$, and so equality holds by Lemma 2.3.

Now consider the general case of $\bar{b} \in \mathrm{acl}(XC)$. By Proposition 2.2($a$), we may assume that some coordinate of $\bar{b}$ is not in $\mathrm{acl}(C)$. Fix $\bar{a} = (a_1, \ldots, a_n) \in X$, algebraically independent over $C$, with $\bar{b} \in \mathrm{acl}(\bar{a}, C)$. Let $k \leq n$ be maximal such that, for some $i_1 < \ldots < i_k \leq n$, we have $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_C (a_{i_1}, \ldots, a_{i_k})$ (it is possible here that $k = 0$). Without loss of generality, assume $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_C (a_1, \ldots, a_k)$. Let $\bar{a}_1 = (a_1, \ldots, a_k)$ and $\bar{a}_2 = (a_{k+1}, \ldots, a_n)$. We show $\bar{a}_2 \in \mathrm{acl}(\bar{b}, \bar{a}_1, C)$. If not, then there is some $k < i \leq n$ such that $a_i \notin \mathrm{acl}(\bar{b}, \bar{a}_1, C)$, and so $a_i \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}_1, C} \bar{b}$ by (†). Together with $\bar{a}_1 \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$, we have $(\bar{a}_1, a_i) \mathop{\smile\!\!\!\!\!\!|}_C \bar{b}$, contradicting the maximality of $k$. Altogether, $\bar{a}_2 \in \mathrm{acl}(\bar{b}, \bar{a}_1, C)$ and $\bar{b} \in \mathrm{acl}(\bar{a}_2, \bar{a}_1, C)$. By the first case above, and Proposition 2.2($b$), $U(\bar{b}/C) = U(\bar{a}_2/C) = \mathrm{wt}(\bar{a}_2/C) = \mathrm{wt}(\bar{b}/C)$. □

**Remark 2.5.** The notion of weight also behaves nicely in simple theories. For example, after replacing all occurrences of $U$-rank with $SU$-rank, the statements of Proposition 2.2, Lemma 2.3, and Proposition 2.4 hold when $T$ is simple (with identical proofs).

We now turn to stable groups. Once again, a *stable group* is a group $G$ definable in (some model of) a stable theory $T$. We will continue to identify the definable set $G$ with its realization $G(\mathbb{M})$ in the monster model. Given a definable group $G$, we let $G^0$ denote the *connected component of* $G$, which is the intersection of all definable subgroups of $G$ of finite index. By stability (see Fact 2.6 below), $G^0$ is the intersection of at most $|T|$ many definable subgroups of $G$ of finite index, and hence is type-definable (over the same parameters used to define $G$). We say $G$ is *connected* if $G = G^0$.

We first recall the following classical results.

**Fact 2.6.** *Let $G$ be a stable group.*

($a$) (Baldwin-Saxl [19, Proposition 1.4]) *Let $\{H_i : i \in I\}$ be a family of uniformly definable subgroups of $G$, and set $H = \bigcap_{i\in I} H_i$. Then $H = \bigcap_{i\in I_0} H_i$ for some finite $I_0 \subseteq I$. In particular, $H$ is definable.*

($b$) (Poizat [19, Theorem 5.17]) *Any type-definable subgroup of $G$ is the intersection of at most $|T|$ many definable subgroups of $G$.*

**Remark 2.7.** Unlike the previous preliminaries, these facts on stable groups do not immediately go through through if $T$ is only assumed to be simple. In fact, there are simple unstable groups where part ($a$) fails [25, Example 1]. On the the other hand, whether part ($b$) holds for groups definable in simple theories is a well known open question.

For the sake of clarity, it is worth making a few remarks concerning weight and $U$-rank of stable groups. In particular, given a definable group $G$ and $A \subset \mathbb{M}$, we let $S_G(A)$ denote the space of complete types, over parameters in $A$, which contain a formula defining $G$. Then, if rk denotes either $U$-rank or weight, we can express $\mathrm{rk}(G)$ as

$$\mathrm{rk}(G) = \sup\{\mathrm{rk}(p) : p \in S_G(A) \text{ for some } A \subset \mathbb{M}\}.$$

We say $G$ *has finite $U$-rank* (resp. *finite weight*) if $U(G) < \omega$ (resp. $\mathrm{wt}(G) < \omega$).

If $G$ is stable then $U(p) = U(G)$ for any generic type $p$ in $G$ (see [3, Lemma III.4.5($i$)]). However, this can fail for weight. For example, it is possible that all types in $G$ have finite weight, but $\mathrm{wt}(G)$ is not finite (e.g. Example 2.9(1) below). Since our focus is on the case that $\mathrm{wt}(G)$ is finite, we will not concern ourselves with this situation.

**Remark 2.8.** When considering examples of stable groups, it is often the case that the group $G$ is the whole structure (i.e. defined by the formula $x = x$). Therefore, given a group $G = (G, \cdot, 1, \ldots)$, when we speak of the $U$-rank or weight of $G$, we continue to mean as calculated in a monster model according the definitions and conventions above.

The following examples illustrate some of the possible variety concerning weight, $U$-rank, and $<_\infty$-chains in stable groups.

**Example 2.9.**

(1) Let $\Pi_2 = \{2^n : n \in \mathbb{N}\}$ and let $G = (\mathbb{Z}, +, 0, \Pi_2)$. Then $G$ is superstable of $U$-rank $\omega$ (see [15], [20]). Therefore, $G$ is strongly stable, but does not have finite weight by Theorem 1.4.

(2) Fix an integer $n > 0$ and let $G = (\mathbb{Q}^n, +, 0, (H_k)_{k<n})$ where, for each $k < n$, $H_k = \mathbb{Q}^k \times \{0\}^{n-k}$. We have a sequence $(E_k)_{k<n}$ of definable equivalence relations, given by $E_k(x,y) \leftrightarrow x - y \in H_k$. Let $\mathbb{M}$ be a monster model. Given $a, b \in \mathbb{M}$, let $d(a,b) = \min\{k < n : E_k(a,b)\}$. Then $d$ is an ultrametric on $\mathbb{M}$, taking values in $\{0, 1, \ldots, n\}$; and nonforking independence is characterized by: $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if, for all $a \in \operatorname{acl}(AC)$, $d(a, \operatorname{acl}(BC)) = d(a, \operatorname{acl}(C))$ (where algebraic closure is the same as in $(\mathbb{Q}^n, +, 0)$). Using this, one may verify that $G$ is superstable of $U$-rank $n$ and weight 1.

(3) Let $G = (\mathbb{Q}^\omega, +, 0, (H_n)_{n<\omega})$ where, for each $n < \omega$, $H_n = \mathbb{Q}^n \times \{0\}^\omega$. Using a similar argument as in part (2), one may show that $G$ is superstable of $U$-rank $\omega$ and weight 1.

(4) Let $G = (\mathbb{Q}^\omega, +, 0, (K_n)_{n<\omega})$ where, for each $n < \omega$, $H_n = \{0\}^n \times \mathbb{Q}^\omega$. Then $G$ is strictly stable of weight 1.

Our final preliminary tools concern indecomposable sets in stable groups.

**Definition 2.10.** Let $G$ be a stable group. A type-definable set $X \subseteq G$ is **indecomposable** if, for all type-definable subgroups $H \leq G$, either $X/H$ is unbounded or $|X/H| = 1$ (where $X/H = \{xH : x \in X\}$).

**Proposition 2.11.** *Let $G$ be a stable group. Fix $A \subset \mathbb{M}$ and a stationary type $p \in S_G(A)$. Let $X = p(\mathbb{M})$. Then $X \subseteq G$ is indecomposable.*

*Proof.* Let $\mathcal{F}$ denote the family of type-definable subgroups $H \leq G$ such that $X/H$ is bounded. Let $H_0$ be the intersection of the elements of $\mathcal{F}$. Using Fact 2.6, it is a standard exercise to show that $H_0$ is a type-definable subgroup of $G$ and $X/H_0$ is bounded (i.e. $H_0 \in \mathcal{F}$). Note also that $A$-invariance of $X$ implies $A$-invariance of $H_0$, and so $H_0$ is type-definable over $A$.

Let $\tilde{p} \in S_G(\mathbb{M})$ be the unique global nonforking extension of $p$. Let $C \subset X$ be a bounded set such that $X/H_0 = \{cH_0 : c \in C\}$, and fix a realization $u \in G$ of $\tilde{p}|_{AC}$. Then $u \in X$, and so $u \in cH_0$ for some $c \in C$, which means $\tilde{p} \models cH_0$. If $f \in \operatorname{Aut}(\mathbb{M}/A)$ then, by $A$-invariance of $H_0$ and $\tilde{p}$, we have $\tilde{p} \models f(c)H_0$, and so $f(cH_0) = f(c)H_0 = cH_0$. Consequently, $cH_0$ is type-definable over $A$, and so $p \models cH_0$. Therefore $X \subseteq cH_0$, which implies $X \subseteq cH$ for all $H \in \mathcal{F}$. □

A well-known result of Berline and Lascar is the Indecomposability Theorem for superstable groups [3, Theorem V.3.1]. In order to use this result without the assumption of superstability, we state the following corollary of its proof.

**Fact 2.12.** *Suppose $G$ is a stable group and $\{X_i : i \in I\}$ is a family of indecomposable type-definable subsets of $G$, each containing 1. Given $n > 0$ and $\sigma = (i_0, \ldots, i_n) \in I^{<\omega}$, let $X_\sigma = X_{i_0} \cdot X_{i_1} \cdot \ldots \cdot X_{i_n}$. Assume that there is a uniform finite bound on $U(X_\sigma)$, where $\sigma$ ranges over $I^{<\omega}$. Then $\bigcup_{i \in I} X_i$ generates a connected type-definable subgroup $H$ of $G$. In particular, there are $i_0, \ldots, i_n$ such that $H = X_{i_0} \cdot \ldots \cdot X_{i_n} \cdot X_{i_n}^{-1} \cdot \ldots \cdot X_{i_0}^{-1}$.*

## 3. Proof of the main result

Toward the proof of Theorem 1.4, the first step is to prove that if $G$ is a stable group of finite weight, then one may construct a type-definable subgroup of $G$ with several nice properties.

**Lemma 3.1.** *If $G$ is an infinite stable group of finite weight then there is an infinite connected type-definable normal subgroup $H \leq G$, with $U(H) = \mathrm{wt}(H)$.*

*Proof.* Fix a stationary type $p \in S_G(A)$, for some $A \subset \mathbb{M}$, such that $U(p) = 1$. For example, choose $p$ minimal in the fundamental order among non-algebraic types in $S_G(A)$ (with $A$ varying over small parameter sets in $\mathbb{M}$), and then replace $p$ by a nonforking extension to a model.

Let $Y = p(\mathbb{M})$. Then $Y \subseteq G$ is indecomposable by Proposition 2.11. Fix some $u \in Y$, and set $X = u^{-1}Y$. Given $g \in G$, let $X^g = gXg^{-1}$. Then $\{X^g : g \in G\}$ is a family of indecomposable type-definable subsets of $G$, each of which contains 1. By Proposition 2.2($c$), $U(X^g) = 1$ for all $g \in G$. Given a sequence $\sigma = (g_0, \ldots, g_n)$ of elements of $G$, we set $X_\sigma = X^{g_0} \cdot \ldots \cdot X^{g_n}$.

We claim that, for any $\sigma \in G^{<\omega}$, $U(X_\sigma) = \mathrm{wt}(X_\sigma)$. Suppose, for a contradiction, that $\mathrm{wt}(X_\sigma) < U(X_\sigma)$, with $\sigma = (g_0, \ldots, g_n)$. Then there is a type $q$ such that $q \models X_\sigma$ and $U(q) > \mathrm{wt}(X_\sigma)$. After taking a nonforking extension, we may assume $q \in S(C)$, where $C \subset \mathbb{M}$ is such that $X^{g_i}$ is type-definable over $C$ for all $0 \leq i \leq n$. Let $a \models q$. Then $a \in X_\sigma$, so we may write $a = b_0 \cdot \ldots \cdot b_n$ with $b_i \in X^{g_i}$. For any $0 \leq i \leq n$, $\mathrm{tp}(b_i/C) \models X^{g_i}$ by choice of $C$, and so $U(b_i/C) \leq 1$. Since $a \in \mathrm{acl}(b_0, \ldots, b_n)$, we have $U(a/C) = \mathrm{wt}(a/C)$ by Proposition 2.4. But $\mathrm{wt}(a/C) \leq \mathrm{wt}(X_\sigma)$, which contradicts $U(a/C) = U(q) > \mathrm{wt}(X_\sigma)$.

We have $U(X_\sigma) \leq \mathrm{wt}(G)$ for all $\sigma \in G^{<\omega}$. Therefore, we may apply Fact 2.12 to conclude that $\bigcup_{g \in G} X^g$ generates an infinite connected type-definable subgroup $H$ of $G$, which is normal by construction. Moreover, $H = X_\sigma$ for some $\sigma \in G^{<\omega}$, and so $U(H) = \mathrm{wt}(H)$. □

Given a group $G$, and a normal subgroup $K$, we let $\rho_{G/K}$ denote the pullback function on subgroups of $G/K$, i.e., given $H \leq G/K$, define $\rho_{G/K}(H) = \{g \in G : gK \in H\} \leq G$. The following are easy observations.

**Proposition 3.2.** *Let $G$ be a group and $K$ a normal subgroup of $G$.*

($a$) *If $H \leq G/K$ then $K$ is a normal subgroup of $\rho_{G/K}(H)$ and $H = \rho_{G/K}(H)/K$.*
($b$) *If $H_1 \leq H_2 \leq G/K$ then $\rho_{G/K}(H_1) \leq \rho_{G/K}(H_2) \leq G$ and $[\rho_{G/K}(H_2) : \rho_{G/K}(H_1)] = [H_2 : H_1]$.*

We now prove the main result.

*Proof of Theorem 1.4.* As mentioned in the introduction, one direction of this result follows from standard facts. Specifically, we have already seen in Lemma 2.3 that finite $U$-rank implies finite weight. Moreover, if $G$ is superstable of finite $U$-rank then it follows from Lascar's inequality for cosets [3, Corollary III.8.2] that, in fact, $G$ has a *uniform* finite bound on the length of a $<_\infty$-chain of definable subgroups.

For the other direction, fix a stable group $G$ of finite weight, with no infinite $<_\infty$-chains of definable subgroups. We want to show that $G$ is superstable of finite $U$-rank. We may clearly assume $G$ is infinite. Suppose, toward a contradiction, that $U(G)$ is infinite (or undefined). We inductively construct sequences $(G_i)_{i<\omega}$ and $(K_i)_{i<\omega}$ of definable groups as follows.

Let $K_0 = \{1\}$ and set $G_0 = G_1 = G$. Fix $n > 0$, and suppose we have constructed $(G_i)_{i \leq n}$ and $(K_i)_{i<n}$ satisfying the following properties.

(1) For all $0 < i < n$, $K_i$ is an infinite definable normal subgroup of $G_i$.
(2) For all $0 < i \leq n$, $G_i$ is infinite and equal to $G_{i-1}/K_{i-1}$.
(3) $U(G) \leq (n-1)\,\mathrm{wt}(G) \oplus U(G_n)$.

Note that $G_n$ is an infinite stable group and, by Proposition 2.2($c$), $\mathrm{wt}(G_n) \leq \mathrm{wt}(G) < \omega$. By Lemma 3.1, we may fix an infinite connected type-definable normal subgroup $H \leq G_n$, with $U(H) = \mathrm{wt}(H)$.

From the assumption on $<_\infty$-chains of definable subgroups of $G$, along with Proposition 3.2, it is easy to see that $G_n$ also has no infinite $<_\infty$-chains of definable subgroups. Since $H$ is type-definable, it follows (with the help of Fact 2.6($b$)) that there is a definable subgroup $K_n$ of $G_n$ such that $H \leq K_n$ and $[K_n : H]$ is bounded. (In fact, the superstable descending chain condition is equivalent to this assertion on type-definable subgroups.) By Fact 2.6($a$) and normality of $H$, we may replace $K_n$ by $\bigcap_{g\in G} gK_ng^{-1}$, and thus assume $K_n$ is normal.

Since $H$ is connected and type-definable, with bounded index in $K_n$, it follows that $H = K_n^0$, which implies $U(H) = U(K_n)$ (see, e.g., [3, Sections III.4, IV,3]). Since $U(H) = \mathrm{wt}(H)$, we must have $U(K_n) = \mathrm{wt}(K_n) \leq \mathrm{wt}(G)$. Let $G_{n+1} = G_n/K_n$. Then, using (3) and Lascar's inequality for cosets [3, Corollary III.8.2],

$$\begin{aligned} U(G) &\leq (n-1)\,\mathrm{wt}(G) \oplus U(G_n) \\ &\leq (n-1)\,\mathrm{wt}(G) \oplus U(K_n) \oplus U(G_n/K_n) \leq n\,\mathrm{wt}(G) \oplus U(G_{n+1}). \end{aligned}$$

It therefore follows from our assumption on $U(G)$ that $G_{n+1}$ must be infinite. Altogether, $(G_i)_{i\leq n+1}$ and $(K_i)_{i\leq n}$ satisfy (1), (2), (3) above.

We have now constructed $(G_n)_{n<\omega}$ and $(K_n)_{n<\omega}$. Given $n < \omega$, let $\rho_n$ denote the pullback function $\rho_{G_n/K_n}$. Let $L_0 = \{1\}$ and, given $n > 0$, set $L_n = \rho_0 \dots \rho_{n-1}(K_n)$. Using Proposition 3.2, it is easy to see that $(L_n)_{n<\omega}$ is an ascending chain of definable subgroups of $G$ and, for any $n > 0$, since $K_n$ is infinite we have $L_{n-1} <_\infty L_n$. Therefore $(L_n)_{n<\omega}$ is an $<_\infty$-chain, which is a contradiction. □

**Remark 3.3.** In the previous proof, note that we needed to assume that $G$ had no infinite *ascending or descending* $<_\infty$-chains of definable groups. Specifically, no infinite descending $<_\infty$-chains allows us to construct the definable group $K_n$ from the type-definable group $H$, which is necessary to complete the induction step of the argument. No infinite ascending $<_\infty$-chains ensures us that the inductive construction halts at some finite stage. The stable group described in Example 2.9(3) also illustrates the necessity of these assumptions.

*Proof of Corollary 1.5.* Let $G$ be a stable group of finite weight, with no infinite $<_\infty$-chains of definable subgroups. Then $G$ has finite $U$-rank by Theorem 1.4 and so, by Lascar's inequality for cosets, there is a uniform finite bound on the length of a $<_\infty$-chain of definable subgroups of $G$. Let $n < \omega$ be the length of the longest $<_\infty$-chain of definable subgroups of $G$. Repeating the construction in the previous proof, it follows that there must be some $m \leq n$ such that $G_{m+1}$ is finite (since otherwise we would construct a $<_\infty$-chain $(L_0, \dots, L_{n+1})$ of length $n+1$). □

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