

Domination and regularity

Anand Pillay*

University of Notre Dame

August 13, 2018

Abstract

We discuss the close relationship between structural theorems in (generalized) stability theory, and graph regularity theorems.

1 Introduction and preliminaries

We point out analogies between domination theorems in model theory and graph regularity theorems in various “tame” contexts, showing that these are essentially the same theorems, modulo compactness and the pseudofinite yoga, and if one is not so concerned with optimal bounds.

The motivation comes partly from our joint works with Conant and Terry [4], [5], where “tame” regularity theorems in a group environment *are* obtained from structural theorems (sometimes new) concerning stable and *NIP* groups.

We will give later precise statements of all theorems (as well as references to other works). But for now we give a heuristic introduction to the notions in this paper.

First on the graph-theoretic side we recall the regularity theorems which specialize the well known Szemerédi regularity theorem. We will focus on bi-partite graphs. Szemerédi regularity concerns *all* finite graphs (V, W, E) . It says that one can partition the vertex sets V, W into a small number of sets $V_1, \dots, V_n, W_1, \dots, W_m$ such that outside a small exceptional set of pairs (i, j) , the induced subgraphs $(V_i, W_j, E|(V_i \times W_j))$ are almost regular, namely

*Supported by NSF grants DMS-1360702, DMS-1665035, and DMS-372746

sufficiently large induced subgraphs have approximately the same density. These are approximate or asymptotic statements in the sense that for every $\epsilon > 0$ there is N_ϵ such that for every finite graph etc.

Tame versions of Szemerédi regularity place restrictions on the class of finite graphs (V, W, E) considered, and try to get stronger conclusions. The kind of restrictions are: omitting a certain induced subgraph, being uniformly definable in some nice structure, or being the collection of finite induced subgraphs of some given graph definable in a nice structure. The improvements in the conclusions typically replace almost regularity by almost homogeneity (and sometimes outright homogeneity so giving a Ramsey-type theorem) and sometimes remove the need for the exceptional set.

On the model theory side, we work with theories T , or formulas $\phi(x, y)$ in a given theory, which are well-behaved in various senses, and we consider a Keisler measure μ on the x -sort over a saturated model \bar{M} , possibly restricted to definable sets in the Boolean algebra generated by instances of $\phi(x, y)$. The domination statements have the form: there is a small model M_0 , suitable space S of types over M_0 , such that if μ_0 is the measure on S induced by μ then we have (generic) *domination* of the x sort X say by S via the tautological map $\pi : X \rightarrow S$ taking $a \in X$ to its type over M_0 : for any suitable formula $\psi(x)$ over \bar{M} , outside a closed subset E_ψ of S of μ_0 -measure 0, each fibre of π cannot meet both $\psi(x)$ and $\neg\psi(x)$ in a “ μ -wide” set.

This is actually closely related to a stationarity statement: μ is the unique nonforking extension of its restriction to M_0 . And we also see an exceptional set E appearing, as in the graph regularity statement.

The work with Conant and Terry mentioned earlier is concerned with regularity (and structure) theorems in the context of finite groups G equipped with a distinguished subset A . These give rise to bipartite graphs of the form (G, G, E) where $(a, b) \in E$ iff $ab \in A$. Under assumptions (k -stable, k -NIP) on the relation E , we obtained strong theorems on the structure of the set A and its translates, where local stable and NIP group theory played a major role. We refer the reader to the preprints [4], [5] and we will not explicitly discuss these group results any further in the current paper.

We will go through three model-theoretic situations where there is a domination statement; smooth measures, generically stable measures in NIP theories, and ϕ -measures where $\phi(x, y)$ is stable. In each environment we will conclude more or less directly, via compactness, the relevant graph-regularity statement for suitable classes of finite graphs. These statements are already in the literature in various forms and we will give full references.

This paper is based partly on seminar talks the author gave at the Institut Henri Poincaré in spring 2018 during the trimester on Model theory, Combinatorics and Valued Fields. Thanks to the IHP for its hospitality and to the organizers of the trimester and the seminars. Thanks to Gabriel Conant and Caroline Terry for many discussions. For the record I would also like to thank Udi Hrushovski who already in 2012 pointed out to me (and our co-author) connections between the Lovasz-Szegedy paper [11] and our paper [9] (in particular generic compact domination). Finally, thanks to Christopher Hawthorne who pointed out a correction needed in the proof of Proposition 4.1.

No additional background is needed on the combinatorial side, as all the relevant statements (rather than proofs) are transparent.

On the model theory side we will make use of Keisler measures in a *NIP* and (formula-by-formula) stable environment. But we will make precise a few things which are not made explicit in the literature although should be considered folklore,

Our model theory notation is standard. T denotes a complete theory in a language L and we will work in a very saturated or monster model \bar{M} of T .

The book [15] is a useful reference for material on the *NIP* side, but we will usually refer to the original sources [7], [8] for Keisler measures, and [9] for generically stable and smooth measures. Insofar as stability is concerned [14] is a reference, although we take our definition of forking to be Shelah's.

For $\phi(x, y)$ an L -formula, by a ϕ -measure μ over M we mean a finitely additive probability measure on the Boolean algebra of ϕ -formulas over M , where by a ϕ -formula over M we mean a (finite) Boolean combination of instances $\phi(x, b)$ of $\phi(x, y)$ with $b \in M$. “Global” means over the monster model.

As usual a ϕ -measure over M can be identified with a regular Borel probability measure on the space $S_\phi(M)$ of complete ϕ -types over M .

A ϕ -measure over M is said to be smooth if it has a unique extension to a ϕ -measure over any larger model.

A characteristic property of ϕ -measures when $\phi(x, y)$ is stable is the following (see also Lemma 1.7 of [10]):

Fact 1.1. *Let $\phi(x, y)$ be stable (for T). Then any ϕ -measure, μ_x , over a model M is of the form $\sum_{i=1,2,\dots} \alpha_i p_i$ where p_i is a complete ϕ -type over M ,*

the α_i are positive real numbers, and $\sum_i \alpha_i = 1$.

Proof. We give a proof, for completeness, as this has not been made so explicit in earlier papers. We use Shelah's ϕ -rank $R_\phi(-)$ from Section 3, Chapter 1, of [14] where its basic properties are given (and where really we mean Δ -rank where $\Delta = \{\phi(x, y), x = z\}$). Let p_1, \dots, p_k be the finitely many complete ϕ -types of maximal ϕ -rank n say. Without loss of generality p_1, \dots, p_r have positive μ -measures, (say $\alpha_1, \dots, \alpha_r$, respectively) and p_{r+1}, \dots, p_k have μ -measure 0.

Working in the space $S_\phi(M)$ let U be the complement of $\{p_1, \dots, p_r\}$, an open set whose μ_x -measure is $\beta = 1 - (\alpha_1 + \dots + \alpha_r)$, which we can assume to be positive (otherwise already $\mu = \alpha_1 p_1 + \dots + \alpha_r p_r$). We can find clopen $U_1 \subset U_2 \dots \subset U_i \subset \dots \subset U$, and positive reals $\beta_1 < \beta_2 < \dots < \beta_i < \dots$ such that $\mu(U_i) = \beta_i$ for all i and $\lim_{i \rightarrow \infty} \beta_i = \beta$.

Now U_1 and each $U_{i+1} \setminus U_i$ are ϕ -definable sets of positive measure and with ϕ -rank $< n$. So we can apply induction, to write each of $\mu|_{U_1}, \dots, \mu|(U_{i+1} \setminus U_i), \dots$ as a suitable $\sum_j \gamma_j q_j$. Putting these together with $\alpha_1 p_1 + \dots + \alpha_r p_r$ gives the required expression of μ . \square

The following is not required, but included for completeness.

Corollary 1.2. *If $\phi(x, y)$ is stable and μ_x is a ϕ -measure over M . Then μ is smooth if and only if μ is a weighted sum of realized ϕ -types, i.e. of the form $\sum_i \alpha_i tp_\phi(a_i/M)$ with $a_i \in M$.*

Finally we discuss pseudofiniteness.

Definition 1.3. *Let M be an L -structure and A an arbitrary (not necessarily definable) subset of a sort X in M . We say that A is pseudofinite in M if for any sentence σ in the language L together with an additional predicate symbol P on sort X , if $(M, A) \models \sigma$ then there is an L -structure M' and finite subset A' of $X(M')$ such that $(M', A') \models \sigma$.*

If M is 1-sorted and A is M itself then we say that M is pseudofinite.

From the definition, finite implies pseudofinite.

Remark 1.4. *Suppose that A happens to be definable by a formula $\phi(x, b)$ in the structure M . Then pseudofiniteness of A in M is equivalent to; for every L -formula $\psi(y) \in tp_M(b)$ there is an L -structure M' and $b' \in M'$ satisfying $\psi(y)$ such that $\phi(x, b')(M')$ is finite.*

The following is routine.

Lemma 1.5. *For M an L -structure and A a subset of a sort X in M , the following are equivalent:*

- (i) A is pseudofinite in M ,
- (ii) Let Σ be the set of $L(P)$ -sentences which are true of every (M', A') where M' is an L -structure, and A' is a finite subset of the interpretation of the sort X in M' . Then $(M, A) \models \Sigma$.
- (iii) (M, A) is elementarily equivalent to some ultraproduct of $L(P)$ -structures (M', A') where A' is finite.

We will now talk about the standard model \mathbb{V} of set theory and saturated elementary extensions \mathbb{V}^* of \mathbb{V} . It doesn't really make so much sense, but really we work with some small fragment of set theory including the natural numbers, the reals and all arithmetic operations on them together with cardinality maps for finite sets. The reader can work out for himself or herself the appropriate rigorous statements.

Proposition 1.6. *Suppose (M, A) is pseudofinite. Then there is a (saturated if you wish) elementary extension \mathbb{V}^* of \mathbb{V} and some (M^*, A^*) in \mathbb{V}^* such that*

- (i) (M^*, A^*) is elementarily equivalent to (M, A) ,
- (ii) A^* is finite in the sense of \mathbb{V}^* , and
- (iii) Whenever ψ is a formula of set theory which is true in \mathbb{V}^* of (M^*, A^*) then there is (M, A) (in the standard model), such that ψ is true of (M, A) and A is finite.

Moreover suppose that (M, A) is a model of the common theory of (M_n, A_n) for $n < \omega$ where A_n is finite and of increasing size, and A is infinite, then (M^*, A^*) can be chosen to satisfy also

- (iii)' Whenever ψ is a formula of set theory true of (M^*, A^*) in \mathbb{V}^* then ψ is true of infinitely many (M_n, A_n) in \mathbb{V} .

Proof. This is a compactness argument. Consider the complete diagram of \mathbb{V} together with set of formulas $\psi(y, z)$ true of every (M, A) in \mathbb{V} where M is an L -structure and A a finite subset of the appropriate sort, as well as the formulas expressing that (y, z) is elementarily equivalent (in $L(P)$) to (M, A) . It is finitely satisfiable (in \mathbb{V}), so has a (saturated if you wish) model. The moreover statement is also clear. \square

Remark 1.7. *Typically we take \mathbb{V}^* to be saturated so (M^*, A^*) will be appropriately saturated, so isomorphic to (M, A) assuming the latter was already*

saturated (assuming some set theory and appropriate choices of degree of saturation).

Given \mathbb{V}^* and (M^*, A^*) as in Proposition 1.6, as A^* is finite in the sense of \mathbb{V}^* , every internal subset Z of A^* has a finite cardinality in the sense of \mathbb{V}^* (i.e. $|Z| \in \mathbb{N}^*$) and we obtain the nonstandard normalized counting measure μ^* on the Boolean algebra of internal subsets of A^* which takes Z to $|Z|/|A^*|$, a number in $[0, 1]^*$. For Z a definable (in M^*) subset of the ambient sort X in which A^* lives, define $\mu^*(Z) = \mu^*(Z \cap A^*)$. So μ^* gives a “nonstandard” Keisler measure on the sort X in M^* , in the sense that the values of μ^* are in the nonstandard unit interval (as well as finite additivity etc). We define μ to be the standard part of μ^* (restricted to definable sets) and we see that that μ is Keisler measure on the sort X is the L -structure M^* , which we call the *pseudofinite* Keisler measure on X given by A^* (and the ambient structure \mathbb{V}^*).

The following is important (and well-known). It can be proved by an adaptation of the material in section 2.2 of [3]. In any case we follow the notation and context of Proposition 1.6 and the above construction.

Proposition 1.8. *Assuming $\text{Th}(M^*)$ is NIP, then the pseudofinite Keisler measure on the sort X is generically stable, namely definable over and finitely satisfiable in some small model M_0 .*

2 The distal case

The distal regularity theorem [1] is an attractive generalization of a result of Fox et al [6] on a strong regularity theorem for semialgebraic graphs.

Our treatment here is related to that of Simon [16], but we make more explicit the connection with compact domination.

The relevant structural theorem concerns arbitrary smooth Keisler measures. Recall that a Keisler measure μ_x over M is smooth if it has a unique extension over any N containing M . A global Keisler measure μ_x is said to be smooth over a small submodel M_0 if μ is the unique extension over \bar{M} of $\mu|_{M_0}$.

Here is the domination theorem for smooth measures. It is more or less tautological.

Proposition 2.1. *Fix T , sort X , saturated model \bar{M} and Keisler measure μ on X over \bar{M} . Let M_0 be a small elementary submodel of \bar{M} and $\pi :$*

$X \rightarrow S_X(M_0)$ the tautological map. Suppose μ is smooth over M_0 . Then for every definable (with parameters from \bar{M}) subset Y of X there is a closed subset E of $S_X(M_0)$ of μ_0 -measure 0 such that for every $p \in S_X(M_0)$ either $\pi^{-1}(p) \subset Y$ or $\pi^{-1}(p) \cap Y = \emptyset$.

Proof. Otherwise the (closed) set E of $p \in S_X(M_0)$ such that $p(x)$ is consistent with each of $x \in Y$ and $x \notin Y$, has μ_0 -measure $= \alpha > 0$. Let $(\mu_0)_E$ be the localization of μ_0 to E . Then $(\mu_0)_E$ has two different extensions to a measure over \bar{M} , one giving Y measure 1 and one giving it measure 0. It follows that μ_0 itself has two different extensions to \bar{M} , contradicting smoothness. \square

The following strong regularity (or Ramsey-type) statement is a simple compactness argument applied to Proposition 2.1.

Corollary 2.2. *Let (V, W, R) be a bipartite graph definable in a structure M . Let μ be a smooth Keisler measure on V over M , and ν an arbitrary Keisler measure on W over M . Let $\epsilon > 0$. Then there are partitions $V = V_1 \cup \dots \cup V_n$ and $W = W_1 \cup \dots \cup W_m$ of V, W respectively into definable sets, and a set Σ of pairs (i, j) of indices such that*

- (i) $(\mu \times \nu)(\cup_{(i,j) \in \Sigma} (V_i \times W_j)) < \epsilon$ and
- (ii) for $(i, j) \notin \Sigma$, $V_i \times W_j$ is homogeneous for R , namely $V_i \times W_j$ is either contained in R or disjoint from R .

Proof. We may assume M to be saturated. Let M_0 be a small elementary submodel of M such that μ is smooth over M_0 and $R(x, y)$ is definable over M_0 . We make use of Proposition 2.1 with $X = V$.

Fix $\epsilon > 0$. For any b , Let E_b the closed μ_0 -measure 0 subset of $S_V(M_0)$ outside of which each fibre of π is either contained in or disjoint from $R(x, b)$. Clearly E_b depends only on $tp(b/M_0)$ and so we write as E_q . Let Z_q be an M_0 -definable set containing E_q and with μ_0 -measure $< \epsilon$. By compactness we can partition $V \setminus Z_q$ into M_0 -definable sets $V_{q,1}, \dots, V_{q,n_q}$ such that for each i , $\pi^{-1}(V_{q,i})$ is either contained in $R(x, b)$ (for some/all b realizing q) or disjoint from $R(x, b)$ (for some/all b realizing q). We can now, by compactness, replace q by a formula (or M_0 -definable set) W_q in q , so that for each $i \leq n_q$ either $V_{q,i}$ is contained in $R(x, b)$ for all $b \in W_q$, or $V_{q,i}$ is disjoint from $R(x, b)$ for all $b \in W_q$.

Doing this for each q and applying compactness gives us a partition W_{q_1}, \dots, W_{q_m} of W into M_0 -definable sets, and for each $j = 1, \dots, m$ a partition $V = V_{q_j,1} \cup V_{q_j,2} \cup \dots \cup V_{q_j,n_{q_j}} \cup Z_{q_j}$, such that $\mu_0(Z_{q_j}) = 0$ for all j , and

for all j, i ,

(*) $\pi^{-1}(V_{q_j, i})$ is either contained in $R(x, b)$ for all $b \in W_{q_j}$ or is disjoint from $R(x, b)$ for all $b \in W_{q_j}$.

Let V_1, \dots, V_t be a common refinement of this finite collection of partitions of V . We claim that this partition, together with the partition W_{q_1}, \dots, W_{q_m} of W is as required. We have to identify the exceptional set E of pairs. So let $E = \{(i, q_j) : V_i \subseteq Z_{q_j}\}$. For each q_j , $\cup\{V_i \times W_{q_j} : V_i \subseteq Z_{q_j}\} = Z_{q_j} \times W_{q_j}$ which has $\mu \times \nu$ measure $< \epsilon \nu(W_{q_j})$. So summing over the q_j we get $(\mu \times \nu)(\cup_{(i, q_j) \in E} (V_i \times W_{q_j})) < \epsilon$. And for $(i, q_j) \notin E$, V_i will be contained in $V_{q_j, i}$ for some i , so by (*) $V_i \times W_{q_j}$ is either contained in or disjoint from R . \square

The notion of a *distal* first order theory T was introduced by Pierre Simon in his thesis (see [15]). One of the characterizations of distality is that T has *NIP* and every generically stable measure is smooth. Among distal theories are *o*-minimal theories (such as *RCF*), the theory of \mathbb{Q}_p , and $Th(\mathbb{Z}, +, <)$.

So here is the distal regularity theorem, stated for suitable families of finite graphs.

Proposition 2.3. *Let $\mathcal{G} = (G_i : i \in I)$ be a family of finite (bipartite) graphs $G = (V, W, R)$ such that one of the following happens:*

(i) *The graphs are uniformly definable in some model M of a distal theory T ,*

(ii) *For some model M of some distal theory T , there is a graph (V, W, R) definable in M such that \mathcal{G} is the family of finite (induced) subgraphs of (V, W, E) , or*

(iii) *Every model (V, W, R) of the common theory of the G_i 's is interpretable in a model of some distal theory.*

THEN, for any ϵ there is N_ϵ , such that for every $(V, W, R) \in \mathcal{G}$, there are partitions $V = V_1 \cup \dots \cup V_n$, and $W = W_1 \cup \dots \cup W_m$, with $n, m < N_\epsilon$ such that for some "exceptional" set E of pairs (i, j) (with $1 \leq i \leq n$ and $1 \leq j \leq m$),

the cardinality of $\cup_{(i, j) \in E} (V_i \times W_j)$ is $< \epsilon |V| |W|$, and for all $(i, j) \notin E$, $V_i \times W_j$ is either contained in R or disjoint from R .

Proof. Context (ii) is the one dealt with in [1] and which generalizes [6]. Note that Context (i) would be vacuous when T is *o*-minimal as we have finite bounds on the cardinalities of uniformly definable finite sets, but for the p -adics and/or Presburger, it is nonvacuous.

The proof of the proposition uses Propositions 1.6 and 1.8 (with a possible variant in Context (iii)). We focus here on Context (i). Suppose the conclusion fails. So for some fixed ϵ , no finite N works. So we can find $N_1 < N_2 < \dots$, and counterexamples G_{N_1} for N_1 , G_{N_2} for N_2 , etc. in \mathcal{G} with the cardinalities of the vertex sets increasing. Proposition 1.6 gives us a (saturated) model M^* of T in some \mathbb{V}^* and definable $G^* = (V^*, W^*, R^*)$ in M^* such that V^*, W^* are finite in the sense of \mathbb{V}^* and the moreover clause (iii)' holds. Let μ^*, ν^* be the nonstandard normalized counting measures on V^* and W^* , given by the construction following Remark 1.7, and let μ and ν be the corresponding pseudofinite Keisler measures. By 1.8 μ is generically stable, so smooth as $Th(M^*)$ is distal. Apply Corollary 2.2 to (V^*, W^*, R^*) with say $\epsilon/2$, to get (definable) partitions V_1^*, \dots, V_n^* of V and W_1^*, \dots, W_m^* of W , and an exceptional set E of pairs (i, j) satisfying the conclusions of 2.2. Choose $\epsilon/2 < \delta < \epsilon$, and we can express the existence of the partitions and that $(\mu^* \times \nu^*)(\cup_{(i,j) \in E} (V_i^* \times W_j^*)) < \delta$ and that for $((i, j) \notin E, V_i^* \times W_j^*$ is homogeneous for R^* by the truth of formula ψ of set theory for M^*, G^* in \mathbb{V}^* . The moreover clause of Proposition 1.6 tells that ψ is true for infinitely many of the (M, G_{N_k}) in \mathbb{V} , and for $N_k > n, m$ we get a contradiction. \square

3 The NIP case

The regularity lemma for finite graphs (V, W, R) where the relation R is “ k – NIP” (or has VC-dimension bounded by approximately k) has a nice and elementary direct proof in [2] (in the greater generality of hypergraphs). However we want to again deduce it just from a domination statement, so we work in the context where such statements are currently available, namely inside a NIP theory.

We first recall the notion “ μ -wide”. If μ_x is a (say global) Keisler measure and $\Sigma(x)$ is a partial type over a small set, we say that $\Sigma(x)$ is μ -wide if every finite conjunction of formulas in Σ has μ measure > 0 .

Again we start with the (generic) domination theorem for generically stable measures.

Proposition 3.1. *Suppose T is NIP. Let μ be a global generically stable measure on the definable set (or sort) X and assume μ does not fork over M_0 (so is definable over M_0). Let $\pi : X \rightarrow S_X(M_0)$ be the tautological map, and let μ_0 be the induced measure on $S_X(M_0)$. Then for every definable (with*

parameters from \bar{M}) subset Y of X there is a closed set $E \subseteq S_X(M_0)$ of μ_0 measure 0 such that for each $p \in S_X(M_0) \setminus E$, not both $p \cup "x \in Y"$ and $p \cup "x \notin Y"$ are μ -wide.

Proof. We deduce this formally from the basic results in [9]. First by Proposition 3.3 of [9], μ is the the unique global nonforking extension of its restriction to M_0 . Let \mathcal{P} the space of global complete types $p(x)$ which do not fork over M_0 , and let π' be the restriction map from \mathcal{P} to $S_X(M_0)$. Then by Theorem 5.4, of [9], \mathcal{P} is dominated by $((S_X(M_0), \pi', \mu)$ in the sense that for any formula $\phi(x)$ over \bar{M} the set E of $p \in S_X(M_0)$ such that $\pi'^{-1}(p)$ intersects both (the clopen determined by) $\phi(x)$ and (the clopen determined by) $\neg\phi(x)$, has μ_0 measure 0. Note that E is closed. Recall that we $\pi : X \rightarrow S_X(M_0)$ takes $a \in X(\bar{M})$ to $tp(a/M_0)$. Now suppose that $p(x) \in S_X(M_0) \setminus E$. If $p(x) \cup \{\phi(x)\}$ is μ -wide, then as μ does not fork over M_0 , $p(x) \cup \{\phi(x)\}$ does not fork over M_0 so extends to some $p' \in \mathcal{P}$. Likewise if $p(x) \cup \{\neg\phi(x)\}$ is μ -wide, it extends to some $p'' \in \mathcal{P}$. So as $p \notin E$, not both can happen. \square

A simple compactness argument applied to Proposition 3.1 again gives a strong regularity theorem.

Corollary 3.2. *Suppose T is NIP and (V, W, R) is a graph definable in a model M of T . Let μ be a generically stable measure on V over M and ν any Keisler measure on W over M . Fix $\epsilon > 0$. Then there are partitions $V = V_1 \cup \dots \cup V_n$ and $W = W_1 \cup \dots \cup W_m$ of V, W into definable sets, and an exceptional set E of pairs (i, j) of indices such that*

- (i) *The $\mu \times \nu$ measure of $\cup_{(i,j) \in E} V_i \times W_j$ is $< \epsilon$, and*
- (ii) *For any $(i, j) \notin E$, either $(\mu \otimes \nu)((V_i \times W_j) \cap R) < \epsilon\mu(V_i)\nu(W_j)$ or $(\mu \otimes \nu)((V_i \times W_j) \setminus R) < \epsilon\mu(V_i)(\nu(W_j))$*

Proof. We follow the proof of Corollary 2.2, but with ϵ -homogeneous in place of homogeneous (using definability over M_0 of μ), and paying slightly more attention to the exceptional set.

Again assume M to be saturated, and suppose μ does not fork over M_0 . Fix $\epsilon > 0$. We use Proposition 3.1 with $X = V$. For each $q \in S_W(M_0)$ we find closed $E_q \subseteq S_V(M_0)$ of μ_0 -measure 0, such that for each $p \in S_V(M_0) \setminus E_q$ and some (any) b realizing q at most one of $p(x) \cup \{R(x, b)\}$, $p(x) \cup \{\neg R(x, b)\}$ is μ -wide. Let Z_q be an M_0 -definable set containing E_q and of μ_0 -measure $< \epsilon/2$. By compactness we can partition $V \setminus E_q$ into M_0 -definable sets $V_{q_1}, \dots, V_{q, n_q}$ such that for each i , either $\mu(V_{q,i} \cap R(x, b)) = 0$ (for some/all b realizing

q), or $\mu(V_{q_i} \setminus R(x, b)) = 0$ for some/all b realizing q). We may assume that $\mu(V_{q_i}) > 0$ for each i (otherwise just add it to Z_q).

Now we use definability of μ over M_0 to find an M_0 -definable set W_q containing q such that for each $i = 1, \dots, n_q$ exactly one of the following holds:

- (i)_q for all $b \in W_q$, $\mu(V_{q,i} \cap R(x, b)) < (\epsilon^2/2)\mu(V_{q,i})$.
- (ii)_q for all $b \in W_q$, $\mu(V_{q,i} \setminus R(x, b)) < (\epsilon^2/2)\mu(V_{q,i})$.

By compactness we can find q_1, \dots, q_m such that W_{q_1}, \dots, W_{q_m} partition W . Again we find a common refinement V_1, \dots, V_r of the finitely many partitions $V_{q_j,1}, \dots, V_{q_j,n_j}, Z_{q_j}$ of V .

Then $V = V_1 \cup \dots \cup V_r$ and $W = W_{q_1} \cup \dots \cup W_{q_m}$ will be the desired partitions. We have to check that it works.

We have to identify the exceptional set of pairs of indices.

To that avail let us fix some q_i and call it q , and we focus on the subgraph $(V, W_q, R|(V \times W_q))$. Let $I = \{i : 1 \leq i \leq n_q : \text{and } (i)_q \text{ above holds}\}$. Let J be the rest of the indices i between 1 and n_q , namely where $(ii)_q$ holds.

Let $B \subseteq V \times W_q$ be $\cup_{i \in I}((V_{q,i} \times W_q) \cap R) \cup \cup_{i \in J}((V_{q,i} \times W_q) \setminus R)$. it is then clear that

Claim 1. $(\mu \otimes \nu)(B) < (\epsilon^2/2)\nu(W_q)$.

Let $\Sigma_{q,1}$ be the set of indices $i = 1, \dots, r$ such that $V_i \subseteq Z_q$, and $\Sigma_{q,2}$ the set of indices i such that $(\mu \otimes \nu)((V_i \times W_q) \cap B) \geq \epsilon\mu(V_i)\nu(W_q)$. Let Σ_q be the (disjoint) union of $\Sigma_{q,1}$ and $\Sigma_{q,2}$.

Then

Claim 2. $\sum_{i \in \Sigma_q} (\mu \otimes \nu)(V_i \times W_q) < \epsilon\nu(W_q)$.

Proof of Claim 2. Note that $\sum_{i \in \Sigma_{q,1}} (\mu \otimes \nu)(V_i \times W_q) \leq (\mu \otimes \nu)(Z_q \times W_q) < (\epsilon/2)\nu(W_q)$. So it suffices to prove that $\sum_{i \in \Sigma_{q,2}} (\mu \otimes \nu)(V_i \times W_q) < (\epsilon/2)\nu(W_q)$. If not then by the definition of $\Sigma_{q,2}$, $(\mu \otimes \nu)(B) \geq \sum_{i \in \Sigma_{q,2}} (\mu \otimes \nu)((V_i \times W_q) \cap B) \geq \sum_{i \in \Sigma_{q,2}} \epsilon\mu(V_i)\nu(W_q) = \epsilon \sum_{i \in \Sigma_{q,2}} (\mu \otimes \nu)(V_i \times W_q) \geq (\epsilon^2/2)\nu(W_q)$, which contradicts Claim 1.

Now suppose $t \notin \Sigma_q$.

Case (i). $V_t \subseteq V_{q,i}$ for some $i \in I$.

So $(V_t \times W_q) \cap B = (V_t \cap W_q) \cap R$ and has $\mu \otimes \nu$ measure $< \epsilon\mu(V_t)\nu(W_q)$.

Case (ii). $V_t \subseteq V_{q,i}$ for some $i \in J$.

Likewise we have that $(\mu \otimes \nu)((V_t \times W_q) \setminus R) < \epsilon\mu(V_t)\nu(W_q)$.

Now let the global exceptional set E be $\{(i, q_j) : i \in \Sigma_{q_j} : i = 1, \dots, r, j = 1, \dots, m\}$, and we see from Claim 2 (as well as the Case (i), Case (ii) discussion

above) that the conclusions (i) and (ii) or Corollary 3.2 are satisfied. \square

The application to families of finite graphs is almost identical to Proposition 2.3, with a similar proof, but we state it anyway.

Proposition 3.3. *Let $\mathcal{G} = (G_i : i \in I)$ be a family of finite (bipartite) graphs $G = (V, W, R)$ such that one of the following happens:*

(i) *The graphs are uniformly definable in some model M of an NIP theory T ,*

(ii) *For some model M of some NIP theory T , there is a graph (V, W, R) definable in M such that \mathcal{G} is the family of finite (induced) subgraphs of (V, W, E) , or*

(iii) *Every model (V, W, R) of the common theory of the G_i 's is interpretable in a model of some NIP theory.*

THEN, for any ϵ there is N_ϵ , such that for every $(V, W, R) \in \mathcal{G}$, there are partitions $V = V_1 \cup \dots \cup V_n$, and $W = W_1 \cup \dots \cup W_m$, with $n, m < N_\epsilon$ such that for some “exceptional” set E of pairs (i, j) (with $1 \leq i \leq n$ and $1 \leq j \leq m$),

(a) *cardinality of $\cup_{(i,j) \in E} V_i \times W_j$ is $< \epsilon |V| |W|$, and*

(b) *for all $(i, j) \notin E$, either $|(V_i \times W_j) \cap R| < \epsilon |V_i| |W_j|$ or $|(V_i \times W_j) \setminus R| < \epsilon |V_i| |W_j|$*

Remark 3.4. (i) *Note that Corollary 3.2 depends only on the Keisler measure μ satisfying the generic domination statement over M_0 in Proposition 3.1, as well as being definable over M_0 (i.e. full generic stability of μ and NIP-ness of T are not needed).*

(ii) *Likewise, if $R(x, y)$ is $\phi(x, y)$, assuming the domination statement and definability for a ϕ -measure μ , Corollary 3.2 will hold.*

4 The stable case

The stable regularity theorem concerns finite graphs (V, W, R) where the edge relation $R(x, y)$ is k -stable, and gives a partition into almost homogeneous subgraphs but without any exceptional set. The original statement and proof are in [13] and involve finite combinatorics in the presence of the Shelah 2-rank and give optimal bounds. A pseudofinite proof making use of local stability theory was given in [12]. The proof we present here is a simplification

of the latter. There is no explicit use of any local ranks, other than ingredients in the proof of Fact 1.1.

We first discuss the methods and relationship with the previous proofs. Fix a complete theory T and a stable formula $\phi(x, y)$, where x is of sort X . Let μ be a ϕ -measure on X over a saturated model say \bar{M} . Let M_0 be a small model such that μ does not fork over M_0 (i.e. every ϕ formula over M_0 with positive measure does not fork over M_0). Now every complete ϕ -type p over M_0 has a unique nonforking extension over \bar{M} (i.e. to a complete ϕ -type over \bar{M}). It follows that for each $p \in S_\phi(M_0)$ and any $b \in \bar{M}$ at most one of $p \cup \{\phi(x, b)\}$, $p \cup \{\neg\phi(x, b)\}$ is μ -wide. So we have domination of X by $S_\phi(M_0)$ (but with no exceptional sets). So we can run the proof of Corollary 3.2, but note that it nevertheless gives a possibly nonempty set of exceptional pairs (i, j) in the regularity statement. So more is needed, and this is precisely Fact 1.1.

Remember that a bipartite graph (V, W, R) (or rather the edge relation R on this graph) is k -stable if there do not exist $a_1, \dots, a_k \in V$, $b_1, \dots, b_k \in W$ such that $R(a_i, b_j)$ iff $i \leq j$. Given an L -structure M and L -formula $\phi(x, y)$ we get a corresponding bipartite graph (X, Y, R) (where X is the x -sort in M , Y the y -sort in M and R the interpretation of $\phi(x, y)$ in M). And the formula $\phi(x, y)$ is stable for $Th(M)$ iff (X, Y, R) is k -stable for some finite k . ϕ^* is the same formula as ϕ except the roles of variable variable and parameter variable are interchanged.

We first give the strong regularity theorem (analogues of Corollaries 2.2 and 3.2) for arbitrary graphs where the edge relation is stable.

Proposition 4.1. *Let (V, W, R) be a graph definable in some structure M where the relation $R(x, y)$ is stable. Identify R with the L -formula defining it. Let μ_x be a Keisler measure on V over M , Then for each $\epsilon > 0$ there are partitions $V = V_1 \cup \dots \cup V_n$ of V and $W = W_1 \cup \dots \cup W_m$ of W into definable sets such for each i, j , either for all $b \in W_j$, $\mu(V_i \setminus R(x, b)) \leq \epsilon\mu(V_i)$, or for all $b \in W_j$, $\mu(V_i \cap R(x, b)) \leq \epsilon\mu(V_i)$. Moreover each V_i can be defined by an R -formula, and each W_j by a R^* -formula.*

Proof. By Fact 1.1, $\mu|_R = \sum_{i \in I} \alpha_i p_i$, for some countable I (which we may assume to be ω or a proper initial segment of ω), complete R -types p_i over M , and α_i with $0 < \alpha_i \leq 1$ such that $\sum_i \alpha_i = 1$.

Note that $\mu(p_i) = \alpha_i > 0$ for $i \in I$. Fix $\epsilon > 0$. For each $i \in I$, let V_i be a formula (clopen) in p_i such that $\mu(V_i) < \alpha_i/(1 - \epsilon)$. Let B be $S_R(M) \setminus \{p_i : i \in I\}$. So B is Borel and $\mu(B) = 0$. Let $\delta = (\alpha_0/(1 - \epsilon)) - \mu(V_0)$,

and let $U \supseteq B$ be open such that $\mu(U) < \delta$. Then U together with the V_i form an open cover of $S_R(M_0)$, so let U, V_0, \dots, V_n be a finite subcover. We may assume that V_0, \dots, V_n are disjoint (and we still have that $V_j \in p_j$ and $\mu(V_j) < \alpha_j/(1 - \epsilon)$).

Let V'_0 be the complement of $(V_1 \cup \dots \cup V_n)$ in $S_R(M)$. So V'_0 is clopen and $p_0 \in V'_0 \subseteq U \cup V_0$. Moreover by the choice of U we have that $\mu(V'_0) < \alpha_0/(1 - \epsilon)$.

Thus, if we replace V_0 by V'_0 , we get that V_0, \dots, V_n are clopen sets partitioning $S_R(M)$ (in other words R -definable sets which partition V), with $p_i \in V_i$, and $\mu(V_i \setminus p_i) < \epsilon\mu(V_i)$.

Now we partition W using the R -definitions of p_0, \dots, p_n : For each $i = 0, \dots, n$ the R -definition of p_i is a R^* -formula $\psi_i(y)$ over M (with the property that for all $b \in M$, $R(x, b) \in p_i$ iff $M \models \psi_i(b)$). For each subset J of $\{1, \dots, n\}$, let W_J be the set defined by $\bigwedge_{i \in J} \psi_i(y) \wedge \bigwedge_{i \notin J} \neg \psi_i(y)$. So the W_J partition W into R^* -definable sets. $V = V_0 \cup \dots \cup V_n$ and $W = \bigcup_J W_J$ will be the desired partitions of V, W .

We have to check that the conclusions hold. Note that for for each $i \in \{0, \dots, n\}$ and $J \subseteq \{1, \dots, n\}$, we have either

- (a) for all $b \in W_J$, $R(x, b) \in p_i$, or
- (b) for all $b \in W_J$, $\neg R(x, b) \in p_i$.

In case (a), for all $b \in W_J$, $\mu(V_i \setminus R(x, b)) \leq \mu(V_i \setminus \{p_i\}) < \epsilon\mu(V_i)$. In case (b) for all $b \in W_J$, $\mu(V_i \cap R(x, b)) \leq \mu(V_i \setminus \{p_i\}) < \epsilon\mu(V_i)$.

So we have the desired partition and the proof is complete.

This completes the proof. □

The stable regularity lemma (or a suitable version) for families of finite graphs now follows as earlier:

Corollary 4.2. *Fix k and let \mathcal{G} be the family of finite graphs (V, W, R) where the relation R is k -stable. Then for any $\epsilon > 0$, there is N such that for each $(V, W, R) \in \mathcal{G}$ there are partitions $V = V_1 \cup \dots \cup V_n$ and $W = W_1 \cup \dots \cup W_m$ with $n, m < N$ such that for each i and j , either $|(V_i \times W_j) \cap R| \leq \epsilon|V_i||W_j|$ (so the induced graph on V_i, W_j is almost empty) or $|(V_i \times W_j) \setminus R| \leq \epsilon|V_i||W_j|$ (so the induced graph on V_i, W_j is almost complete).*

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