The free group has the dimensional order property

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Abstract

We prove that the common theory T_{fg} of nonabelian groups has the *dimensional order property*, or DOP, implying, for example, that there is no reasonable structure theorem for $\aleph_1$-saturated models of T_{fg}.

1 Introduction

By the work of Sela [12] and Kharlampovich-Myasnikov [4], all noncommutative free groups are elementarily equivalent (as structures in the group language) and we denote the common (complete) first order theory by T_{fg}. For some time we have been suggesting that "algebraic geometry over the free group" should be the study of the category $Def(T_{fg})$ of definable sets in the free group. In a major piece of work [13], Sela proved that T_{fg} is a *stable theory*. This gives a really new kind of stable theory (or group), and there are a host of notions, properties, and invariants, that one can ask about. The issue has been raised of what groups are definable (or more generally interpretable) in a free group [8],[1]. The latter paper succeeds in provnig the conjecture that no infinite field is definable in the free group. The same paper also shows that centralizers in a free group are cyclic groups with no additional induced structure. The latter statement and other results from [1] will be heavily used in the current paper.

A basic invariant of a (complete) first order theory T is the category $Mod(T)$ of models of T (with elementary embeddings) and the focus of much work especially in [14] was the problem of computing the possible functions $I(\lambda, T)$ = number of models of T of cardinality λ, up to isomorphism, as T varies, and where possible *describing* the models of T. Now T_{fg} being unsuperstable has maximum spectrum function, namely $I(\lambda, T) = 2^\lambda$ for any $\lambda > \aleph_0$. But within the class of (countable) stable, unsuperstable, theories, there is also the possibility of describing or classifing the $\aleph_1$-saturated

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models. The *dimensional order property* (DOP), which will be formally defined in section 2, rules out such a classification, and implies that for typical uncountable λ, T has 2^λ $\aleph_1$-saturated models of cardinality λ. So we prove:

Theorem 1: *T_{fg} has the dimensional order property.*

It is worth noting that there *is* a classification of the $\aleph_1$-saturated models of the theory $Th(\mathbb{Z},+)$ of the free group on 1 generator: any such model is of the form $\widehat{\mathbb{Z}} + \mathbb{Q}^{(\kappa)}$, where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and $\kappa \geq \aleph_1$. In particular for $\kappa > 2^{\aleph_0}$ there is a unique $\aleph_1$-saturated model of cardinality κ. We had wondered for some time about whether there is a reasonable description of the $\aleph_1$-saturated models of the theory T_{fg} of the free group on 2 generators, and our main theorem implies a negative answer.

We will prove that T_{fg} has the DOP by showing
(*) if e_1, e_2 are independent generics then the centralizers $C(e_1)$, $C(e_2)$ are orthogonal.

It is well-known that (*) implies the DOP but we nevertheless give details of the reduction (and more) in Section 2.

The proof of (*) makes use of recent results in [1]. In fact there is a further reduction, using some geometric stability, to proving that the centralizers $C(e_1)$, $C(e_2)$ are not definably isomorphic, and the latter statement is what is actually proved in section 3.

The question of whether T_{fg} has the DOP was raised by the first author when the second author was his Ph.D. student in Leeds.

In the remainder of the introduction we recall some general facts about the model theory of the free group. In section 2 we discuss stability and the DOP property. In Section 3 the main technical result is proved and we will introduce there the required machinery.

We will assume some familiairity with model theory and stability, although in section 2 we will recall details of some classification-theoretic notions.

As above, T_{fg} denotes the common theory of finitely generated non-abelian free groups, and is complete. We typically let $\mathbb{F}$ denote a standard model, namely a free group $\mathbb{F}_k$ on k generators with $k \geq 2$. T_{fg} being stable, stable group theory applies. We let $\bar{M}$ be a saturated mode. We recall some facts and results.

Fact 1.1: *(i) The free group is connected.*
(ii) Denoting by p_0 the unique generic type of the free group over $\emptyset$, $p_0 = tp(a/\emptyset)$ when a is any primitive element of $\mathbb{F}$ (i.e. member of a free basis). Moreover, if $a_1,...,a_n \in \mathbb{F}$ extends to a basis of $\mathbb{F}$ then $a_1,..,a_n$ are independent (in the sense of nonforking) realizations of p_0.
(iii) The proper nontrivial definable (with parameters) subgroups of $\mathbb{F}$ are precisely the cyclic subgroups, and hence are the finite index subgroups of centralizers.
(iv) Let $a \in \mathbb{F} \setminus \{1\}$. Then $(C(a),.)$ as a subgroup of the saturated model,

is "stably embedded" in the sense that any set of n-tuples from $C(a)$ which is definable (with parameters) in the ambient structure $\bar{M}$ is definable (with parameters) in the structure $(C(a),.)$. In particular $C(a)$ has U-rank 1 and is locally modular (or 1-based), as a definable group in $\bar{M}$.

References. (i) and (ii) appears in [10] (although a quick proof follows from results of Poizat [11] on $\mathbb{F}_\omega$), (iii) is Theorem 3 of [8], and (iv) is Corollary 6.27 of [1].

2 Stability

The aim of this section is to give more precise details about the DOP and how we will plan to prove it in the case at hand.

The book [14] is the basic reference for classification theory and associated notions. But we also refer the reader to Section 4 of Chapter 1 of [9], where there is an account of notions such as a-model, domination, weight, etc., which we summarize here.

T will denote a countable, complete, stable theory, and we often work in a saturated model $\bar{M}$.

What we call (following Makkai [7]) an a-model in [9] is what Shelah calls an $F^a_{\kappa(T)}$-saturated model. When T is superstable $\kappa(T) = \aleph_0$, and an a-model is just a model in which all strong types over finite sets are realized. When T is not superstable, as is the case for T_{fg}, $\kappa(T) = \aleph_1$, and an a-model is precisely a model in which all strong types over countable sets are realized, which just amounts to being $\aleph_1$-saturated.

If p, q are stationary types over A, B, respectively, they are said to be *orthogonal* if for any $C \supseteq A \cup B$, the nonforking extensions $p|C$ and $q|C$ of p, q over C, are almost orthogonal in the sense that if a realizes $p|C$ and b realizes $q|C$ then a is independent from b over C. Note that almost orthogonality for the stationary types $p|C$, $q|C$ is equivalent to saying that $(p|C)(x) \cup (q|C)(y)$ determines a complete type $r(x, y)$ over C.

There is also the notion of orthogonality to a *set*. The stationary type $p(B) \in S(A)$ is said to be orthogonal to a set A if p is orthogonal to every strong type over A. This is also characterized as follows: Let B' realize $stp(B/A)$ such that B' is independent from B over A. Then p is orthogonal to the copy p' of p over B.

There are prime models in the category of a-models, and the corresponding notion of isolation is closely related to domination. For example, suppose M is an a-model, c a tuple, and M_1 is a-prime over Mc. Then c *dominates* M_1 over M, namely whenever c is independent from a set B over M, then M_1 is independent from B over M.

Definition 2.1: *T has the dimensional order property, or DOP, if there are a-models M_0, M_1, M_2, M_3 and $p(x) \in S(M_3)$ such that*
(i) $M_0 \subseteq M_1$, $M_0 \subseteq M_2$, and M_1 is independent from M_2 over M_0.

(ii) M_3 is a-prime over $M_1 \cup M_2$, and
(iii) p is orthogonal to both M_1 and M_2.

For superstable T the DOP is a Shelah-style dividing line for a-models, in the sense that assuming DOP gives a nonstructure theorem (many a- models) and assuming NDOP gives a structure theorem (any a-model is a-prime over a suitable tree of small models). This leads to the so-called Main Gap for a-models of superstable theories, see [14], [3]. For stable, nonsuperstable theories, we have the nonstructure theorem [14]:

Fact 2.2: *Suppose T is non superstable and has DOP. Then for any uncountable λ such that $\lambda = \lambda^{\omega}$, T has 2^{λ} $\aleph_1$-saturated models of cardinality λ.*

However there is in general no nice structure theorem for $\aleph_1$-saturated models of nonsuperstable theories with NDOP, and the "Main Gap" for $\aleph_1$-saturated models remains open.

We now aim towards reducing the proof of the Theorem in the introduction to a concrete nondefinability statement about a standard model $\mathbb{F}$.

Proposition 2.3: *Fix a free group $\mathbb{F}$ with free generators $(e_1, e_2, .., ..)$, and assume that the unique isomorphism between $\langle e_1 \rangle$ and $\langle e_2 \rangle$ (taking e_1^n to e_2^n, for $n \in \mathbb{Z}$) is not definable in $\mathbb{F}$. Then T_{fg} has the DOP.*

Proof. We consider $\mathbb{F}$ as an elementary substructure of a very saturated model $\bar{M}$. Let q_1 be the generic type of $C(e_1)^0$, a stationary type over e_1. Likewise for q_2.
Claim 1. q_1 is orthogonal to q_2.
Proof of Claim 1. Suppose otherwise. So for some model M containing the data, there are a realizing $q_1 M$ and b realizing $q_2|M$, a forks with b over M. As q_1,. q_2 have U-rank 1, a and b are interalgebraic over$M$, and $U(tp(a, b/M)) = 1$. But using Fact 1.1, the group $C(e_1) \times C(e_2)$ is 1-based hence byChapter 4, Section 4, of [9], $tp(a, b/M)$ is the generic type of a coset of a connected type-definable subgroup H of $C(e_1) \times C(e_2)$ of U-rank 1, where moreover H is type-defined over $acl(e_1, e_2)$, so over $\mathbb{F}$. As $H \leq C(e_1)^0 \times C(e_2)^0$ which is torsion free divisible, it is clear that H is the graph of an isomorphism between $C(e_1)^0$ and $C(e_2)^0$ defined over $acl(e_1, e_2)$. By compactness, there are definable finite index subgroups G_1 of $C(e_1)$ and G_2 of $C(e_2)$ and a definable isomorphism f between G_1 and G_2 with everything defined over $\mathbb{F}$. Looking at points in the model $\mathbb{F}$, f restricts to an isomorphism between $G_1(\mathbb{F})$ and $G_2(\mathbb{F})$ which we still call f. But $G_1(\mathbb{F})$, being a finite index subgroup of the cyclic group $\langle e_1 \rangle$ is precisely $\langle e_1^k \rangle$ for some $k > 0$, and likewise $G_2(\mathbb{F})$ is $\langle e_2^{\ell} \rangle$ for some $\ell > 0$, and f takes e_1^k to e_2^{ℓ}.

By precomposing with the isomorphism between $\langle e_1 \rangle$ and $\langle e_1^k \rangle$ obtained by raising to the kth power, and postcomposing with the inverse of the analogous isomorphism between $\langle e_2 \rangle$ and $\langle e_2^{\ell} \rangle$, gives an isomorphism between

$\langle e_1 \rangle$ and $\langle e_2 \rangle$ definable in $\mathbb{F}$, contradicting our assumption. The claim is proved.

Now let M_0 be an a-model independent from e_1, e_2. Then e_1 is independent from e_2 over M_0. Let M_1 be a-prime over (M_0, e_1) and M_2 a-prime over (M_0, e_2). Finally let M_3 be a-prime over $M_1 \cup M_2$. Let $c = e_1 + e_2$, so $c \in M_3$. Let q_c be the generic type (over c) of $C(c)^0$, and r_c its nonforking extension over M_3.

Claim 2. r_c is orthogonal to each of M_1, M_2.

Proof of Claim 2. We will just show orthogonality to M_1. Let α be an automorphism of $\bar{M}$ fixing M_1 pointwise, such that $M_3' = \alpha(M_3)$ is independent with M_3 over M_1. Let $c' = \alpha(c)$, and $q_{c'}$ (over c'), $r_{c'}$ (over M_3') be the copies under α of q_c and r_c respectively.

So by the earlier characterizationof orthogonality to a set, we have to show that r_c is orthogonal to $r_{c'}$. As r_c is the unique nonforking extension over M_3 of q_c, and $r_{c'}$ the unique nonforking extension of $q_{c'}$ over M_3', this is equivalent to showing that q_c and $q_{c'}$ are orthogonal. But it is easy to see that c and c' are independent realizations of the generic type p_0, whereby the group they generate is an elementary substructure of $\bar{M}$ isomorphic to $\mathbb{F}_2$, so by Claim 1, q_c is orthogonal to $q_{c'}$.

$\square$

3 Cyclic towers, and the proof of the main theorem

We start this section with the notion of an *amalgamated free product*, we refer the reader to [5, Chapter IV] or to [6, Section 4.4] for more details and motivation. We fix two groups A, B a subgroup C of B and an embedding $f : C \to A$. Then the *amalgamated free product* $G := A *_C B$ is the group $\langle A, B | c = f(c),\ c \in C \rangle$. Note that G can be viewed as the free product $A * B$ quotiened by the normal subgroup containing $\{cf(c)^{-1} \mid c \in C\}$. This construction naturally arises in the context of algebraic topology for example in the Seifert - van Kampen theorem (see [2, Section 1.2]).

For the rest of the section we fix a non abelian finitely generated free group $\mathbb{F} := \langle \bar{e} \rangle$. For notational purposes, when an infinite cyclic group is denoted by a capital letter, say C, its generator will be denoted by the corresponding small letter c.

Definition 3.1: *Let C be an infinite cyclic group. Then a cyclic tower over $\mathbb{F}$ is the amalgamated free product $\mathbb{F} *_C (C \oplus \mathbb{Z})$ where C embeds isomorphically onto a maximal abelian subgroup of $\mathbb{F}$.*

Remark 3.2: *A cyclic tower $G := \mathbb{F} *_C (C \oplus \mathbb{Z})$ over $\mathbb{F}$ (with $f : C \hookrightarrow \mathbb{F}$) has an obvious group presentation. Suppose f embeds C isomorphically onto $C_{\mathbb{F}}(a)$. Assume, without loss of generality, that $a \in \mathbb{F}$ is an element such that $C_F(a) = \langle a \rangle$, i.e. an element wihout proper roots. Then G has the following* *presentation:* $\langle \mathbb{F}, z \mid [z, a] \rangle$.

Definition 3.3: *Let* $G := \mathbb{F} *_C (C \oplus \mathbb{Z})$ *be a cyclic tower over* $\mathbb{F}$*. Let* D *be an infinite cyclic group and* $f : C \oplus \mathbb{Z} \hookrightarrow C \oplus D$ *be an injective morphism that is the identity on* C*, i.e.* $f(c) = c$*. Then the closure of* G *with respect to* f*,* $Cl_f(G)$*, is the amalgamated free product* $\mathbb{F} *_C B$ *where* B *is the group* $C \oplus \mathbb{Z} \oplus D$ *quotiented by the (normal) subgroup generated by* $f(z)z^{-1}$*.*

Remark 3.4: *Let* $G := \mathbb{F} *_C (C \oplus \mathbb{Z})$ *be a cyclic tower over* $\mathbb{F}$ *with presentation* $\langle \mathbb{F}, z \mid [z, a] \rangle$*. Let* $f : C \oplus \mathbb{Z} \hookrightarrow C \oplus D$ *be an injective morphism with* $f(c) = c$ *and* $f(z) = c^m d^k$*. Suppose* $Cl_f(G)$ *is the closure of* G *with respect to* f*. Then:*

- *the injectivity of* f *implies that* k *must be non-zero;*
- *the closure of* G *with respect to* f*, has an obvious presentation:* $\langle \mathbb{F}, z, d \mid [d, a], a^m d^k z^{-1} \rangle$*;*
- *the group* G *can be indentified with the subgroup generated by* $\mathbb{F}, z$ *in its closure.*

Definition 3.5: *Let* G *be a cyclic tower over* $\mathbb{F}$ *with presentation* $\langle \mathbb{F}, z \mid [z, a] \rangle$*. Then a test sequence with respect to* G *is a sequence of morphisms* $(h_n)_{n<\omega} : G \to \mathbb{F}$ *with the following properties:*

- $h_n \restriction \mathbb{F} = Id$ *for all* $n < \omega$*;*
- $h_n(z) = a^{k_n}$ *with* $(k_n)_{n<\omega}$ *strictly increasing.*

Definition 3.6: *Let* G *be a cyclic tower over* $\mathbb{F}$ *and* $Cl_f(G)$ *be a closure (with respect to some* f*). We say that a test sequence* $(h_n)_{n<\omega} : G \to \mathbb{F}$ *extends to* $Cl_f(G)$ *if for all but finitely many* n*,* h_n *extends to a morphism* $h'_n : Cl_f(G) \to \mathbb{F}$*.*

Remark 3.7: *Let* G *be a cyclic tower over* $\mathbb{F}$ *with presentation* $\langle \mathbb{F}, z \mid [z, a] \rangle$*. Then:*

- *a morphism from* G *to* $\mathbb{F}$ *that is the identity on* $\mathbb{F}$ *is determined by the value it gives to* z*, which in turn must commute with* a*;*
- *a test sequence with respect to* G *can be identified with a sequence* $(a^{k_n})_{n<\omega}$ *of strictly increasing powers of* a*;*
- *if* $f : C \oplus \mathbb{Z} \hookrightarrow C \oplus D$ *is an injective morphism with* $f(z) = c^m d^k$ *and* $Cl_f(G)$ *a closure of* G *with respect to* f*. Then a test sequence* $(a^{k_n})_{n<\omega}$ *with respect to* G *extends to* $Cl_f(G)$ *if and only if for all but finitely many* n*,* $k_n \in m + k\mathbb{Z}$*.*

The following theorem is a special case of Theorem 6.33 in [1].

Theorem 3.8: *Let $G := \langle \mathbb{F}, z \mid [z, a] \rangle$ be a cyclic tower over $\mathbb{F}$. Let $\phi(x, y)$ be a formula over $\mathbb{F}$ such that $\mathbb{F} \models \forall y \exists^{<\infty} x \phi(x, y)$.*

Suppose there exists a test sequence $(h_n)_{n<\omega} : G \to \mathbb{F}$ with respect to G and a sequence $(b_n)_{n<\omega}$ of elements of $\mathbb{F}$ such that $\mathbb{F} \models \phi(b_n, h_n(z))$ for all n.

Then there exists a closure $Cl_f(G) := \langle \mathbb{F}, z, d \mid [d, a], z^{-1}f(z) \rangle$ and a word $w = w(d, z, \bar{e})$ in $Cl_f(G)$ such that an infinite subsequence of $(h_n)_{n<\omega}$ extends to a sequence of morphisms $(h'_n)_{n<\omega} : Cl_f(G) \to \mathbb{F}$. Moreover, the extended sequence gives values to the couple (w, z) that satisfy the formula $\phi(x, y)$, i.e. $\mathbb{F} \models \phi(h'_n(w), h'(z))$.

We can now prove as a corollary that the diagonal subgroup of $C_{\mathbb{F}}(e_1) \times C_{\mathbb{F}}(e_2)$, i.e. the cyclic group $\langle (e_1, e_2) \rangle$ is not definable in $\mathbb{F}$.

Corollary 3.9: *The subgroup $\Gamma := \langle (e_1, e_2) \rangle$ of $C_{\mathbb{F}}(e_1) \times C_{\mathbb{F}}(e_2)$ is not definable in $\mathbb{F}$.*

Proof. Suppose for a contradiction that the formula $\phi(x, y)$ over $\mathbb{F}$ defines Γ. We apply Theorem 3.8 to the cyclic tower $G := \langle \mathbb{F}, z \mid [z, e_2] \rangle$ and the formula $\phi(x, y)$. We first see that $\mathbb{F} \models \forall y \exists^{<\infty} x \phi(x, y)$. Moreover, for the test sequence $(e_2^n)_{n<\omega}$ (with respect to G) there exists a sequence of elements of $\mathbb{F}$, namely $(e_1^n)_{n<\omega}$, such that $\mathbb{F} \models \phi(e_1^n, e_2^n)$ for all n. Thus, there exists a closure $Cl_f(G) := \langle \mathbb{F}, z, d \mid [d, e_2], z^{-1}f(z) \rangle$ of G and a word $w = w(d, z, \bar{e})$ such that a subsequence $(e_2^{k_n})_{n<\omega}$ of $(e_2^n)_{n<\omega}$ extends to $Cl_f(G)$ and moreover if $(h'_n)_{n<\omega} : Cl_f(G) \to \mathbb{F}$ is the extended sequence, then $\mathbb{F} \models \phi(h'_n(w(d, z, \bar{e})), e_2^{k_n}))$ for all n.

We observe that, since d and z commute with e_2 in G, for each n, $h'_n(d)$ and $h'_n(z)$ must be powers of e_2. On the other hand, in the word $w(d, z, \bar{e})$, the letter e_1 appears finitely many times. Since the only solution of $\phi(x, e_2^{k_n})$ is $e_1^{k_n}$, and by definition $h'_n(z) = e_2^{k_n}$, we must have that $h'_n(w(d, z, \bar{e}) = w(e_2^{l_n}, e_2^{k_n}, \bar{e}) = e_1^{k_n}$. But for n large enough this is impossible.

□

Theorem 1 follows directly from the Corollary above and Proposition 2.3.

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