Strong types, Galois groups, dynamics

Notre Dame Model Theory seminar

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Introduction

- This is a background talk to a reading or working seminar on Hrushovski's "Definability patterns and their symmetries" (on arXiv, January 2020, version 2).
- Hrushovski's paper describes, among other things, a certain compact Hausdorff group attached to a complete first order theory T, which maps onto the various Galois groups associated to T.
- The existence of such a group was established in earlier papers, using topological dynamical methods, but the group there lives on objects (type spaces) associated to saturated models of T.
- Hrushovski's group lives on the type spaces over arbitrary models of T, which is one of the improvements.
- Among the aims of the general endeavour is to give a mathematical account of Lascar strong types and the Lascar group, attached to T.
- So here I will set the scene, in terms of the problematic and earlier work.

Strong types I

- We typically take T to be a complete theory in a language L and there is no harm in assuming that T has QE, so is the model companion of its universal part.
- And as usual we feel free to work inside a very saturated model $\bar{M}$ of T (which may be many sorted). Also in general tuples maybe be infinite, i.e. indexed by an infinite ordinal or cardinal.
- Tuples a, b have the same Shelah strong type, $E_{Sh}(a, b)$, if $E(a, b)$ for every $\emptyset$-definable equivalence relation E with finitely many classes.
- Tuples a, b have the same KP-strong type, $E_{KP}(a, b)$, if $E(a, b)$ for any type-definable over $\emptyset$ equivalence relation E with boundedly many ($\leq 2^{|T|}$) classes.
- And tuples a, b have the same Lascar strong type, $E_L(a, b)$, if $E(a, b)$ for every $Aut(\bar{M})$-invariant equivalence relation E with boundedly many classes (in the above sense). These equivalence relations refine each other.

Strong types II

- So what?
- Well the most general i.e. Lascar, strong types, so all the others, are obstructions to type amalgamation:
- For example (for some of you), assume $p(x)$ is a complete type over $\emptyset$ which doesn't fork over $\emptyset$.
- Let a realize p and let M be any model, i.e. el. substructure of $\bar{M}$, then there is b such that $E_L(a,b)$ and $tp(b/M)$ is a nonforking extension of p.
- But $tp(b/M)$ determines the Lascar strong type of b over $\emptyset$.
- Hence, if M_1, M_2 are models and q_1, q_2 are nonforking extensions of p over M_1, M_2 respectively, which determine different Lascar strong types of p, then there will not be a common extension of q_1, q_2 to a larger model N.
- Various theorems (FERT, Independence Theorem) say that these are the only obstructions to type amalgamation (in stable, simple theories ....)

Examples

- Consider RCF and the interval $[0, 1]$ in a saturated model. The relation that $d(a, b)$ is infinitesimal is precisely E_{KP} on this sort and this is NOT an intersection of $\emptyset$-definable finite equivalence relations.
- Consider the many sorted theory with sorts S_n where S_n is the circle with the betweenness relation (circular ordering) and with a function for clockwise rotation by $2\pi/n$ degrees.
- Consider the sort consisting of ω-tuples $(a_n)_n$ where a_n is in S_n.
- Then the relation between $(a_n)_n$ and $(b_n)_n$ that for some k, $d_n(a_n, b_n) \leq k/n$ for all n, is precisely E_L on this sort of suitable ω-tuples, and is NOT a type-definable equivalence relation.
- In fact E_{KP} on this sort is trivial.

Galois groups

- For each of E_{Sh}, E_{KP}, and E_L we can consider the group of permutations of the classes (as the sorts vary) induced by automorphisms of $\bar{M}$.
- For E_{Sh} we obtain a profinite group $Gal_{Sh}(T)$. (Example of ACF_0)
- For E_{KP} we obtain a compact, Hausdorff, group $Gal_{KP}(T)$, whose maxmal profinite quotient is $Gal_{Sh}(T)$.
- For E_L we get an abstract group $Gal_L(T)$, the status of which is unclear and whose clarification is one of the main aims of the whole endeavour.
- $Gal_L(T)$ can also be described (Lascar) as the quotient of $Aut(\bar{M})$ by its normal subgroup of "Lascar strong" automorphisms, generated by the fixators of small elementary submodels.
- This description reflects that E_L, on a given sort, can be described as the transitive closure of the relation that a and b have the same type over some model (el. substructure of $\bar{M}$).

Interlude

- All the data above (equivalence relations etc.) are over $\emptyset$. One can relativise to a set A of parameters. But if we work over a model M, then all these strong types are the same as the types, and the Galois groups above are trivial.
- There is an analogue for *definable* groups in place of automorphism groups.
- Fix a group G definable over a set A of parameters. Then we have the "connected components" G^0_A, G^{00}_A, G^{000}_A.
- The quotients G/G^0_A, G/G^{00}_A and G/G^{000}_A are analogues of Gal_{Sh}, Gal_{KP} and Gal_L.
- The compact Hausdorff group G/G^{00} plays a big role in model-theoretic approaches to approximate subgroups and "arithmetic regularity".
- Basically if G is pseudofinite then definable sets of positve pseudofinite counting measure are controlled by G/G^{00}.

Borel equivalence relations

- We return to the original context and let us assume T to be countable.
- One of the first attempts to describe the Lascar group, was as a quotient of a Polish space by a Borel, in fact K_σ, equivalence relation, and to ask about the complexity of this equivalence relation. See [CLPZ] where the example above also appears, as well as [KPS].
- Namely, fix a countable model M, let $\bar{m}$ enumerate M, and let $S_{\bar{m}}(M)$ be the space of extensions of $tp(\bar{m}/\emptyset)$ to complete types over M.
- For $\sigma \in Aut(\bar{M})$, the image of σ in $Gal_L(T)$ depends only on $tp(\sigma(\bar{m})/M)$, so we have a map $S_{\bar{m}}(M) \to Gal_L(T)$, and using facts above, this is a quotient of $S_{\bar{m}}(M)$ by a K_σ-equivalence relation.
- It was proved ([KMS], later [KPR]) that smooth implies closed, confirming conjectures in [CPLZ] and [KPS].

Ellis semigroup I

- Let us start to explain more recent work which uses topological dynamics machinery, namely [KPR].
- Let M be a saturated model of T, and $\bar{m}$ an enumeration of M and again we consider the space $S_{\bar{m}}(M)$ of complete types over M extending $p_0 = tp(\bar{m}/\emptyset)$.
- $S_{\bar{m}}(M)$ is a compact Hausdorff space and is acted on continuously by the topological group $Aut(M)$.
- Consider the collection C of maps, in fact homeomorphisms, from $S_{\bar{m}}(M) \to S_{\bar{m}}(M)$ given by elements of $Aut(M)$.
- Then the Ellis semigroup $E = E(S_{\bar{m}}(M))$ of the flow is the closure of C in $S_{\bar{m}}(M)^{S_{\bar{m}}(M)}$ where the latter is equipped with the product topology.
- The semigroup structure on E is just composition of maps. And E is also a $Aut(M)$-flow under composition of maps.

Ellis semigroup II

- We sometimes write $*$ for the product operation in E. It is continuous on the left.
- Namely for each $q \in E$, the map $E \to E$ taking p to $p * q$ is continuous.
- One reason for denoting elements of E by p, q, etc is that E is naturally a closed subspace of the space of extensions of $tp(\bar{m})$ to complete types over an even bigger saturated model N, say, which are finitely satisfiable in M. More about this later.
- Minimal closed $Aut(M)$-subflows of E are the same thing as minimal left ideals and they exist.
- Let us fix one, $\mathcal{M}$. Then there is an idempotent $r \in \mathcal{M}$ (i.e. $r * r = r$) and in fact $\mathcal{M} = E * r$.
- Finally $G = (r * \mathcal{M}, *)$ is a group, which we sometimes (incorrectly) refer to as the Ellis group attached to the original $Aut(M)$-flow $S_{\bar{m}}(M)$.

Ellis group and Lascar group

- We claim that there is a surjective homomorphism from the "Ellis group" G onto $Gal_L(T)$. How, why, what, who, .?
- Well, we first get a surjective semigroup map from E to $Gal_L(T)$ as follows:
- Given $p \in E$, let $p(tp(\bar{m}/M)) = tp(\bar{m}'/M)$. Then $\bar{m}' = \sigma(\bar{m})$ for some automorphism σ of the monster model.
- As mentioned three slides earlier, the image of σ in $Gal_L(T)$ depends only on $tp(\sigma(\bar{m})/M)$, so this gives us a map, f, i.e. p under f goes to "σ modulo strong automorphisms", which can be checked to be a semigroup map from E to $Gal_L(T)$.
- As $G = r * E * r$ and r is an idempotent, it folllows that already $f|G$ is a surjective homomorphic map to $Gal_L(T)$.
- QED.

τ-topology I

- But so far G is only an abstract group, rather than a compact Hausdorff group. For example, in general G is NOT a closed subset of $\mathcal{M}$.
- There are various definitions of Ellis' τ-topology on G. I will give one of them, suitable for our purposes, as it makes sense independently of the ambient flow.
- The first observation is that G, acting by $*$ on the *right*, is precisely the group of automorphisms of $\mathcal{M}$, as a $Aut(M)$-flow.
- For each $f \in G$, consider the graph Γ_f of f as a subset of $\mathcal{M} \times \mathcal{M}$.
- For a subset K of G, define $cl_\tau(K)$, the closure of K in G in the τ-topology, to be the set of $\gamma \in G$ such that Γ_γ is contained in the closure, in $\mathcal{M} \times \mathcal{M}$, of $\Gamma_K = \cup_{f \in K} \Gamma_f$.

τ-topology II

- The τ-topology on G is not necessarily Hausdorff, but is T_1, and (quasi) compact.
- T_1 means that for every pair p, q of distinct points in G there is an open neighbourhood of p not containing q, and an open neighbourhood of q not containing p.
- With respect to τ, the group operation on G is separately continuous.
- G has a maximal Hausdorff quotient, namely its quotient by the normal subgroup H which is the intersection of all τ-closures of open neighbourhoods of the identity, and G/H is a compact Hausdorff topological group.
- Finally one proves that if $f : G \to Gal_L(T)$ is the surjective homomorphism defined earlier, then $H \subseteq ker(f)$, whereby f induces a surjective homomorphism from the compact group G/H to $Gal_L(T)$.

τ-topology III

- Moreover the induced surjective homomorphism from G/H to $Gal_{KP}(T)$ is continuous.
- It is proved in [KNS] that G with its τ-topology, is independent of the choice of the saturated model M, and therefore so is the compact group G/H.
- Finally, there are a couple of things to mention from the topological dynamics literature:
- First, when $\mathcal{M}$ is the universal minimal flow of a topological group T then the compact group G/H is called, by Glasner, the generalized Bohr compactification of T.
- Secondly, again when $\mathcal{M}$ is the universal minimal flow of a topological group T, then the τ-topology on G, as originally introduced by Ellis, is related to a certain Galois theory of minimal flows, which may have interesting connections with the model-theoretic context.

References

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- KPR, K. Krupinski, A. Pillay. T. Rzepecki, Topological dynamics and the complexity of strong types, Israel J. Math, 228, (2018), 863-932.
- KPS, K. Krupinski, A. Pillay, S. Solecki, Borel equivalence relations and Lascar strong types, J. Math. Logic, vol 13 (2013).
- Also for topological dynamics, various books, such as Proximal Flows, by Glasner, Lectures on Topological Dynamics, by Ellis, and Minimal Flows and Their Extensions, by J. Auslander.