On function field Mordell-Lang and Manin-Mumford

Franck Benoist*
Université Paris-Sud, Orsay

Elisabeth Bouscaren*
CNRS - Université Paris-Sud, Orsay

Anand Pillay†
University of Notre Dame

April 27, 2014

Abstract

We give a reduction of the function field Mordell-Lang conjecture to the function field Manin-Mumford conjecture, in all characteristics, via Model theory, but avoiding recourse to the dichotomy theorems for (generalized) Zariski geometries. Additional ingredients include the "Theorem of the kernel", and a result of Wagner on commutative groups of finite Morley rank without proper infinite definable subgroups. In characteristic 0 the methods also yield another account of the local modularity of $A^\sharp$ for A a traceless simple abelian variety. In positive characteristic, where the main interest lies, we require another result to make the strategy work: so-called quantifier-elimination for the corresponding $A^\sharp = p^\infty A(\mathcal{U})$ where $\mathcal{U}$ is a saturated separably closed field, which we prove here for simple abelian varieties.

*Partially supported by ANR MODIG (ANR-09-BLAN-0047) Model theory and Interactions with Geometry and ANR ValCoMo (ANR-13-BS01-0006) Valuations, Combinatorics and Model Theory

†Partially supported by EPSRC grants

1 Introduction and preliminaries

1.1 Introduction

This paper concerns relationships between "known" results. The original motivation was to supply a transparent account of function field Mordell-Lang in positive characteristic. After the third author gave a talk on the topic of reducing Mordell-Lang to Manin-Mumford, in Paris, December 2010, Damian Rössler picked up the theme, and eventually with Corpet, produced a successful algebraic-geometric account of such a reduction, in positive characteristic ([22], [8]). In the current paper we outline our original strategy, where the model-theoretic notion of "quantifier-elimination" for certain type-definable groups, as well as some other nice but soft stable-group theoretic results, play a role. And we prove that it works, including supplying the so far missing ingredient in characteristic p, the "quantifier elimination" result, but just for (Cartesian powers of) simple abelian varieties.

The subtext is Hrushovski's proof of function field Mordell-Lang [11], which depends on a dichotomy theorem for (generalized) Zariski geometries. In the characteristic 0 case, it is classical (strongly minimal) Zariski geometries which are relevant and the dichotomy theorem is proved in [13] and in [25], although all proofs are complicated, to say the least. But in positive characteristic, type-definable Zariski geometries are the relevant objects. In [13] there is an axiomatic treatment of "how to construct a field" which does not presuppose that one is working in a strongly minimal set. In [11] arguments are given for how to prove that the axioms are satisfied in the particular case of the minimal types we are interested in. Nevertheless, in all cases, this very important model-theoretic proof of function field Mordell-Lang has some "black boxes", and the current authors have been preoccupied for some years about seeing what is really going on, in particular avoiding the recourse to (generalized) Zariski geometries, and/or recovering the "black box" results by more direct arguments.

In [19] this issue was taken up, and an approach using differential jet spaces was developed. This succeeded in characteristic 0, but not entirely in positive characteristic due to inseparability issues, although the approach recovered some cases due to Abramovich and Voloch [1]. It is still open whether the approach can be tweaked so as to work in general in the characteristic p case.

In [20] Pink and Rössler gave a reasonably transparent algebraic-geometric proof of function field Manin-Mumford in positive characteristic with *all* torsion points in place of prime-to-p torsion points. This suggested to us to try to reduce function field Mordell-Lang to function field Manin-Mumford. The current paper is devoted to explaining a certain strategy, and how it works. Exploring connections between the methods of Rössler and Corpet and either the jet space ideas in [19], or the current paper, would be interesting.

Of course we could also consider the full or absolute Mordell-Lang conjecture in characteristic 0 (proved by Faltings, McQuillan,..) and ask whether there is a "soft" reduction to absolute Manin-Mumford. We doubt that this is the case as these two theorems seem to us (maybe incorrectly) to be of different orders of difficulty. That such a reduction is possible in the function field case has additional interest.

Acknowledgements. We would like to thank both Université Paris-Sud, Orsay, where the third author was a Professeur Invité in March-April 2010 and the Mathematical Sciences Research Institute, Berkeley, where the authors participated in the Spring 2014 model theory program. The first author would like to thank Françoise Delon for very useful discussions about the quantifier elimination question.

1.2 Preliminaries

Let us now state the function field statements in the precise form that we will prove them. We restrict to function fields of one variable, although the higher-dimensional case also works.

Statement of function field Mordell-Lang in characteristic 0**.** Let $K = k(t)^{alg}$, the algebraic closure of $k(t)$, where $k = \mathbb{C}$. Let A be an abelian variety over K with k-trace 0. Let X be an irreducible subvariety of A (defined over K), and let Γ be a "finite-rank" subgroup of $A(K)$, namely Γ is contained in the division points of a finitely generated subgroup of $A(K)$. Suppose $X \cap \Gamma$ is Zariski-dense in X. Then X is a translate of an abelian subvariety of A.

Statement of function field Manin-Mumford in characteristic 0**.** As above, except that the hypothesis on Γ is strengthened to: Γ is contained in the group of torsion points of A.

Statement of function field Mordell-Lang in characteristic $p > 0$. Let K be the separable closure of $k(t)$ where k is an algebraically closed field of characteristic p which we will be taking to be $\mathbb{F}_p^{alg}$. Let A be an abelian variety over K with k-trace 0. Let X be an irreducible subvariety of A, defined over K. And let Γ be a subgroup of $A(K)$ contained in the prime-to-p-division points of a finitely generated subgroup. Suppose that $X \cap \Gamma$ is Zariski-dense in X. Then X is a translate of an abelian subvariety of A.

The formulation involving prime-to-p division points is due to Abramovich and Voloch [1]. One could also ask what happens when K is the algebraic closure of $k(t)$ and $\Gamma < A(K)$ is the group of *all* division points of some finitely generated subgroup. No obstacle is currently known to this.

Statement of function field Manin-Mumford in characteristic p. As above, except that Γ is assumed to be contained in the group of all torsion points of A.

We will write MM for Manin-Mumford and ML for Mordell-Lang. The general idea is:

Basic strategy: MM + "Theorem of the kernel" + "structure of g-minimal groups of finite Morley rank" (Wagner) implies ML.

We will explain these ingredients (including the truth of the "Theorem of the Kernel") as well as the object $A^\sharp$ shortly. In sections 2 and 3 respectively we prove:

Theorem 1.1. *(i) In characteristic* 0 *the Basic Strategy holds.*
(ii) In characteristic p, the Basic Strategy holds, assuming that the type-definable group $A^\sharp(\mathcal{U})$ has "quantifier-elimination", where $\mathcal{U}$ is a saturated elementary extension of K.

The proof of (i) will use additional relatively soft ingredients from the approaches of Hrushovski (as well as Buium [6]), namely embedding the data into a differential algebraic framework, and the (weak) socle theorem. Moreover, assuming MM for the the universal vectorial extension of A, we will also deduce the local modularity (or 1-basedness) of $A^\sharp$. The proof of (ii) will be easier and more direct.

In Section 4, we complete the paper by proving the main result:

Theorem 1.2. *(i) In characteristic $p > 0$, for A simple, $A^\sharp(\mathcal{U})$* does *have quantifier-elimination, and so $(A^\sharp)^n$ (Cartesian power) also has quantifier elimination for any n.*
(ii) Hence by Theorem 1.1(ii), the Basic Strategy works in positive characteristic too for Cartesian powers of simple abelian varieties.

Let us explain the remaining ingredients. We assume, both above and below, familiarity with model theory, basic stability, as well as differentially and separably closed fields. The book [14] is a reasonable reference, as well as [17] for more on stability theory. Definability means with parameters unless we say otherwise.
We fix some more notation.

In characteristic 0, K has a unique derivation ∂ extending d/dt on $k(t)$ and K^{diff} denotes a differential closure of (K, ∂), in the language of differential fields. It is convenient sometimes to work in a saturated elementary extension $\mathcal{U}$ of K^{diff}. In any case DCF_0, the first order theory of K^{diff} is ω-stable with quantifier-elimination.

In characteristic p, $\mathcal{U}$ denotes a saturated elementary extension of K in the language of fields. It will be crucial to pass to $\mathcal{U}$. The first order theory of K in the language of fields is known as or denoted by $SCF_{p,1}$, the theory of separably closed fields of characteristic p and degree of imperfection 1. It is stable, but not superstable, and has quantifier-elimination after either adding symbols for so-called λ-functions, or symbols for a strict iterative Hasse derivation.

Definition 1.3. *(i) In characteristic 0, $A^\sharp$ is the "Kolchin closure of the torsion", namely the smallest definable subgroup of $A(\mathcal{U})$ which contains the torsion subgroup (so note $A^\sharp$ is definable over K.)*
(ii) In positive characteristic, $A^\sharp$ denotes $p^\infty(A(\mathcal{U})) =_{def} \bigcap_n p^n(A(\mathcal{U}))$.

Remark 1.4. *In positive characteristic $A^\sharp$ is the maximal divisible subgroup of $A(\mathcal{U})$. Moreover $A^\sharp(K)$ is also the maximal divisible subgroup of $A(K)$ and coincides with $\bigcap_n p^n(A(K))$.*

Statement of The Theorem of the Kernel. In all characteristics, $A^\sharp(K)$ is contained in the group of torsion points of A.

This is a differential algebraic or model-theoretic theorem of the kernel. In Corollary K3 of [5], the characteristic 0 case is given, where it is deduced from Chai's strengthening of a theorem of Manin. In [21], the positive characteristic case is proved.

In characteristic 0, function field MM as stated is clearly a special case of function field ML. And it follows from the absolute case of MM, of which there are many proofs. In positive characteristic MM (with all torsion points) is proved by Pink and Rössler [20]. The proof uses a variety of methods, including Dieudonné modules, but is accessible. A proof was also given by Scanlon [23] using the dichotomy theorem in $ACFA_p$ (algebraically closed fields of characteristic p with a generic automorphism), which itself depends on an even more generalized notion of Zariski geometries than used in separably closed fields.

We now pass to the other model-theoretic ingredients.

Definition 1.5. *Let G be a group with additional structure, which has finite Morley rank and is commutative and connected. We say that G is g-minimal if it has no proper nontrivial connected definable subgroup (equivalently, no proper infinite definable subgroup).*

Let us remark that g-minimality of G passes to saturated elementary extensions (in groups of finite Morley rank, there is a bound on the cardinality of uniformly definable families of finite subgroups, as they do not have the "finite cover property").

The following appears in [24]:

Theorem 1.6. *Suppose that G is g-minimal. Then any infinite algebraically closed subset of G is (the universe of) an elementary substructure of G.*

So a g-minimal group behaves a bit like a strongly minimal set. By Zilber's indecomposability theorem a g-minimal group is "almost strongly minimal" namely in the algebraic closure of a strongly minimal definable subset. But parameters may be required, so one cannot immediately deduce Theorem 1.6 from the strongly minimal case.

We now fix an ambient saturated stable structure $\mathcal{U}$. A subset X of $\mathcal{U}^n$ is called *type-definable* if it is the intersection of a small collection of definable sets, namely defined by a small partial type. In the case of interest X will be a countable intersection of definable sets.

Suppose X is type-definable over the small set of parameters A. By a relatively definable subset Y of X^m we mean the intersection of some definable (with parameters) subset Z of a suitable Cartesian power of $\mathcal{U}$ with X^m. We will say that Y is relatively A-definable, or relatively definable over A, if Z can be chosen to be A-definable.

Let us fix a set X, type-definable over some small set of parameters A.

Lemma 1.7. *Let Y be a relatively definable subset of X (or some Cartesian power of X) which is invariant under automorphisms of $\mathcal{U}$ fixing A pointwise. Then Y is relatively A-definable.*

Proof. Suppose Y is relatively definable by formula $\phi(x, b)$ (namely Y is the set of solutions in X of $\phi(x, b)$), where we are exhibiting the required parameters b. By our assumptions, we have that if $tp(b_1/A) = tp(b/A)$ then $\phi(x, b_1)$ relatively defines the same subset of X as does $\phi(x, b)$. We can apply compactness to find a formula $\psi(y) \in tp(b/A)$ such that Y is relatively defined by the formula $\exists y(\psi(y) \wedge \phi(x, y))$.

□

Let us denote by $\mathcal{X}_A$ the structure with universe X and predicates for relatively definable over A, subsets of X^n for all n. With this notation:

Definition 1.8. *We will say that X_A has* quantifier elimination *or QE, if $Th(\mathcal{X}_A)$ has quantifier elimination in the language above.*

It is clear from this definition that:

Remark 1.9. *X_A has QE if and only if whenever Y is a relatively A-definable subset of X^{n+1} then the projection of Y to X^n is relatively A-definable.*

Lemma 1.10. *(i) Suppose $A \subseteq B$. Then X_A has QE iff X_B has QE (in which case we just say that X has QE).*
(ii) X has QE just if the projection of any relatively definable subset of any X^{n+1} to X^n is relatively definable (noting that in general it is only type-definable).
(iii) X having QE is equivalent to $\mathcal{X}_A$ being a saturated structure.

Proof. (i) Right implies left follows from Lemma 1.7 as a projection of an A-invariant set is also A-invariant. For left to right: Suppose $Y \subset X^{n+1}$ is relatively definable over B, by $\phi(x_1, .., x_{n+1}, b)$ where we witness the parameters $b \in B$, which may live outside X. Let Y_1 be the projection of Y

to X^n. By Lemma 1.7 it suffices to prove that Y_1 is relatively definable (as clearly it is invariant under automorphisms of $\mathcal{U}$ fixing b pointwise). Now by definability of types we may find some L- formula $\psi(x_1,..,x_{n+1},z)$ and c from X such that Y is relatively definable by $\psi(x_1,..,x_{n+1},c)$. Suppose z is an m-tuple of variables. Let Y' be the subset of X^{n+1+m} relatively defined by ψ. By our assumption that X_A has QE, the projection Y'' of Y obtained by existentially quantifying out x_{n+1}, is a relatively A-definable subset of X^{n+m}, (relatively) defined by a formula $\chi(x_1,..,x_n,z)$ say. So $\chi(x_1,..,x_n,c)$ relatively defines Y_1, as required.
(ii) By (i) and Lemma 1.7.
(iii) The point is that to say that X_A has QE means precisely that if b_1, b_2 are n-tuples from X, then $tp(b_1/A) = tp(b_2/A)$ in the sense of $\mathcal{U}$ iff $tp(b_1) = tp(b_2)$ in the sense of $\mathcal{X}_A$.

□

Note that in general $\mathcal{X}_A$ is only quantifier-free saturated (and homogeneous) and is sometimes referred to as a Robinson structure.

Of course when X is definable (rather than type-definable) over A, then X_A always has QE and is sometimes referred to as "X with its induced structure".

Using Theorem 1.6 applied to the G_i's with their induced structure, we deduce easily:

Corollary 1.11. *Suppose G is a (saturated) commutative, connected, group of finite Morley rank (with additional structure) which is a sum of finitely many g-minimal $\emptyset$-definable subgroups G_i. Then any algebraically closed subset of G which meets each G_i in an infinite set, is an elementary substructure of G.*

So in positive characteristic we will suppose that $A^\sharp$ has QE, and use it to show that we can apply Wagner's theorem (Theorem 1.6 and Corollary 1.11). Now the (generalized) Zariski geometry arguments from [11] give the dichotomy theorem for minimal "thin" types in separably closed fields, implying that if A is simple (and with k-trace 0), then $A^\sharp$ is minimal (i.e. U-rank 1), connected, and 1-based, from which it easily follows that $A^\sharp$ has QE. For arbitrary traceless abelian varieties A, $A^\sharp$ will be a sum of such minimals, hence also 1-based and so we also have QE. So the QE hypothesis is true, after the fact so to speak.

In section 1.2 we give a direct proof of QE for $A^\sharp$ for the case of A a simple abelian variety, which of course implies the same thing for Cartesian powers of $A^\sharp$.

On the face of it, the QE hypothesis is substantially weaker than the 1-based hypothesis. In any case our method of proving QE is related to proofs that minimal thin types in SCF are Zariski, but does not use the dichotomy theorem for (type-definable) Zariski geometries (nor in fact even the so called dimension theorem).

On the other hand we can define $A^\sharp$ in the same way for semiabelian varieties, and building on work in [2], Alexandra Omar Aziz, in her Ph.D. thesis [15] gave semiabelian examples G for which $G^\sharp$ does not have QE. In any case Theorem 1.2(ii) solves to some extent a problem that we have been grappling with for a long time.

2 Proof of Theorem 1.1: characteristic 0

We first deal with the characteristic 0 case, namely prove Theorem 1.1(i). with notation as in Section 1. We want to avoid reference to the dichotomy theorem for minimal types in differentially closed fields and/or the 1-basedness and strong minimality of $A^\sharp$ when A is simple with k-trace 0. But we will use relatively softer ingredients of Hrushovski's proof, among them the socle theorem, which we now recall. Let G be a commutative connected group of finite Morley rank, definable in some ambient stable structure M. The model-theoretic (or stability-theoretic) socle $s(G)$ of G, is the greatest connected definable subgroup of G which is generated (abstractly) by strongly minimal definable subsets of G. For X a definable subset of G with Morley degree 1, define $Stab_G(X)$ to be $\{g \in G : RM(X \cap (X+g)) = RM(X)\}$ (so the stabilizer of the generic type of X over M). In this context, Hrushovski's socle theorem is:

Lemma 2.1. *Suppose that $s(G)$ is "rigid" in the sense that (passing to a saturated model) all connected definable subgroups of $s(G)$ are defined over $acl(B)$ (where G is defined over B). Suppose that $Stab_G(X)$ is finite. Then some translate of X is contained in $s(G)$.*

A is our abelian variety over K with k-trace 0. K^{diff} is a the differential closure of the differential field $(K, d/dt)$ and $\mathcal{U}$ a saturated elementary extension. We defined $A^\sharp = A^\sharp(\mathcal{U})$ to be be smallest definable subgroup of $A(\mathcal{U})$ containing the torsion subgroup $A_{torsion}$ of A (which exists as DCF_0 is ω-stable). Note that $A^\sharp$ is K-definable. The following is well-known except maybe (vi). But we give references for completeness.

Lemma 2.2. *(i) $A^\sharp$ is also the unique smallest Zariski-dense definable subgroup of $A(\mathcal{U})$.*
(ii) $A^\sharp$ is connected with finite Morley rank.
(iii) $A(\mathcal{U})/A^\sharp$ embeds definably in (the group of $\mathcal{U}$-points of) a unipotent algebraic group over $\mathcal{U}$.
(iv) If A is a simple abelian variety then $A^\sharp$ is g-minimal.
(v) If A is the sum of simple A_i then $A^\sharp$ is the sum of the $A_i^\sharp$.
(vi) If H is a finite Morley rank definable subgroup of $A = A(K^{diff})$, containing $A^\sharp$ then $s(H) = A^\sharp$.
(vii) $A^\sharp$ is rigid.

Proof. We will use repeatedly the following fact, due to Buium, and also proved in [16]:
(*) Suppose $G = G(\mathcal{U})$ is a commutative connected algebraic group over $\mathcal{U}$ and H is a Zariski-dense definable subgroup. Then G/H definably embeds in a unipotent algebraic group, namely $(\mathcal{U}, +)^d$ for some d.
(i) By (*) any Zariski dense definable subgroup of $A(\mathcal{U})$ contains $A_{torsion}$, which suffices.
(ii) Maybe the simplest way of seeing it is via so-called algebraic ∂-groups, as in [7] and outlined again in [5]. The universal vectorial extension $\tilde{A}$ of A, also an algebraic group over K, is equipped with a unique structure of an algebraic ∂-group, namely an extension of the derivation ∂ to a derivation of the structure sheaf of $\tilde{A}$, commuting with co-multiplication. Equivalently a rational homomorphic section s, defined over K, of a certain shifted tangent bundle $T_\partial(\tilde{A})$. Then $\tilde{A}^\partial = \{x \in \tilde{A}(\mathcal{U}) : \partial(x) = s(x)\}$ is a finite Morley rank connected definable subgroup of $\tilde{A}$ and $A^\sharp$ is its image under the canonical surjective homomorphism $\tilde{A} \to A$.
(iii) is given by (*).
(iv) If A is simple, any infinite subgroup is Zariski-dense, so it follows from (i).
(v) is immediate.
(vi). This reduces to the case where A is simple. Assuming the local modularity of $A^\sharp$ gives an easy account, but of course we don't want to make that assumption. Now $A^\sharp$ is g-minimal, hence as observed earlier, generated by

some strongly minimal subset, so contained in $s(H)$. Suppose by way of contradiction that $s(H)$ properly contains $A^\sharp$. Then by (*) $s(H)/A^\sharp$ is a finite dimensional vector space over the constants $\mathcal{C}$ of $\mathcal{U}$. Now on general grounds $s(H)$ is an almost direct sum of mutually orthogonal definable groups H_i (where H_i is generated by a collection of mutually nonorthogonal strongly minimal sets). If $A^\sharp$ is nonorthogonal to $s(H)/A^\sharp$ then $A^\sharp$ is nonorthogonal to the constants, which implies easily that A is defined over $\mathcal{C}$ and thus k, contradiction. Hence we have orthogonality, and by the statement above $s(H) = A^\sharp + V$ where V is definable and has trivial intersection with $A^\sharp$. This contradicts part (i).

(vii) Again there are various ways of seeing this. Let $\tilde{A}$ be as in the proof of (ii), equipped with its unique algebraic ∂-group structure.We again refer to [5] for background and details. Let U be the maximal unipotent algebraic ∂-subgroup of $\tilde{A}$. Then the canonical projection from $\tilde{A}/U$ to A induces an isomorphism between $(\tilde{A}/U)^\partial$ and $A^\sharp$. But the connected definable subgroups of $(\tilde{A}/U)^\partial$ are precisely the intersections with $(\tilde{A}/U)^\partial$ of connected algebraic ∂-subgroups of $\tilde{A}/U$ and by the choice of U these are preimages of abelian subvarieties of A, so defined over the algebraic closure of the base set of parameters. □

Proof of Theorem 1.1 (i). First, quotienting by $Stab_A(X)$ we obtain another abelian variety over K with k-trace 0. In order to show that X is the translate of an abelian subvariety of A, we will assume that $Stab_A(X)$ is finite, and look for a contradiction. To be consistent with earlier notation we work in the saturated elementary extension $\mathcal{U}$ of K^{diff} with field of constants $\mathcal{C}$ (although it is not really necessary). In any case all data we discuss will be defined over K or at the most K^{diff}. By (*) in the proof of Lemma 2.2, $A/A^\sharp$ definably embeds via some μ in a vector group. So $\mu(\Gamma)$ is contained in a finite-dimensional vector space over $\mathcal{C}$, the preimage of which we call H: a connected definable finite Morley rank subgroup of A containing both $A^\sharp$ and Γ, and defined over K. Now $X^\sharp = X \cap H$ is Zariski-dense in X. For simplicity we assume $X^\sharp$ irreducible as a differential algebraic variety. It follows that $Stab_{A^\sharp}(H)$ is finite, so by Lemma 2.1 and Lemma 2.2 (vi), (vii), after replacing $X^\sharp$ (and so X) by a suitable translate (which can be assumed to be defined over K^{diff}), $X^\sharp$ is contained in $A^\sharp$. We now consider the K-definable group $A^\sharp$ with all its induced structure over K, namely equipped with predicates for K-definable subsets of Cartesian powers. Now A is a sum of K-definable g-minimal subgroups A_i. And we likewise have the $A_i^\sharp$

with the induced structure. Now $A_i^\sharp(K)$ is clearly infinite as it contains the torsion. So by Corollary 1.11, $A^\sharp(K)$ is an elementary substructure of $A^\sharp$.
Claim. $A^\sharp(K) = A^\sharp(K^{diff})$.
Proof of claim. If $b \in A^\sharp(K^{diff})$ then $tp(b/K)$ is isolated in DCF_0. But then $tp(b/A^\sharp(K))$ is isolated in the structure $A^\sharp$. As $A^\sharp(K)$ is an elementary substructure of $A^\sharp$, $b \in A^\sharp(K)$.

Note that X is now defined over K^{diff}, and $X \cap A^\sharp(K^{diff})$ is Zariski-dense in X. By the claim and the Theorem of the kernel, $A^\sharp(K^{diff}) = A_{torsion}$, hence by Manin-Mumford, X is a translate of an abelian subvariety of A. The proof is complete.

We now briefly discuss how to deduce the 1-basedness and strong minimality of $A^\sharp$ when A is a simple abelian variety with k-trace 0. (Aside: The reader might think that this follows as in [18] where the truth of Mordell-Lang is shown to be equivalent to the 1-basedness of Γ in the expansion $(\mathbb{C}, +, \times, \Gamma)$ by the relevant finite rank subgroup Γ of the relevant (semi) abelian variety. However we are working here with $A^\sharp$ with its induced structure from an ambient differentially closed field, so additional arguments are needed.) We already discussed algebraic ∂-groups in the proof of Lemma 2.2 above. The universal vectorial extension $\pi : \tilde{A} \to A$ has a unique ∂-group structure. We mentioned that $A^\sharp$ is the image of $\tilde{A}^\partial$. Let U be the maximal unipotent algebraic ∂-subgroup of $\tilde{A}$, and let $\overline{A}$ be $\tilde{A}/U$ (which also has a unique algebraic ∂-group structure). Then π factors through $\pi_1 : \overline{A} \to A$ and $\pi_2 : \tilde{A} \to \overline{A}$. Moreover π_1 induces an *isomorphism* between $\overline{A}^\partial$ and $A^\sharp$. (See [5].)

Recall that a definable connected (commutative) group G (in a stable theory) is called 1-based if every definable subset of G^n is a Boolean combination of translates of definable subgroups. Assuming G defined over an algebraically closed set of parameters B, it suffices that every B-definable subset of G^n is of the appropriate form.

When we say that a commutative algebraic group G satisfies Manin-Mumford we mean that if X is an irreducible subvariety of G such that the torsion points in X are Zarisk-dense then X is a translate of an algebraic subgroup. In fact Manin-Mumford for commutative algebraic groups is known. (See for example [12].) But in the proof of the proposition below we will only be using its truth for Cartesian powers of $\tilde{A}$ where A is an abelian variety over K with k-trace 0. In any case, keeping to our earlier notation we have:

Proposition 2.3. *Let A be an abelian variety over K with k-trace 0. Assume*

Manin-Mumford for Cartesian powers of $\tilde{A}$. Then
(i) $A^\sharp$ is 1-based.
(ii) If moreover A is simple, then $A^\sharp$ is strongly minimal.

Proof. (i) It is enough to work in the model K^{diff} and show that every K-definable subset of $(A^\sharp)^n$ is a Boolean combination of translates of subgroups. We use the maps π_1, π_2 discussed above, and their properties. As in the proof of 1.1 (i), $A^\sharp(K^{diff})$ and thus also $\overline{A}^\partial(K^{diff})$ consists just of the torsion points of the relevant groups. Moreover π_2 induces an isomorphism between the torsion subgroups of of $\tilde{A}$ and $\overline{A}$. All this also works for Cartesian powers. Now the K-definable subsets of Cartesian powers of $\overline{A}^\partial$ are just the intersections with K-subvarieties of Cartesian powers of $\overline{A}$. The assumption of Manin-Mumford for Cartesian powers of $\tilde{A}$ and above observations implies that such an intersection is a finite union of translates of subgroups, proving 1-basedness.
(ii) If A is simple, then $A^\sharp$ is g-minimal. By part (i) every definable subset of $A^\sharp$ is a finite Boolean combination of translates of definable subgroups, so by g-minimality has to be finite or cofinite.

□

3 Theorem 1.1: Characteristic p

We prove Theorem 1.1(ii). This will be substantially simpler than the characteristic 0 case, avoiding any recourse to the socle theorem or to any Buium homomorphism. But, of course, assuming that $A^\sharp$ has QE (which will be proved for Cartesian powers of simple A in the next section). We again work with notation as in Section 1: A, X are over $K = k(t)^{sep}$, Γ is contained in the prime-to-p division points of a finitely generated subgroup of $A(K)$ (so $\Gamma < A(K)$ too), $X \cap \Gamma$ is Zariski-dense in X, and we take $\mathcal{U}$ to be a saturated elementary extension of K. As before, $A^\sharp$ denotes $A^\sharp(\mathcal{U})$ and A is a sum of simple abelian subvarieties $A_1, .., A_n$, all defined over K. By Lemma 3.6 of [2], for each i, $A_i^\sharp$ is the connected component of $A_i \cap A^\sharp$ and $A^\sharp$ is the sum of the $A_i^\sharp$. Moreover A is assumed to have k-trace 0.

We refer the reader to [2] where we give a precise account of relative Morley rank for type-definable sets in a stable structure (called "internal Morley dimension" in [11]).

Fact 3.1. *Both $A^\sharp$ and the $A_i^\sharp$ are connected, with finite relative Morley rank.*

Moreover, $A_i^\sharp$ is relatively definable in $A^\sharp$ and $A_i^\sharp$ has no proper relatively definable infinite subgroup.

Comments. The fact that $A^\sharp$ has finite relative Morley rank is claimed in Remark 2.19 of [11]. However it is also implicitly claimed there that $G^\sharp = p^\infty(G(\mathcal{U}))$ also has finite relative Morley rank, whenever G is *semi-abelian*, and this is actually wrong, as pointed out in [2]. So we refer the reader rather to the proof of Fact 3.8 from [2]. It is worth remarking that the semiabelian counterexample from [2] is shown in [15] not to have QE. But in general there is no reason why a type-definable group of finite relative Morley rank should have QE. Once we know that $A^\sharp$ has relative finite Morley rank, as, for each i, $A_i^\sharp$ is the connected component of $A_i \cap A^\sharp$, it follows that $A_i^\sharp$ is relatively definable in $A^\sharp$, but again this is no longer true in the semiabelian counterexample.
The fact that for a simple abelian variety A_i, $A_i^\sharp$ has no proper infinite relatively definable subgroup was originally proved in [11] (see also [4]).

We now consider $A^\sharp$ and the $A_i^\sharp$ with their induced structure over K; by assumption $A^\sharp$ has quantifier elimination. By Fact 3.1, each $A_i^\sharp$ is relatively definable in $A^\sharp$ so has quantifier elimination too. It follows, also from 3.1, that each of the $A_i^\sharp$ (with its induced structure) remains a g-minimal group. Noting that $A_i^\sharp(K)$ is infinite (it contains all the prime-to-p torsion of A_i) and using Corollary 1.11, we conclude:

Corollary 3.2. *The groups $A^\sharp$ and the $A_i^\sharp$ are connected groups of finite Morley rank and $A^\sharp(K)$ is an elementary substructure of $A^\sharp$.*

We now complete the proof. Note that $p^{n+1}\Gamma$ has finite index in $p^n\Gamma$ for all n. As Γ meets X in a Zariski-dense set, we find cosets D_i of $p^i\Gamma$ in Γ such that $i < j$ implies $D_j \subseteq D_i$, and each D_i meets X in a Zariski-dense set. Hence we obtain a descending chain of cosets C_i of $p^iA^\sharp(K)$ in $A^\sharp(K)$, each meeting X in a Zariski-dense set. Passing to the saturated model $\mathcal{U}$, let C be the intersection of the C_i. Then by compactness $X \cap C$ is Zariski-dense in X. Note that C is type-definable over K, although $C(K)$ may be empty.

We consider the two sorted structure $M = (A^\sharp, C)$ with all K-induced structure. It easily follows from Corollary 3.2 that $Th(M)$ has finite Morley rank and moreover that the sort $A^\sharp$ with induced structure has $A^\sharp(K)$ as an elementary substructure. Let $M_0 \prec M$ be prime (so atomic) over $A^\sharp(K)$. It follows that M_0 is of the form $(A^\sharp(K), C_0)$ for some elementary substructure C_0 of C. Note that C_0 is definably (without parameters) a PHS (principal

homogeneous space) for $A^\sharp(K)$.
Claim. $X \cap C_0$ is Zariski-dense in X.
Proof of claim. Suppose not. Then there is a proper subvariety Z of X (defined over some field) such that $X \cap C_0 \subseteq Z$. We may replace Z by the Zariski closure of $X \cap C_0$, and so we may assume that Z is defined over C_0. We now have that $X \cap C_0 = Z \cap C_0$. Now $Z \cap C$ viewed as a set definable in the structure C (or M) is defined over C_0, so as $C_0 \prec C$ we easily conclude that $X \cap C = Z \cap C$, contradicting Zariski-denseness of $X \cap C$ in X. This completes the proof of the claim.

Let $a \in X \cap C_0$. Let $X_1 = X - a$. Then $X_1 \cap A^\sharp(K)$ is Zariski-dense in X_1. In particular X_1 is defined over K. Moreover using the theorem of the kernel, $X_1 \cap A_{torsion}$ is Zariski-dense in X_1, so by MM, X_1 is a translate of an abelian subvariety of A. This completes the proof of Theorem 1.1(ii).

4 QE for $A^\sharp$

Here we prove Theorem 1.2. in slightly more generality.

So we assume A to be a simple abelian variety over any separably closed field K of finite degree of imperfection, $\mathcal{U}$ is a saturated extension of K. We consider $A^\sharp = p^\infty A(\mathcal{U})$, and we denote by $\mathcal{A}$ the structure $A^\sharp$ with relatively definable sets (with parameters from K). By Lemma 1.10 it suffices to prove that $Th(\mathcal{A})$ has quantifier elimination, which we will accomplish here. For each n, let $\Pi_n A$ be obtained from A by Weil's restriction of the scalars from K to K^{p^n}. Recall ([3]) that there are definable bijective homomorphisms $\phi_n : A(\mathcal{U}) \to \Pi_n A(\mathcal{U}^{p^n})$ (they are compatible sections for the projections $\pi_{m,n} : \Pi_m A \to \Pi_n A$ for $m \geq n$) and algebraic subgroups $A_n \subset \Pi_n A$ over K^{p^n}, with A_n isogenous to A, such that for $x \in A(\mathcal{U})$, $x \in p^n A(\mathcal{U})$ iff $\phi_n(x) \in A_n$. Looking at $A(\mathcal{U})$ affine chart by affine chart, $A(\mathcal{U})^d$ and $(A^\sharp)^d$ are equipped with the λ -topology (or equivalently, the Hasse differential algebraic topology), the basic closed sets are of the form $X = \langle X_n \rangle := \{x \in A(\mathcal{U})^d : \phi_n(x) \in X_n\}$ for some integer n and some algebraic subvariety X_n of $\Pi_n A$ (note that X_n can be chosen as the Zariski closure of $\phi_n(X(\mathcal{U}))$, hence defined over $\mathcal{U}^{p^n}$). By quantifier elimination for separably closed fields in the language with the λ-functions or the Hasse derivations, in $\mathcal{U}$, relatively definable subsets of $A(\mathcal{U})^d$ are Boolean combinations of sets of the form $\langle X_n \rangle$ (using preimages by the projection $\pi_{m,n}$, we may assume n being the same for each of the pieces). It follows from the characterization of $p^i A(\mathcal{U})$ that

relatively definable subsets of $(A^\sharp)^d$ are Boolean combinations of sets $\langle X_n \rangle$, for X_n algebraic subvarieties of A_n.
We will also use the following consequence of the Zilber indecomposability theorem (see Fact 3.8 in [2]): as A is simple, $A^\sharp$ is g-minimal and hence, there exists a minimal type in $A^\sharp$ whose set of realizations, which will be denoted by Q in the following, is such that $A^\sharp = Q + \ldots + Q$ (m times for some m).

Lemma 4.1. *For every $n \geq 1$, the topology on $(A^\sharp)^n$ is Noetherian of finite dimension.*

Proof. It is shown in [11] that the trace of the λ-topology on each Q^k is Noetherian and has finite dimension, if Q is a *thin* minimal type, i.e. one with finite transcendence rank, which is the case here. Now the noetherianity and finiteness of dimension for $A^\sharp$ follows as we have a continuous relatively definable surjective map from $Q^{md} \to {A^\sharp}^d$. □

Note, as a corollary, that each closed set of $(A^\sharp)^d$ is actually relatively definable.

If $\mathcal{U}_0 \prec \mathcal{U}$, and $a \in \mathcal{U}$, $\mathcal{U}_0\{a\}$ denotes the ring generated by a and its images by the λ-functions over $\mathcal{U}_0$ and $\mathcal{U}_0(\{a\})$, the fraction field of $\mathcal{U}_0\{a\}$.

Lemma 4.2. *Let X and Y be closed sets in $(A^\sharp)^d$, with X irreducible, and $pr : X \times Y \to X$ the projection. Let $G \subsetneq F$ be closed subsets of $X \times Y$, such that $\overline{pr(F)} = X$ and F is irreducible. Let a be a topological generic of X over some small model $\mathcal{U}_0$ of definition for F and G, we denote $F(a) = \{y \in Y \mid (a, y) \in F\}$, and similarly for G. Then $G(a) \subsetneq F(a)$. Moreover $F(a)$ is irreducible as a closed set over $\mathcal{U}_0(\{a\})$ (note however that it may be reducible as a closed set over $\mathcal{U}$, or even over $\mathcal{U}_0(\{a\})^{sep}$).*

Proof. We denote by $\mathcal{U}_0\{X\} := \mathcal{U}_0[\lambda_i T_j]/I(X)$ the ring of λ-coordinates of X over $\mathcal{U}_0$, and $\mathcal{U}_0\{Y\}$ in a similar way. By irreducibility, $\mathcal{U}_0\{X\}$ is an integral domain, and by choice of a, $\mathcal{U}_0\{X\} \simeq \mathcal{U}_0\{a\}$; we denote by $\mathcal{U}_0(\{a\})$ the fraction field of the latter.
We denote by $I(F)$ and $I(G)$ the separable ideals in $\mathcal{U}_0\{a\}\{Y\} \simeq \mathcal{U}_0\{X \times Y\}$ corresponding to F and G. Since $\overline{pr(F)} = X$, $I(F) \cap \mathcal{U}_0\{a\} = 0$. Now $I(F(a))$ is the ideal generated by $I(F)$ in $\mathcal{U}_0(\{a\})\{Y\}$, and similarly for $I(G(a))$. We claim that $I(F) \subsetneq I(G)$ implies that $I(F(a)) \subsetneq I(G(a))$. If $I(F(a)) = I(G(a))$, then for every $P \in I(G)$, there is some non zero $d \in \mathcal{U}_0\{a\}$ such that $dP \in I(F)$, which implies that $P \in I(F)$ since $d \notin I(F)$, which is prime. That contradicts $I(F) \neq I(G)$. We get that $I(F(a))$ is prime by the same

kind of argument. This means that $F(a)$ is irreducible as a closed set over $\mathcal{U}_0(\{a\})$. $\square$

We will now use completeness of abelian varieties to show that the family of closed subsets is closed under projection.

Lemma 4.3. *Let F be a definable closed set of A^d and $pr : (A^\sharp)^d \to (A^\sharp)^{d-1}$ be the projection on the first coordinates. Then $pr(F \cap (A^\sharp)^d)$ is closed and relatively definable.*

Proof. Let $\overline{x} = (x_1, \dots, x_{d-1})$ be in $(A^\sharp)^{d-1}$. By saturation of $\mathcal{U}$, $\overline{x} \in pr(F \cap (A^\sharp)^d)$ if and only if, for every m (big enough), there is $y_m \in p^m A(\mathcal{U})$ such that $(\overline{x}, y_m) \in F$. We know that $F = \langle X_n \rangle$ for some (closed) subvariety X_n of $\Pi_n A$. Note that by letting $X_m = \pi_{m,n}^{-1}(X_n) \subseteq \Pi_m A$ for $m \geq n$, we also have $F = \langle X_m \rangle$. Now we use the fact that, in $\mathcal{U}$, there is $y_m \in p^m A(\mathcal{U})$ such that $(\overline{x}, y_m) \in F$ if and only if $\phi_m(\overline{x}) \in pr(X_m \cap A_m^d)$. Since A_m is an abelian variety, $Y_m := pr(X_m \cap A_m^d)$ is a closed subvariety of A_m^{d-1}, hence, by Noetherianity, $pr(F \cap (A^\sharp)^d) = \cap_{m \geq n} \langle Y_m \rangle$ is a relatively definable closed set. $\square$

In order to go from the case of closed sets to the case of constructible sets, we really follow the lines of the proof of quantifier elimination for one-dimensional Zariski geometries given in [25] or [13] (note that QE for one dimensional Zariski geometries is a basic consequence of the axioms and does not involve the deep dichotomy result). We know that if A is a simple abelian variety, then $A^\sharp = Q + \dots + Q$, where Q is the set of realizations of a minimal type (that is, a type of U-rank one). It would be convenient to work with the relatively definable closed set $\overline{Q}$, the closure of Q in the sense of the λ-topology, but it is not clear a priori why it should be of topological dimension 1. A more general fact would be that the U-rank of a type t coincides with the topological dimension of $\overline{t}$ (the closure of the set of its realizations, or equivalently, the closed set given by the type ideal corresponding to t). It is true for types in $A^\sharp$, actually, but we know it only a posteriori, via the dichotomy theorem, which we do not want to use. We know however that U-rank $(t) \leq dim(\overline{t})$: because of the correspondance between closed type-definable sets and prime separable ideals in a suitable polynomials algebra, $dim(\overline{t})$ is given by the separable depth of the corresponding ideal I_t, and the separable depth is a stability rank, hence greater than or equal to the U-rank (see [9]).

It follows that if $F \subset A^\sharp$ is irreducible closed of topological dimension one, then F also has a unique type of U-rank one, which is of maximal rank and is also its topological generic.

So, rather than Q, we will consider a suitable relatively definable irreducible closed set H of dimension 1. We proceed as follows. Recall that $A^\sharp$ is a g-minimal group. Let H be a relatively definable irreducible closed subset of $A^\sharp$ of dimension 1 (it exists since we know that the topology is Noetherian, and since translations are bicontinuous, we are allowed to replace H by any of its translates in the following). By the comparison between the U-rank and the topological dimension above, it is clear that the generic type (in the topological sense) of H is minimal, hence, by g-minimality of $A^\sharp$, we can apply the indecomposability theorem to get $A^\sharp = H + \ldots + H$ (m times for some m).
Now for any formula $\phi(x, \overline{a})$ with parameters in $A^\sharp$, $\mathcal{A} \models \exists x\, \phi(x, \overline{a})$ if and only if $\mathcal{A} \models \exists x_1 \in H \ldots \exists x_m \in H\, \phi(\sum_i x_i, \overline{a})$. Hence it is sufficient to consider projections of the form $pr : H \times (A^\sharp)^d \to (A^\sharp)^d$.From Lemma 4.3 and the fact that H is closed, we get that pr takes closed sets to closed sets. From quantifier elimination in the separably closed field $\mathcal{U}$ and Noetherianity in $\mathcal{A}$, we just have to consider projections of definable sets $F \setminus G$, where $G \subsetneq F \subseteq H \times (A^\sharp)^d$ are closed relatively definable sets, with F irreducible.

Proposition 4.4. *The projection $pr(F \setminus G)$ is constructible in $\mathcal{A}$.*

Proof. We proceed by induction on $dim(F)$. The case $dim(F) = 0$ is obvious since it implies that F is a singleton.
Now $dim(F) = k + 1$. We consider the closed sets $F_1 = pr(F)$, $G_1 = pr(G)$, $F_0 = \{\overline{y} \in (A^\sharp)^d \mid \forall x \in H, (x, \overline{y}) \in F\} = \bigcap_{x \in H} F_x$, where $F_x = \{\overline{y} \in (A^\sharp)^d \mid (x, \overline{y}) \in F\}$ is closed (note that we allow parameters in $A^\sharp$ in the definition of the topology), and $G_0 = \{\overline{y} \in (A^\sharp)^d \mid \forall x \in H, (x, \overline{y}) \in G\}$. There are three cases:

1. if $F_0 = F_1$, we see easily that $pr(F \setminus G) = F_0 \setminus G_0$, hence is constructible.

2. if $G_1 \subsetneq F_1$, we have a proper closed subset $(H \times G_1) \cap F \subsetneq F$, hence $dim((H \times G_1) \cap F) < dim(F)$ since F is irreducible. But we can write $pr(F \setminus G) = F_1 \setminus G_1 \cup pr(((H \times G_1) \cap F) \setminus G)$, and the result comes from the induction hypothesis.

3. if $F_0 \subsetneq F_1 = G_1$, we consider a generic point a of F_1 over $\mathcal{U}_0$, a model of definition for F and G (note that F_1 is irreducible since F is). In

particular $a \notin F_0$, which implies that the fiber $F(a)$ is finite, as it is a proper closed set of the irreducible dimension one H. Furthermore, by Lemma 4.2, $G(a) \subsetneq F(a)$, and $F(a)$ is irreducible as a closed set over $\mathcal{U}_0(\{a\})$. It follows that $F(a)$ is the orbit of any of its points under $\mathrm{Aut}(\mathcal{U}/\mathcal{U}_0(\{a\}))$ (note that $\mathcal{U}_0(\{a\})$ is definably closed in $\mathcal{U}$, see [10]), and a fortiori, it is the orbit of $G(a)$. Hence $F(a) = G(a)$, a contradiction.

□

Corollary 4.5. *The structure $\mathcal{A}$ has QE.*

So we have proved Theorem 1.2, and this completes the paper.

References

[1] D. Abramovich and F. Voloch, Towards a proof of the Mordell-Lang conjecture in positive characteristic, International Math. Research Notices 5 (1992), 103-115.

[2] F. Benoist, E. Bouscaren, and A. Pillay, Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences, preprint (2013).

[3] F. Benoist and F. Delon, Questions de corps de définition pour les variétés abéliennes en caractéristique positive, Journal de l'Institut de Mathématiques de Jussieu, vol. 7 (2008), 623–639.

[4] E. Bouscaren and F. Delon, Minimal groups in separably closed fields, J. Symbolic Logic 67 (2002), no. 1, 239259.

[5] D. Bertrand and A. Pillay, A Lindemann-Weierstrass theorem for semi-abelian varieties over function fields, Journal AMS, 23 (2010), 491-533.

[6] A. Buium, Intersections in jet spaces and a conjecture of S. Lang, Annals of Math., 136 (1996), 557-567.

[7] A. Buium, *Differential algebraic groups of finite dimension*, LNM 1506, Springer, 1992.

[8] C. Corpet, Around the Mordell-Lang and Manin-Mumford conjectures in positive characteristic, preprint 2012.

[9] F. Delon, Idéaux et types sur les corps séparablement clos, *Supplément au Bulletin de la SMF*, T116 (1988).

[10] F. Delon, Separably closed fields, in *Model theory and algebraic geometry*, edited by E. Bouscaren, Lecture Notes in Mathematics 1696, Springer, 1996.

[11] E. Hrushovski, The Mordell-Lang conjecture for function fields, Journal AMS 9(1996), 667-690.

[12] E. Hrushovski, The Manin-Mumford conjecture and the model theory of difference fields, Annals of Pure and Applied Logic, 112 (2001), 43-115.

[13] E. Hrushovski and B. Zilber, Zariski geometries, Journal AMS 9(1996), 1-56.

[14] D. Marker, M. Messmer, and A. Pillay, *Model theory of Fields*, Lecture Notes in Logic 5, second edition, ASL-AK Peters, 2003.

[15] A. Omar Aziz, *Type definable stable groups and separably closed fields*, Ph. D. thesis, Leeds, 2012.

[16] A. Pillay, Differential algebraic groups and the number of countable models, in *Model theory of Fields* above.

[17] A. Pillay, *Geometric Stability Theory*, Oxford University Press, 1996.

[18] A. Pillay, The model-theoretic content of Lang's conjecture, *Model theory and algebraic geometry*, edited by E. Bouscaren, Lecture Notes in Mathematics 1696, Springer, 1996.

[19] A. Pillay and M. Ziegler, Jet spaces of varieties over differential and difference fields, Selecta Math. 9 (2003), 579-599.

[20] R. Pink and D. Rössler, On ψ-invariant subvarieties of semiabelian varieties and the Manin-Mumford conjecture, J. Algebraic Geometry 13 (2004), 771-798.

[21] D. Rössler, Infinitely p-divisible points on abelian varieties defined over function fields of characteristic $p > 0$, Notre Dame Journal of Formal Logic 54 (2013), no. 3-4, 579-589

[22] D. Rössler, On the Manin-Mumford and Mordell-Lang conjectures in positive characteristic, Algebra and Number Theory 7 (2013), 2039-2057.

[23] T. Scanlon, A positive characteristic Manin-Mumford theorem, Compositio Math. 141 (2005),1351-1364.

[24] F. Wagner, Fields of finite Morley Rank. Journal of Symbolic Logic, 66(2001), 703-706

[25] B. Zilber, *Zariski geometries*, LMS Lecture series, CUP, 2011.