Universal covers of commutative finite Morley rank groups

Martin Bays, Bradd Hart, and Anand Pillay

Wed 19 Mar 09:51:27 PDT 2014

Abstract

We give an algebraic description of the structure of the analytic universal cover of a complex abelian variety which suffices to determine the structure up to isomorphism. More generally, we classify the models of theories of "universal covers" of rigid divisible commutative finite Morley rank groups.

1 Introduction

1.1 Characterising universal covers of abelian varieties

Let $\mathbb{G}$ be an abelian variety defined over $\mathbb{C}$. Considering $\mathbb{G}(\mathbb{C})$ as a complex Lie group, the exponential map provides a surjective analytic homomorphism

$$\rho : T_0(\mathbb{G}(\mathbb{C})) \twoheadrightarrow \mathbb{G}(\mathbb{C}).$$

Let $\mathcal{O} := \{\eta \in \mathrm{End}(T_0(\mathbb{G}(\mathbb{C}))) \mid \eta(\ker \rho) \subseteq \ker \rho\} \cong \mathrm{End}(\mathbb{G})$ be the ring of $\mathbb{C}$-linear endomorphisms which induce endomorphisms of $\mathbb{G}(\mathbb{C})$; these are precisely the algebraic endomorphisms of $\mathbb{G}$. Consider $T_0(\mathbb{G}(\mathbb{C}))$ as an $\mathcal{O}$-module.

In this paper, we use model theoretic techniques and Kummer theory to give a purely algebraic characterisation of the algebraic consequences of this analytic picture. To state this precisely, we first define a notion of isomorphism which captures these "algebraic consequences".

Let $k_0 \leq \mathbb{C}$ be a field over which $\mathbb{G}$ and the action on $\mathbb{G}$ of each $\eta \in \mathcal{O}$ are defined. Say that a surjective $\mathcal{O}$-homomorphism $\rho' : V \twoheadrightarrow \mathbb{G}(\mathbb{C})$ from an $\mathcal{O}$-module V is *algebraically isomorphic* to ρ if there is an $\mathcal{O}$-module isomorphism $\sigma : T_0(\mathbb{G}(\mathbb{C})) \to V$ and a field automorphism $\tau \in \mathrm{Aut}(\mathbb{C}/k_0)$ of $\mathbb{C}$ fixing k_0 such that $\tau\rho'\sigma = \rho$.

Say that such an isomorphism is *over* a submodule of V if σ fixes the submodule pointwise.

Let $\langle \ker \rho \rangle_{\mathbb{Q}} \leq T_0(\mathbb{G}(\mathbb{C}))$ be the divisible subgroup generated by $\ker \rho$, so $\langle \ker \rho \rangle_{\mathbb{Q}} = \rho^{-1}(\mathrm{Tor}(\mathbb{G}))$. We prove:

Theorem 1.1. *Suppose k_0 is a number field. Then $\rho : T_0(\mathbb{G}(\mathbb{C})) \twoheadrightarrow \mathbb{G}(\mathbb{C})$ is, up to algebraic isomorphism over $\langle \ker \rho \rangle_{\mathbb{Q}}$, the unique surjective $\mathcal{O}$-homomorphism with kernel $\ker \rho$ which extends $\rho{\restriction}_{\langle \ker \rho \rangle_{\mathbb{Q}}}$.*

We require here that k_0 is a number field in order to have Kummer theory available. We have a corresponding result in the case that $\mathbb{G}$ is a split semiabelian variety defined over a number field, but general semiabelian varieties are problematic due to failure of Kummer theory.

We prove Theorem 1.1 by classifying the models of the first order theory of ρ. Our proof can be split into three stages:

(i) Kummer theory for abelian varieties (due to Faltings, Ribet, Serre, Bogomolov) explains the behaviour for finite extensions of k_0, and suffices to show uniqueness of the restriction of ρ to $\rho^{-1}(\mathbb{G}(\bar{\mathbb{Q}}))$;

(ii) a function-field analogue of this Kummer theory allows us to extend the uniqueness to $\mathbb{G}(F)$ for F an algebraically closed field of cardinality $\leq \aleph_1$;

(iii) we extend to arbitrary cardinals (in particular the continuum, which without assuming the continuum hypothesis is not covered by (ii)) using arguments involving independent systems, based on arguments involved in Shelah's Main Gap theorem.

In [BGH11], it was found that the geometric Kummer theory of (ii) actually follows from a general model-theoretic principle, Zilber's Indecomposability Theorem, and hence holds in the generality of rigid (see below) commutative divisible finite Morley rank groups.

This also turns out to be a natural level of generality for (iii), and it is in this context that we will actually work for most of this paper. We correspondingly obtain an analogue of Theorem 1.1 in this generality, Theorem 3.11 below - although since there is no analogue of (i) in such generality we get a correspondingly weaker result.

This does allow us to remove the restrictions in Theorem 1.1 and still get a uniqueness result: if $\mathbb{G}$ is an abelian variety over a field $k_0 \leq \mathbb{C}$, then the exponential map $\rho : T_0(\mathbb{G}(\mathbb{C})) \twoheadrightarrow \mathbb{G}(\mathbb{C})$ is, up to algebraic isomorphism over $\rho\restriction_{k_0^{\mathrm{alg}}}$, the unique surjective $\mathrm{End}(\mathbb{G})$-homomorphism with kernel $\ker\rho$ which extends $\rho\restriction_{k_0^{\mathrm{alg}}}$. We obtain an analogous result for semiabelian varieties, but it admits no statement of this form.

We also obtain similar results for complex tori which are not abelian varieties, and for semiabelian varieties in positive characteristic.

1.2 Pro-algebraic covers and the model-theoretic setup

Let us now explain the model-theoretic setup used in the remainder of the paper from the point of view of algebraic groups.

The algebraic group $\mathbb{G}$ can be viewed as a definable group in ACF_0, and as such inherits the structure of a finite Morley rank group. Explicitly, we consider $\mathbb{G}(K)$ for K an algebraically closed extension of k_0 as a structure in the language with a predicate for each k_0-Zariski-closed subset of each Cartesian power $\mathbb{G}^n(K)$. This structure is bi-interpretable with the field $(K; +, \cdot, (c)_{c\in k_0})$ with parameters for k_0, and is a finite Morley rank group of rank $\dim(\mathbb{G})$.

As in Grothendieck's construction of the étale fundamental group, we may consider the inverse limit of finite étale covers of $\mathbb{G}$ as an "étale universal cover" of $\mathbb{G}$. Any finite étale cover of $\mathbb{G}$ is dominated by a multiplication-by-n homomorphism $[n] : \mathbb{G} \twoheadrightarrow \mathbb{G}$, meaning that it suffices to consider the projective limit of copies of $\mathbb{G}(K)$ with these maps between them, $\widehat{\mathbb{G}}(K) := \varprojlim [n] : \mathbb{G} \twoheadrightarrow \mathbb{G}$. This can be considered as a purely algebraic substitute for the analytic universal cover; we will see below one justification for this: in an appropriate language, the latter is an elementary submodel of the former.

We can parallel this construction for an abstract commutative divisible finite Morley rank group $\mathbb{G}$, defining the "pro-definable universal cover" $\widehat{\mathbb{G}} := \varprojlim [n] : \mathbb{G} \twoheadrightarrow \mathbb{G}$. If we consider definable finite group covers as a substitute for étale maps, again any is dominated by some $[n]$, so this is the analogue of the étale universal cover.

The results described in the previous subsection result from classifying the models of the theory of this structure in an appropriate language, which in the algebraic case corresponds to the notion of "algebraic isomorphism" defined above.

1.3 The literature

We discuss the previous work on which this work builds. For $\mathbb{G} = \mathbb{G}_m$ the multiplicative group, Theorem 1.1 was proven in [ZCovers, BZCovers]. It was proven for $\mathbb{G}$ an abelian variety in [Gav06] under the assumption of the continuum hypothesis, i.e. with only the first two of the three steps described above. A path toward the full result was discussed in [Zil02], and for $\mathbb{G}$ an elliptic curve the full result was obtained in [Bay09]. These previous proofs of (iii) use algebraic techniques analogous to but substantially more complicated and limited than the model theoretic techniques of the present work.

In previous work, the problem was considered one of categoricity in infinitary logic, and correspondingly the techniques applied were those of Shelah's theory of excellent classes, and more specifically Zilber's adaptation to Quasiminimal Excellent (QME) classes. It was key to the developments in this paper to instead consider the problem in terms of first-order classification theory; the argument which allows us to get (iii) in the generality we do is an adaptation of Shelah's "NOTOP" argument, which reduces the condition of excellence in the first-order case to a simpler condition.

In fact, while the current paper was in preparation, it was found that this same idea applies in the context of QME classes [BHH$^+$13]. For the benefit of any readers familiar with that paper, we mention how it relates to this paper. Our main results do not fit into the definition of QME, even if we assume the kernel to be countable: we consider finite Morley rank groups which are not necessarily almost strongly minimal; correspondingly, the covers are not even almost quasiminimal. In the case of a semiabelian variety $\mathbb{G}$ discussed above, however, the covers structure can be seen as almost quasiminimal - and moreover it is bi-interpretable with the quasiminimal structure induced on the inverse image in the cover of a Kummer-generic (in the sense of [BGH11]) curve in $\mathbb{G}$ which generates $\mathbb{G}$ as a group. So in this case, (iii) above could be deduced from the main result of [BHH$^+$13].

1.4 Acknowledgements

This paper grew out of the thesis of the first author, supervised by Boris Zilber, and many of the ideas are due eventually to Zilber. The first author was also strongly influenced by the work of Misha Gavrilovich. We would like to thank

Udi Hrushovski for first suggesting the relevance of NOTOP/PMOP. The first author would also like to thank John Baldwin, Juan Diego Caycedo, Martin Hils and Jonathan Kirby for helpful pointers and discussion at various stages of this project.

2 First-order theory of pro-definable universal covers of divisible commutative finite Morley rank groups

If $\mathbb{G}$ is a commutative group and $[n]$ is the multiplication-by-n map, let $\widehat{\mathbb{G}} := \varprojlim [n] : \mathbb{G} \to \mathbb{G}$. Let $\rho_n : \widehat{\mathbb{G}} \twoheadrightarrow \mathbb{G}$ be the corresponding maps, so $[n]\rho_{nm} = \rho_m$. Let $\rho := \rho_1$. We often write elements of $\widehat{G}$ in the form $\gamma = (g_n)_n$, and then $\rho_n(\gamma) = g_n$.

If $\theta : G \to H$, define $\widehat{\theta} : \widehat{G} \to \widehat{H}$ by $\widehat{\theta}((g_n)_n) = (\theta(g_n)_n$.

Say a commutative group G is *divisible-by-finite* if its divisible hull $G^o := \bigcap_n nG \leq G$ has finite index in G. We first note some elementary homological algebraic properties of the functor $\widehat{\cdot}$ on divisible-by-finite groups:

Lemma 2.1. *Suppose* $0 \to A \to B \to C \to 0$ *is an exact sequence of divisible-by-finite groups. Then*

(I) $0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$ *is exact.*

(II)
$$0 \to \pi_1(A) \to \pi_1(B) \to \pi_1(C) \to \pi_0(A) \to \pi_0(B) \to \pi_0(C) \to 0$$
is exact, where $\pi_1(A) := \rho^{-1}(0) \leq \widehat{A}$ *and* $\pi_0(A) := A/A^o$

Proof. (I) The only difficulty is the surjectivity of $\widehat{B} \to \widehat{C}$. We may assume $A \to B$ is an inclusion. Factoring θ via $B/(A^o)$, we see that it suffices to prove the surjectivity of $\widehat{B} \to \widehat{C}$ under the assumption that A is divisible or finite.

(a) Suppose A is divisible. We first show that given any $n \in \mathbb{N}$, $b \in B$ and $c' \in C$ such that $\theta(b) = [n]c'$, there is $b' \in B$ such that $[n]b' = b$ and $\theta(b') = c'$. Say $\theta(b'') = c'$; then $\theta([n]b'') = [n]c' = \theta(b)$, so $b - [n]b'' \in A$. Say $a' \in A$ with $[n]a' = b - [n]b''$. Then $b' := b'' + a'$ is as required.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "{[n]}"] & B \arrow[r] \arrow[d, "{[n]}"] & C \arrow[r] \arrow[d, "{[n]}"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Given $\widehat{c}$, we can therefore inductively define $b_{n!}$ such that $[n]b_{(n+1)!} = b_{n!}$ and $\theta(b_{n!}) = \rho_{n!}(\widehat{c})$. Easily, there is a unique $\widehat{b} \in \widehat{B}$ such that $\rho_{n!}(\widehat{b}) = b_{n!}$, and $\widehat{\theta}(\widehat{b}) = \widehat{c}$.

(b) Suppose A is finite, say $[n]A = 0$. Then θ factors $[n]$ - indeed, define ϕ such that the left triangle in the following diagram commutes, then note that the right triangle does:

$$\begin{tikzcd} B \arrow[rr, "{[n]}"] \arrow[dr, "\theta"'] & & B \arrow[dr, "\theta"] & \\ & C \arrow[ur, "\phi"] \arrow[rr, "{[n]}"'] & & C \end{tikzcd}$$

But $\widehat{[n]}$ is surjective, hence so is $\widehat{\theta}$.

(II) Snake Lemma applied to the diagram

$$\begin{tikzcd} & \widehat{A} \arrow[r] \arrow[d, "\rho"] & \widehat{B} \arrow[r] \arrow[d, "\rho"] & \widehat{C} \arrow[r] \arrow[d, "\rho"] & 0\ , \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

noting that each column has kernel π_1 and cokernel π_0.

$\square$

Now let $\mathbb{G}$ be a connected commutative finite Morley rank group, and suppose moreover that it is divisible. Then $[n] : \mathbb{G} \twoheadrightarrow \mathbb{G}$ has finite kernel, and it follows that any definable subgroup $A \leq \mathbb{G}$ is divisible-by-finite, and its divisible hull A^o is its connected component in the model-theoretic sense (the smallest subgroup of finite index).

Let $T := \mathrm{Th}(\mathbb{G})$; we assume (by appropriate choice of language) that T has quantifier elimination. We also assume that the language of T is countable.

Let $\widehat{T}$ be the theory of $(\widehat{\mathbb{G}}, \mathbb{G})$ in the two-sorted language consisting of the maps ρ_n for each n, the full T-structure on $\mathbb{G}$, and, for each $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable connected subgroup H of $\mathbb{G}^n$, a predicate $\widehat{H}$ interpreted as the subgroup $\widehat{H} = \{x \mid \bigwedge_n \rho_n(x) \in H\}$ of $\widehat{\mathbb{G}}^n$.

For quantifier elimination purposes, we actually assume (by expanding T by constants if necessary) that every $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable connected subgroup of $\mathbb{G}^n$ is $\emptyset$-definable.

We say that T is *rigid* if for $\mathbb{G}$ a saturated model of T, every definable connected subgroup of $\mathbb{G}^n$ is defined over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Although the results of this section do not require rigidity, our language is chosen with it in mind.

Notation 2.1. If $(\widetilde{M}, M) \models \widehat{T}$ and $\widetilde{a} \in \widetilde{M}$ is a tuple, then we will write a_n for $\rho_n(\widetilde{a})$, and a for $\rho(\widetilde{a})$, and $\widehat{a}$ for $(a_n)_n$.

We will often just write $\widetilde{M} \models \widehat{T}$ rather than $(\widetilde{M}, M) \models \widehat{T}$; we recover M as $\rho(\widetilde{M})$. We say in this situation that $\widetilde{M}$ is *over* M.

$\widehat{G}$ and $\widehat{H}$ will always refer to the predicates corresponding to $\emptyset$-definable connected subgroups G and H of a cartesian power of $\mathbb{G}$. $\widehat{C}$ will refer to a coset of some $\widehat{H}$.

$\widehat{G}(\widetilde{a})$ is the definable set $\{\widetilde{x} \mid (\widetilde{x}, \widetilde{a}) \in \widehat{G}\}$, a coset of $\widehat{G}(0)$.

ker is the definable set $\ker(\rho)$.

$\ker^0$ is the $\bigwedge$-definable set $\bigwedge_n \rho_n(x) = 0$.

Abusively, ker and $\ker^0$ also refer to the corresponding sets in cartesian powers of $\mathbb{G}$.

$\widehat{H}_0 := \widehat{H} \cap \ker^0$, a $\mathbb{Q}$-subspace of the $\mathbb{Q}$-vector space $\ker^0$.

Proposition 2.2. *$\widehat{T}$ has quantifier elimination (QE), and is axiomatised by $\widehat{T}'$:*

(A1) T

(A2) if Γ_+ is the graph of the group operation on $\mathbb{G}$, then $\widehat{\Gamma_+}$ is the graph of a commutative divisible torsion free group operation, which we write as "+" and work with respect to in the following axioms;

(A3) if Δ is the diagonal subgroup (the graph of equality), then $\widehat{\Delta}$ is the respective diagonal subgroup;

(A4) each $\widehat{H}$ is a subgroup;

(A5) $[m]\rho_{nm} = \rho_n$*;*

(A6) $\rho_n(\widehat{H}) = H$*;*

(A7) $H \subseteq G$ *iff* $\widehat{H} \subseteq \widehat{G}$*;*

(A8) If $H \subseteq G$ *and* $\mathrm{Tor}(H) = \mathrm{Tor}(G)$*, then* $\widehat{H} \cap \ker = \widehat{G} \cap \ker$*;*

(A9) if a co-ordinate projection pr *induces a map* $\mathrm{pr} : G \twoheadrightarrow H$ *with kernel* K *then*

(I) the corresponding co-ordinate projection induces a map $\mathrm{pr} : \widehat{G} \twoheadrightarrow \widehat{H}$ *with kernel* $\widehat{K^o}$*;*

(II) if K/K^o *has exponent* N*, then* $\mathrm{pr} : \widehat{G} \cap \ker \twoheadrightarrow N(\widehat{H} \cap \ker)$.

In particular, $\mathrm{Th}(\widehat{\mathbb{G}})$ *does not depend on the choice of* $\mathbb{G} \models T$*.*

Remark 2.1. Note that to express these axioms, we require our assumption that connected subgroups are $\emptyset$-definable.

Proof. That $\widehat{\mathbb{G}}$ satisfies (A1)-(A8) is easily checked; by Lemma 2.1, it also satisfies (A9). We proceed to show completeness and QE.

Note that by (A6) applied to the graphs of the group operations, the ρ_n are homomorphisms, and it follows from (A6) and torsion-freeness that each $\widehat{H}$ is divisible.

Suppose given ω-saturated models $\widetilde{M}, \widetilde{N} \models \widehat{T}'$, tuples $\widetilde{m} \equiv_{qf} \widetilde{n}$ from each, and a point $\widetilde{m}'$.

Let $\widehat{H}$ be the *group locus* of $\widetilde{m}$ over 0, the smallest $\widehat{H}$ containing $\widetilde{m}$. Such exists by ω-stability of T, since $G \mapsto \widehat{G}$ is a lattice isomorphism by (A7).

Let $\widehat{G}$ be the group-locus of $(\widetilde{m}, \widetilde{m}')$ over 0, let pr be the corresponding co-ordinate projection.

$\widehat{T}' \models \mathrm{pr}(\widehat{G}) = \widehat{\mathrm{pr}(G)}$ by (A9), and we deduce that $\mathrm{pr} : \widehat{G} \twoheadrightarrow \widehat{H}$ and $\mathrm{pr} : G \twoheadrightarrow H$.

Claim 2.2.1. $\mathrm{pr} : \widehat{G}_0(\widetilde{N}) \twoheadrightarrow \widehat{H}_0(\widetilde{N})$

Proof. Let K be the kernel of $\mathrm{pr} : G \twoheadrightarrow H$, and suppose K/K^o has exponent N, so by (A9)

$$\widehat{T}' \models \mathrm{pr} : \widehat{G} \cap \ker \twoheadrightarrow N(\widehat{H} \cap \ker).$$

Then for each m:

$$\widehat{T}' \models \mathrm{pr} : m(\widehat{G} \cap \ker) \twoheadrightarrow mN(\widehat{H} \cap \ker).$$

But then by ω-saturation,

$$\widetilde{N} \models \mathrm{pr} : \bigcap_m m(\widehat{G} \cap \ker) \twoheadrightarrow \bigcap_m mN(\widehat{H} \cap \ker)$$

i.e.

$$\widetilde{N} \models \mathrm{pr} : \widehat{G}_0 \twoheadrightarrow \widehat{H}_0.$$

□

Now by QE in T and ω-saturation, we can find $\widetilde{n}'$ such that

$$(\widehat{m}, \widehat{m}') \equiv_{qf} (\widehat{n}, \widehat{n}')$$

in particular, $\rho_n(\widetilde{n}, \widetilde{n}') \in G$, so $(\widetilde{n}, \widetilde{n}') \in \widehat{G}(\widetilde{N}) + \ker^0(\widetilde{N})$. By the claim, we can then find such $\widetilde{n}'$ with $(\widetilde{n}, \widetilde{n}') \in \widehat{G}(\widetilde{N})$.

Now suppose the group locus of $(\widetilde{n}, \widetilde{n}') = \widehat{G}' < \widehat{G}$. Then $\rho_n(\widetilde{m}, \widetilde{m}') \in G'$ for each n, so $(\widetilde{m}, \widetilde{m}') \in \widehat{G}' + \zeta$ for some $\zeta \in \widehat{G}_0 \setminus \widehat{G}_0'$. So $\widehat{G}_0'(\widetilde{M}) < \widehat{G}_0(\widetilde{M})$, so, by (A8), $\mathrm{Tor}(\mathbb{G}') < \mathrm{Tor}(\mathbb{G})$, and so by divisibility $\widehat{G}_0'(\widetilde{N}) < \widehat{G}_0(\widetilde{N})$. Since $\mathrm{pr}(\widehat{G}') = \widehat{H} = \mathrm{pr}(\widehat{G})$, we have a corresponding strict inclusion $\widehat{G}_0'(0) < \widehat{G}_0(0)$ in $\widetilde{N}$ for the fibres above $0 \in \widehat{H}$. Hence by translating, we can find $\widetilde{n}'$ such that $(\widetilde{n}, \widetilde{n}') \notin \widehat{G}'$. Now $\widehat{G}_0(0)$ is not covered by any finitely many such $\widehat{G}_0'(0)$, since they are proper $\mathbb{Q}$-subspaces. So we can avoid any finitely many such proper subgroups simultaneously, and so, applying saturation, we find $\widetilde{n}'$ such that $(\widetilde{n}, \widetilde{n}')$ has group locus $\widehat{G}$.

Then, using (A3), it follows that $(\widetilde{m}, \widetilde{m}') \equiv_{qf} (\widetilde{n}, \widetilde{n}')$ as required. □

In Corollary 4.2.1 below we see that the analytic universal cover of an abelian variety, as discussed in the introduction, is, in the appropriate language, a model of $\widehat{T}$.

Definition 2.2. Let $\widetilde{A} \subseteq \widetilde{M} \models \widehat{T}$, and $\widetilde{a} \in \widetilde{M}$. Then $\mathrm{grploc}(\widetilde{a}/\widetilde{A})$, the *group locus* of $\widetilde{a}$ over $\widetilde{A}$, is the smallest coset definable over $\widetilde{A}$ containing $\widetilde{a}$.

Corollary 2.2.1. *$\widehat{T}$ is superstable of finite rank, and* $\mathrm{tp}(\widetilde{a}/\widetilde{B})$ *forks over* $\widetilde{A} \subseteq \widetilde{B}$ *iff* $\mathrm{tp}(a/\widehat{B})$ *forks over* $\widehat{A}$ *or* $\mathrm{grploc}(\widetilde{a}/\widetilde{B}) \neq \mathrm{grploc}(\widetilde{a}/\widetilde{A})$.

Proof. By the QE, $\mathrm{tp}(\widetilde{a}/\widetilde{A})$ is determined by $\mathrm{tp}(\widehat{a}/\widehat{A})$ and $\mathrm{grploc}(\widetilde{a}/\widetilde{A})$. For the former, there are only finitely many possibilities for $\mathrm{tp}(a_i/\widehat{A}a)$, and the latter is determined by a choice of coset over $\widetilde{A}$. So by ω-stability of T, if $|\widetilde{A}| = \lambda \geq 2^{|T|}$ then $|S(\widetilde{A})| \leq (\lambda 2^{\aleph_0})(|T|\lambda) = \lambda$. So $\widehat{T}$ is superstable.

Now suppose $\mathrm{tp}(\widetilde{a}/\widetilde{B})$ forks over $\widetilde{A}$; say $\phi(x, \widetilde{b}) \in \mathrm{tp}(\widetilde{a}/\widetilde{B})$ divides over $\widetilde{A}$. Let $\widehat{C} := \mathrm{grploc}(\widetilde{a}/\widetilde{B})$. WLOG, $\phi(x, \widetilde{b}) \models x \in \widehat{C}$.

Suppose $\widehat{C}$ is over $\widetilde{A}$. Then also $\phi(x,\widetilde{b}') \models x \in \widehat{C}$ for any $\widetilde{b}' \equiv_{\widetilde{A}} \widetilde{b}$. Now by the QE,

$$\phi(x) \leftarrow x \in (\widehat{C} \setminus \bigcup_i \widehat{C}'_i) \wedge \psi(\rho_n(x))$$

where $\widehat{C}'_i \not\leq \widehat{C}$ and $\psi(x)$ is a T-formula over $\widehat{B}$, WLOG implying $x \in \rho_n(\widehat{C} \setminus \bigcup_i \widehat{C}'_i)$. So since ϕ divides over $\widetilde{A}$, ψ must divide over $\widehat{A}$. So $\mathrm{tp}(a_n/\widehat{B})$ forks over $\widehat{A}$, and since a is algebraic over a_n, so does $\mathrm{tp}(a/\widehat{B})$.

For the converse: if $\widehat{C}$ is not over $\widetilde{A}$, then, since distinct conjugates of $\widehat{C}$ are cosets of the same subgroup and hence disjoint, $\widehat{C}$ forks over $\widetilde{A}$.

Finite rankedness of $\widehat{T}$ follows from finite rankedness of T and Noetherianity of the subgroups in T. □

Remark 2.2. As in the proof of Lemma 2.1, any definable finite group cover of $\mathbb{G}$ is dominated by some $[n]$.

Note that this argument uses divisibility - for example, the Artin-Schreier map $x \mapsto x^p - x$ is a finite definable group cover of the additive group in ACF_p which isn't handled by our setup (c.f. [BGH11] where this issue is discussed).

Remark 2.3. It follows from the QE that $\ker^0$ is indeed the connected component of the kernel in the model-theoretic sense, and more generally that $\widehat{H} + \ker^0$ is the connected component of $\rho^{-1}(H) = \widehat{H} + \ker$.

Lemma 2.3. *Let $\widetilde{A} \subseteq \widetilde{M} \models \widehat{T}$, and $\widetilde{a} \in \widetilde{M}$. Let $\widehat{C} := \mathrm{grploc}(\widetilde{a}/\widetilde{A})$.*

Suppose $\ker(\widetilde{M}) \subseteq \widetilde{A}$.

Then $p' := \mathrm{tp}(\widehat{a}/\widehat{A}) \cup \{\widetilde{a} \in \widehat{C}\} \models \mathrm{tp}(\widetilde{a}/\widetilde{A})$

Proof. By the QE, we need only see that if $\widetilde{b} \models p'$ in an elementary extension, then for all $\widehat{H}$ and all $\widetilde{c} \in \widetilde{A}$, $\widetilde{a} \in \widehat{H}(\widetilde{c})$ iff $\widetilde{b} \in \widehat{H}(\widetilde{c})$.

Now $\widetilde{a} \in \widehat{H}(\widetilde{c})$ iff $\widehat{C} \leq \widehat{H}(\widetilde{c})$, so the forward direction is clear.

For the converse, suppose $\widetilde{b} \in \widehat{H}(\widetilde{c})$. Then $b \in H(c)$, hence $a \in H(c)$. So $(\widetilde{a},\widetilde{c}) \in \widehat{H} + \ker(\widetilde{M})$, i.e. $\widetilde{a} \in \widehat{H}(\widetilde{c} + \zeta) + \xi$ for some $\zeta, \xi \in \ker(\widetilde{M})$. But $\ker(\widetilde{M}) \subseteq \widetilde{A}$, so $\widehat{C} \leq \widehat{H}(\widetilde{c} + \zeta) + \xi$. So $\widetilde{b} \in \widehat{H}(\widetilde{c}) \cap (\widehat{H}(\widetilde{c} + \zeta) + \xi)$; but this is an intersection of cosets of $\widehat{H}(0)$, so they are equal, and so $\widetilde{a} \in \widehat{H}(\widetilde{c})$. □

2.1 Further remarks on $\widehat{T}$

Proposition 2.4. *Let $\widetilde{\mathbb{M}} \models \widehat{T}$ be a monster model.*

(i) $\ker^0$ *is stably embedded, in the sense that every relatively definable set is relatively definable with parameters from* $\ker^0$*. Consider* $\ker^0(\widetilde{\mathbb{M}})$ *as a structure with the $\emptyset$-relatively-definable sets as predicates, and let $\widehat{T}^0 := \mathrm{Th}(\ker^0(\widetilde{\mathbb{M}}))$. Then $\widehat{T}^0$ is an ω-stable 1-based group of finite Morley rank bounded above by the Morley rank of T.*

In particular, $\ker^0$ *has finite relative Morley rank in the sense of [BBP09].*

(ii) Every type in $\widehat{T}^{\mathrm{eq}}$ is analysable in $\ker^0$ *and* $\mathrm{im}(\rho)$*.*

(iii) $\ker^0$ *is orthogonal to* $\mathrm{im}(\rho)$*.*

(iv) A regular type in $\widehat{T}^{\mathrm{eq}}$ is non-orthogonal to one of

(a) *a strongly minimal type in* T^{eq}*;*

(b) ${}^{\widehat{G}_0}/_{\widehat{H}_0}$ *for* $H \leq G$ *connected subgroups of* $\mathbb{G}$ *with no intermediate connected subgroup.*

Proof. (i) By the QE, the only structure on $\ker^0$ is the abelian structure given by the $\widehat{H}$. Stable embeddedness and 1-basedness follow easily.

Since $\ker^0$ is torsion-free and $\widehat{H} \cap \widehat{G} = (\widehat{H \cap G})^o$, the definable subgroups are precisely those of the form $\widehat{H}$. So there is is no infinite chain of definable subgroups of $\ker^0$, so $\widehat{T}^0$ is of finite Morley rank. The rank is bounded by the longest length of such a chain, which is bounded by the rank of T.

(ii) Consider a strong type $q = \mathrm{stp}(\widetilde{a}/\widetilde{A})$. If $\widetilde{b} \in \widetilde{\mathbb{M}}$ is an independent realisation of $q_1 = \mathrm{stp}(\widetilde{a}/a\widetilde{A})$, then since $\widehat{a} \subseteq \mathrm{acl}(a)$, we have $\widetilde{a} - \widetilde{b} \in \ker^0$. So q_1 is internal to $\ker^0$, and clearly $stp(a/\widetilde{A})$ is internal to T.

(iii) By the QE, there are no non-constant definable functions $(\ker^0)^n \to \mathrm{im}(\rho)$.

(iv) By (i), the types in (b) are minimal, and $\ker^0$ is analysed in them. So this follows from (ii).

□

3 Classification of models of $\widehat{T}$

In this section, we prove the main model-theoretic result of this paper, Theorem 3.11 below, which classifies the models of $\widehat{T}$. The classification proceeds as follows. First, recall the classification of models of T: by [Las85, Theorem 6], T is almost $\aleph_1$-categorical, so there is a finite set of mutually orthogonal strongly minimal sets D_i defined over the prime model such that any model M is primary (i.e. constructible) and minimal over M_0B where $M_0 \subseteq M$ is the prime model and B is the union of arbitrary bases for the $D_i(M)$, and conjugate strongly minimal sets have the same dimension [Bue96, 7.1]. So the models are determined by the dimensions of the strongly minimal sets.

We will show that this picture lifts to $\widehat{T}$. We will show that a model $\widetilde{M} \models \widehat{T}$ is primary and minimal over $\widetilde{M}_0B$ where $\widetilde{M}_0 = \rho^{-1}(M_0)$ and B is as above. So models of $\widehat{T}$ are determined by a choice of model of T and a choice of lift of the prime model M_0 (which in particular involves a choice of kernel).

3.1 Preliminaries

We make use of l-isolation, a technique due to Lachlan [Lac73].

Definition 3.1. A type p is l-isolated if for each ϕ there exists $\psi \in p$ such that $\psi \models p|\phi$. l-atomicity and l-primariness are then defined by analogy with the usual notions.

Lemma 3.1. *(a) Work in a monster model* $\mathcal{M}$ *of a complete stable theory* T'*.*

(i) l-primary models exist over arbitrary $A \subseteq \mathcal{M}^{\mathrm{eq}}$*.*

(ii) If $M \preceq \mathcal{M}$ and ϕ is a formula over M and $\phi(M) \subseteq A \subseteq \mathcal{M}^{\mathrm{eq}}$ and $\mathrm{dcl}^{\mathrm{eq}}(A) \cap \phi(\mathcal{M}) = \phi(M)$ and b is l-isolated over A and $\models \phi(b)$, then $b \in \phi(M)$.

(b) If $\widetilde{M} \models \widehat{T}$ and $N \models T$, $M \preceq N$, and $\widetilde{N}$ is l-primary over $A := \widetilde{M} \cup N$, then $\ker(\widetilde{N}) = \ker(\widetilde{M})$ and $\rho(\widetilde{N}) = N$, and so $\widetilde{N}$ is minimal over A.

Proof. (a) (i) [She90, IV.2.18(4), IV.3.1(5)]

(ii) If $b \notin \phi(M)$, then by l-isolation, there is a formula $\psi \in \mathrm{tp}(b/A)$ such that

$$\psi(x) \models \phi(x) \wedge x \notin \phi(M).$$

By stable embeddedness of ϕ, we may take ψ to be over $\mathrm{dcl}^{\mathrm{eq}}(A) \cap \phi(\mathcal{M}) \subseteq M$. But then ψ is realised in M, which is a contradiction.

(b) Follows from (a)(ii), since by the QE there are no non-constant definable functions from $\ker \rho$ to $\mathrm{im}\, \rho$ or vice-versa, and $\widehat{N}$ is a model of $\widehat{T}$ with $\rho(\widehat{N}) = N$.

Minimality is clear, since ρ is a homomorphism.

□

For getting primariness when the kernel is uncountable:

Lemma 3.2. *Suppose $\widetilde{M} \models \widehat{T}$ and $A \subseteq \widetilde{M}^{\mathrm{eq}}$ with $\ker(\widetilde{M}) \subseteq A$, suppose M is countable, and suppose $\widetilde{M}$ is (l-)atomic over A. Then $\widetilde{M}$ is (l-)primary over A.*

Proof. Take an arbitrary section of S of $\pi : \widetilde{M} \to M$. Then S is countable and (l-)atomic hence (l-)primary over A, and $\widetilde{M} = S + \ker$ is primary over S. □

3.2 ω-stability over models

From now on, in order to make the following lemma work, we make the following additional assumption.

Assumption: T is *rigid* - for $\mathbb{G}$ a saturated model of T, every connected definable subgroup of $\mathbb{G}^n$ is defined over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ (and hence has in the language of $\widehat{T}$ a predicate $\widehat{H}$ corresponding to it).

Lemma 3.3. *Suppose $M \models T$ and b is a tuple in an elementary extension, and let $M(b)$ be the prime model over Mb. Let $\widetilde{M} \models \widehat{T}$ over M, and suppose $\widetilde{M}$ has a kernel-preserving extension $\widetilde{M}(b) \models \widehat{T}$ over $M(b)$. Then $\widetilde{M}(b)$ is atomic over $\widetilde{M}b$. If M is countable, $\widetilde{M}(b)$ is primary over $\widetilde{M}b$.*

Furthermore, such an $\widetilde{M}(b)$ exists.

Proof. Existence is by Lemma 3.1(b).

Let $\widetilde{c} \in \widetilde{M}(b)$. Let $\widehat{H} + \widetilde{d} = \mathrm{grploc}(\widetilde{c}/\widetilde{M})$. Replacing $\widetilde{c}$ with $\widetilde{c} - \widetilde{d}$, we may assume $\widetilde{d} = 0$.

Since $\widetilde{M}$ contains $\ker(\widetilde{M}(b))$ and T is rigid, c is *free* in H over M, i.e. in no proper coset defined over M. By [BGH11, 6.4], for some n,

$$\mathrm{tp}(c_n/M) \cup \{\widetilde{c} \in \widehat{H}\} \cup \{c_i = \rho_i(\widetilde{c}) \mid i\} \models \mathrm{tp}(\widehat{c}/M).$$

Now by ω-stability, $\mathrm{tp}(b/Mc)$ has finite multiplicity, i.e. finitely many extensions to $\mathrm{acl}^{\mathrm{eq}}(Mc) \supseteq \widehat{c}$. Hence $\mathrm{tp}(\widehat{c}/M) \cup \mathrm{tp}(c/Mb)$ has only finitely many extensions to Mb. So again, for some n,

$$\mathrm{tp}(c_n/Mb) \cup \{\widetilde{c} \in \widehat{H}\} \cup \{c_i = \rho_i(\widetilde{c}) \mid i\} \models \mathrm{tp}(\widehat{c}/Mb).$$

So by Lemma 2.3,

$$\mathrm{tp}(c_n/Mb) \cup \{\widetilde{c} \in \widehat{H}\} \models \mathrm{tp}(\widetilde{c}/\widetilde{M}b).$$

But $c_n \in M(b)$, so $\mathrm{tp}(c_n/Mb)$ is isolated, so $\mathrm{tp}(\widetilde{c}/\widetilde{M}b)$ is isolated.

This proves atomicity. Primariness assuming countability follows by Lemma 3.2.

$\square$

Remark 3.1. Note that $\widetilde{M}(b)$ will **not** be primary, or prime, over $\widetilde{M} \cup M(b)$: indeed, if $\widetilde{a} \in \widetilde{M}(b) \setminus \widetilde{M}$, then each $a_n \in M(b) \setminus M$, so easily $\mathrm{tp}(\widetilde{a}/\widetilde{M} \cup M(b))$ is not isolated.

Remark 3.2. If we don't assume rigidity, there could be subgroups definable over $M(b)$ which aren't definable over M, which could cause a failure of atomicity.

Remark 3.3. Lemma 3.3 implies that we have ω-stability over models in the following sense: if $\widetilde{M} \models \widehat{T}$ is countable, then there are only countably many types over $\widetilde{M}$ realised in kernel-preserving extensions of $\widetilde{M}$. Indeed, by the Lemma any such type is isolated over $\widetilde{M}b$ for some b, and by ω-stability of T there are only countably many possible types $\mathrm{tp}(b/\widetilde{M}) \dashv \mathrm{tp}(b/M)$.

3.3 Independent systems

Countability of M was crucial to get primariness in Lemma 3.3. For such primariness of extensions in higher cardinals, we require primariness over independent systems of models. [She90, XII] and [Har87] are the sources for the techniques used here.

Definition 3.2. If I is a downward-closed set of sets, an *I-system* in a stable theory T is a collection $(M_s \mid s \in I)$ of elementary submodels of a model of T, such that for $s \subseteq t$, M_s is an elementary submodel of M_t. For $J \subseteq I$, define $M_J := \bigcup_{s \in J} M_s$.

Define $< s := P^-(s) := P(s) \setminus \{s\}$, and $\not\geq s := I \setminus \{t \mid t \supseteq s\}$.

The system is *(l-)primary* if M_s is (l-)primary over $M_{<s}$ for all $s \in I$ with $|s| > 1$.

The system is *independent* if $M_s \underset{M_{<s}}{\perp} M_{\not\geq s}$ for all $s \in I$.

I is *Noetherian* if each $s \in I$ is finite.

$|n| := \{0, ..., n-1\}$.

An *enumeration* of I is a sequence $(s_i)_{i \in \lambda}$ such that $I = \{s_i | i \in \lambda\}$ and $s_i \subseteq s_j \to i \leq j$. We write $s_{<i}$ for $\{s_j \mid j < i\}$.

Definition 3.3. Let M be a (possibly multi-sorted) structure. If $A \subseteq B \subseteq M$, we say A is *Tarski-Vaught* in B, $A \subseteq_{TV} B$, if every formula over A which is realised in B is realised in A.

Lemma 3.4. *Suppose $C \subseteq_{TV} B$.*

(i) If a type $\text{tp}(a/C)$ *is l-isolated, then* $\text{tp}(a/C) \models \text{tp}(a/B)$.

(ii) If A *is (l-)primary over* C *then* A *is (l-)primary over* B.

(iii) If A *is l-atomic over* C*, then* $A \underset{C}{\perp} B$.

Proof. (i) is straightforward, and (ii) and (iii) follow. □

The following is [She90, Lemma XII.2.3(2)], to which we refer for the proof.

Lemma 3.5 (TV Lemma). *If* $(M_s)_s$ *is an independent* I*-system in a stable theory, if* $J \subseteq I$*, and if* $\forall s \in I.(s \subseteq \bigcup J \to s \in J)$*, then* $M_J \subseteq_{TV} M_I$.

Lemma 3.6. *Let* $(M_s)_s$ *be an (l-)primary Noetherian independent* I*-system. Let* S *be the set of singletons* $S := \{\{i\} \mid i \in \bigcup I\}$*. Then* M_I *is (l-)primary over* $A := M_S$.

If moreover each $M_{\{i\}}$ *is (l-)primary over some* $M_\emptyset B_i$*, then* M_I *is (l-)primary over* $A' = M_\emptyset \bigcup_{i \in \bigcup I} B_i$.

Proof. Let $(s_i)_{i<\lambda}$ be an enumeration of I.

If $|s_i| > 1$, then M_{s_i} is (l-)primary over $M_{<s_i}$; but $M_{<s_i} \subseteq_{TV} M_{s_{<i} \cup S}$ by the TV Lemma, so M_{s_i} is (l-)primary over $M_{s_{<i} \cup S} = AM_{s_{<i}}$. This also holds when $|s_i| \leq 1$, trivially, so we conclude by induction on i.

For the moreover clause, the same proof works, using the given assumption when $|s_i| = 1$. □

Lemma 3.7. *Suppose* $\bigcup I$ *is finite.*

An l-primary I*-system is independent iff for each* $i \in \bigcup I$,

$$M_{\{i\}} \underset{M_\emptyset}{\perp} M_{\not\geq \{i\}}.$$

Proof. Suppose inductively that for any downward closed proper subset J of I, the restriction of the I-system to a J-system is independent.

So it suffices to show that for $s \in I$ maximal, $M_s \underset{M_{<s}}{\perp} M_{I \setminus \{s\}}$.

If $|s| = 1$, this holds by assumption.

If $|s| > 1$, then if $t \subseteq s$ and $t \in I \setminus \{s\}$ then $t \in < s$, so by the TV Lemma applied to the restricted independent $(I \setminus \{s\})$-system,

$$M_{<s} \subseteq_{TV} M_{I \setminus \{s\}}.$$

But M_s is l-atomic over $M_{<s}$, so we conclude the independence by Lemma 3.4(iii). □

Now let $M \models T$ extending the prime model M_0, and say B_i is an acl-basis for $D_i(M)$ over M_0. Let $B := \bigcup_i B_i$, and let $\mathcal{P}^{\text{fin}}(B)$ be the set of finite subsets of B. Let $M_\emptyset = M_0$, and for $s \in \mathcal{P}^{\text{fin}}(B)$ inductively let $M_s \subseteq M$ be a copy of the prime model over $M_{<s} \cup s$.

Lemma 3.8. $(M_s \mid s \in \mathcal{P}^{\text{fin}}(B))$ *is a primary independent system, and* $\bigcup_s M_s = M$.

Proof. $\bigcup_s M_s$ is an elementary submodel of M which contains M_0B, but M is minimal over M_0B, so $\bigcup_s M_s = M$.

For independence, by finite character of forking and Lemma 3.7 it suffices to see that $M_{\{b\}} \underset{M_0}{\perp} M_s$ when $b \notin s \in \mathcal{P}^{\text{fin}}(B)$.

We may assume inductively that the restriction of the system to s is independent. So by Lemma 3.6, M_s is primary over $M_0 s$.

Now $b \notin M_s$ since (by orthogonality of the D_i) $\text{tp}(b/M_0 s)$ is not algebraic and hence not isolated.

So $b \underset{M_0}{\perp} M_s$. So by the coheir property of forking independence, $bM_0 \subseteq_{TV} bM_s$. By atomicity, it follows (as in the proof of Lemma 3.7) that $M_{\{b\}} \underset{bM_0}{\perp} M_s$, and hence $M_{\{b\}} \underset{M_0}{\perp} M_s$ by symmetry and transitivity. □

Definition 3.4. An I-$\sim$-system is an I-system $(\widetilde{M_s})_s$ in $\widehat{T}$ such that

- setting $M_s := \rho(\widetilde{M_s}) \models T$, $(M_s)_s$ is an independent primary I-system in T
- $\widetilde{M_s} = \rho^{-1}(M_s)$ in $\widetilde{M_n}$ (i.e. the submodels contain all the kernel, i.e. the embeddings preserve the kernel)

Lemma 3.9. *An I-$\sim$-system $(\widetilde{M_s})_s$ is an independent l-primary I-system in $\widehat{T}$.*

Proof.

- Independence: We want to see $\widetilde{M_s} \underset{\widetilde{M}_{<s}}{\perp} \widetilde{M}_{\ngeq s}$ for all $s \in I$.

  If $\widetilde{a} \in \widetilde{M_s}$ and $\widehat{C} = \text{grploc}(\widetilde{a}/\widetilde{M}_{\ngeq s})$, then $\rho(\text{grploc}(\widetilde{a}/\widetilde{M}_{<s})) = \rho(\widehat{C})$ by the TV lemma applied to T, and so since there is no new kernel, $\text{grploc}(\widetilde{a}/\widetilde{M}_{<s}) = \widehat{C}$. So we conclude by Corollary 2.2.1.

- l-primariness: By Lemma 3.1(a), there is an l-primary model over $\widetilde{M}_{<s}$. But $\widetilde{M}_{<s}$ contains M_s and the kernel, so $\widetilde{M_s}$ is minimal over $\widetilde{M}_{<s}$.

□

The key point is that we can strengthen "l-primary" in the previous lemma to "primary":

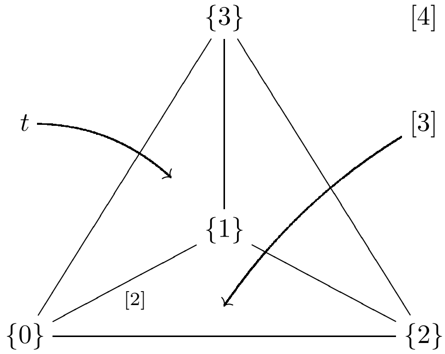
Proposition 3.10. *Let $(\widetilde{M_s})_s$ be an I-$\sim$-system with each M_s countable and I Noetherian. Then the system is primary.*

Proof. We may assume that $I = \mathcal{P}(|n|)$, $n > 1$, and assume inductively the result for $1 < n' < n$.

We show that $\widetilde{M}_{|n|}$ is atomic over $\widetilde{M}_{<|n|}$, hence primary by Lemma 3.2.

Claim 3.10.1. $(\widetilde{M_s})_s$ *extends to a $\mathcal{P}(|n+1|)$-$\sim$-system such that $\widetilde{M}_{|n|}$ is isomorphic over $\widetilde{M}_{|n-1|}$ to $\widetilde{M}_{|n-1|\cup\{n\}}$, say by an isomorphism σ.*

Proof. Let $t := |n-1| \cup \{n\}$.



We define an enumeration s_i of $\mathcal{P}(|n+1|)$, and define $\widetilde{M}_{s_i}$ such that

$$M_{s_i} \underset{M_{<s_i}}{\perp} M_{s_{<i}}$$

and M_{s_i} is primary over $M_{<s_i}$, and the embeddings preserve the kernel.

Begin with an enumeration of $\mathcal{P}(|n|)$; the corresponding $\widetilde{M}_{s_i}$ are already given.

Continue with an enumeration of $\mathcal{P}(t)$. Let $\widetilde{M}_t$ be an independent realisation of $\mathrm{tp}(\widetilde{M}_{[n]}/\widetilde{M}_{[n-1]})$, and for $n \in s_i \subseteq t$, let $\widetilde{M}_{s_i} \preceq \widetilde{M}_t$ be the image of $\widetilde{M}_{(s_i \setminus \{n\}) \cup \{n-1\}}$. The independence conditions follow from $M_t \underset{M_{[n-1]}}{\perp} M_{[n]}$.

For the remaining s_i: let M'_{s_i} be a primary model over $M_{<s_i}$, let $\widetilde{M}'_{s_i}$ be an l-primary model over $\widetilde{M}_{<s_i} M'_{s_i}$, and let $\widetilde{M}_{s_i}$ be a realisation of $\mathrm{tp}(\widetilde{M}'_{s_i}/\widetilde{M}_{<s_i})$ independent from $\widetilde{M}_{s_{<i}}$ over $\widetilde{M}_{<s_i}$.

The resulting system is a $\mathcal{P}(|n+1|)$-$\sim$-system (c.f. [ShCT Lemma XII.2.3(1)]). ☐

Define

$$\begin{aligned}
\widetilde{\Delta} &:= \widetilde{M}_{|n|} & \widetilde{\Delta}' &:= \widetilde{M}_{|n+1|} \\
\mathrm{d}_i \widetilde{\Delta} &:= \widetilde{M}_{|n| \setminus \{i-1\}} & \mathrm{d}_i \widetilde{\Delta}' &:= \widetilde{M}_{|n+1| \setminus \{i-1\}} \\
\mathrm{d} \widetilde{\Delta} &:= \bigcup_{1 \le i \le n} \mathrm{d}_i \widetilde{\Delta} & \mathrm{d} \widetilde{\Delta}' &:= \bigcup_{1 \le i \le n} \mathrm{d}_i \widetilde{\Delta}' \\
\widetilde{\Lambda} &:= \bigcup_{1 \le i < n} \mathrm{d}_i \widetilde{\Delta} & \widetilde{\Lambda}' &:= \bigcup_{1 \le i < n} \mathrm{d}_i \widetilde{\Delta}' \\
\mathrm{d}\,\mathrm{d}_i \widetilde{\Delta} &:= \bigcup_{j \in |n| \setminus \{i-1\}} \widetilde{M}_{|n| \setminus \{i-1, j\}}
\end{aligned}$$

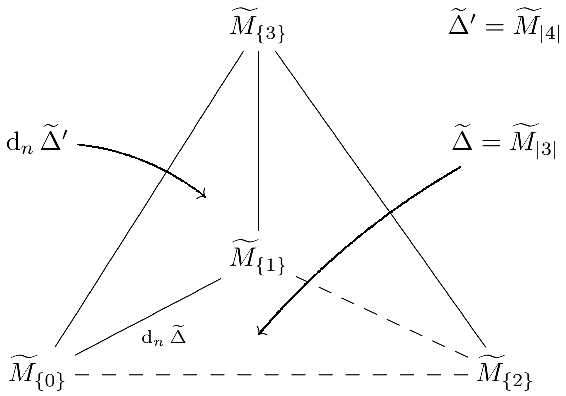
We also define the corresponding sets in T, e.g. $\Lambda := \rho(\Lambda) = \bigcup_{i<n-1} M_{|n| \setminus \{i\}}$. In this notation, the isomorphism of the previous claim is

$$\sigma : \widetilde{\Delta} \xrightarrow{\cong}_{\mathrm{d}_n \widetilde{\Delta}} \mathrm{d}_n \widetilde{\Delta}'.$$

Note that it induces an isomorphism

$$\sigma : \Delta \xrightarrow{\cong}_{\mathrm{d}_n \Delta} \mathrm{d}_n \Delta'.$$

A diagram for $n = 3$:



the dashed lines indicate $\widetilde{\Lambda}$, and the faces above them form $\widetilde{\Lambda}'$.

Let $\widetilde{a} \in \widetilde{\Delta}$ be a tuple; we want to show that $\mathrm{tp}(\widetilde{a}/\mathrm{d}\,\widetilde{\Delta})$ is isolated.

Claim 3.10.2. *There exists $b_0 \in \mathrm{d}_n \Delta$ such that, setting $A := \mathrm{acl}^{\mathrm{eq}}(\mathrm{d}\,\mathrm{d}_n \Delta b_0)$,*

$$\mathrm{tp}(\widehat{a}/A\Lambda) \models \mathrm{tp}(\widehat{a}/\mathrm{d}\,\Delta).$$

Proof. Let $b_0 \in \mathrm{d}_n \Delta$ such that $\mathrm{tp}(a/\mathrm{d}\,\Delta) \dashv \mathrm{tp}(a/b_0\Lambda)$. First note that any extension of $\mathrm{tp}(a_m/b_0\Lambda)$ to $\mathrm{d}\,\Delta$ doesn't fork. Indeed, that holds for $m = 1$ by isolation, and hence for any m by interalgebraicity of a_m with a. So it suffices to see that $\mathrm{tp}(a_m/A\Lambda)$ has a unique non-forking extension to $\mathrm{d}\,\Delta$. So suppose c_1, c_2 realise two such extensions. Then $\mathrm{d}_n \Delta \mathop{\smile\!\!\!\!\!\mid}_{A\Lambda} c_i$. Now $\mathrm{tp}(\mathrm{d}_n \Delta/A)$ is stationary, and since $\mathrm{d}_n \Delta$ doesn't fork from $A\Lambda$ over A, also $\mathrm{tp}(\mathrm{d}_n \Delta/A\Lambda)$ is stationary. So $c_1 \equiv_{\mathrm{d}\,\Delta} c_2$. ☐

Claim 3.10.3.

$$\mathrm{tp}(\widehat{a}/\sigma(\widehat{a})\Lambda' b_0) \models \mathrm{tp}(\widehat{a}/A\Lambda)$$

Proof. Say $\models \phi(a_n, b, e)$ where $b \in A$ and $e \in \Lambda$.

Say θ is an algebraic formula isolating $\mathrm{tp}(b/\mathrm{d}\,\mathrm{d}_n \Delta b_0)$.

Let

$$\psi(x) := \forall y \in \theta.(\phi(x, y, e) \leftrightarrow \phi(\sigma a_n, y, \sigma e)),$$

a formula in $\mathrm{tp}(a_n/\sigma a_n, \Lambda', b_0)$.

Then $\psi(x) \models \phi(x, b, e)$, since $\models \phi(\sigma a_n, b, \sigma e)$, since $b \in \mathrm{d}_n \Delta$ and $\sigma : \Delta \xrightarrow{\cong}_{\mathrm{d}_n \Delta} \mathrm{d}_n \Delta'$. ☐

Now $\mathrm{d}\,\widetilde{\Delta} \subseteq_{TV} \mathrm{d}\,\widetilde{\Delta}'$ by the TV lemma, and $\mathrm{tp}(\widetilde{a}/\mathrm{d}\,\widetilde{\Delta})$ is l-isolated, so $\mathrm{tp}(\widetilde{a}/\mathrm{d}\,\widetilde{\Delta}) \models \mathrm{tp}(\widetilde{a}/\mathrm{d}\,\widetilde{\Delta}')$.

Let $\widetilde{b}_0 \in \rho^{-1}(b_0) \subseteq \mathrm{d}_n \widetilde{\Delta}$, and let $\widetilde{b}_0 \subseteq \widetilde{b}_0' \in \mathrm{d}_n \widetilde{\Delta}$ be such that $\mathrm{grploc}(\widetilde{a}/\mathrm{d}\,\widetilde{\Delta})$ is over $\widetilde{b}_0'\widetilde{\Lambda}$. Then by Lemma 2.3 and the above Claims, we have:

$$\mathrm{tp}(\widetilde{a}/\mathrm{d}\,\widetilde{\Delta}) \dashv\vdash \mathrm{tp}(\widetilde{a}/\sigma(\widetilde{a})\widetilde{\Lambda}'\widetilde{b}_0').$$

So it suffices to see that the latter type is isolated. If $n = 2$, it follows from Lemma 3.3 that $\mathrm{tp}(\widetilde{a}\sigma(\widetilde{a})\widetilde{b}'_0/\widetilde{\Lambda}'b'_0)$ is isolated, since $\widetilde{\Lambda}' = \widetilde{M}_{\{1,2\}}$ and for an appropriate embedding of the prime model $M_{\{1,2\}}(b'_0)$ into Δ', we have $a, \sigma(a) \in M_{\{1,2\}}(b'_0)$. Meanwhile if $n > 2$, we have $\mathrm{tp}(\widetilde{a}\sigma(\widetilde{a})\widetilde{b}'_0/\widetilde{\Lambda}')$ is isolated by the inductive hypothesis applied to the $\mathcal{P}(|n-1|)$-$\sim$-system $(\widetilde{M}'_s)_s$ defined by $\widetilde{M}'_s := \widetilde{M}_{s\cup\{n-1,n\}}$, since $\widetilde{\Lambda}' = \widetilde{M}'_{<|n-1|}$ and $\widetilde{M}'_{|n-1|} = \widetilde{M}_{|n+1|} = \widetilde{\Delta}'$. □

3.4 Classification

Theorem 3.11 (Classification). *Let $M \models T$, and let $\widetilde{M_0} \models \widehat{T}$ with $M_0 \subseteq M$ prime over $\emptyset$. Let B_i be an* acl*-basis over M_0 for $D_i(M)$; let $B := \bigcup_i B_i$.*

Then there is $\widetilde{M} \models \widehat{T}$ primary over $B\widetilde{M_0}$, with $\ker(\widetilde{M_0}) = \ker(\widetilde{M})$.

Hence any model $\widetilde{M}$ of $\widehat{T}$ is primary and minimal over such a $B\widetilde{M_0}$, and so is determined up to isomorphism by the isomorphism types of $M = \rho(\widetilde{M}) \models T$ and $\widetilde{M_0} = \rho^{-1}(M_0)$ where $M_0 \subseteq M$ is prime.

Proof. Let $I := \mathcal{P}^{\mathrm{fin}}(B)$.

Let $(M_s \mid s \in I)$ be as given by Lemma 3.8.

Let $\widetilde{M}$ be an l-primary model over $M\widetilde{M_0}$, and let $\widetilde{M_s} = \rho^{-1}(M_s) \subseteq \widetilde{M}$, so $(\widetilde{M_s})_s$ is an I-$\sim$-system.

By Proposition 3.10, $(\widetilde{M_s})$ is a primary independent system, so, by Lemma 3.3 and Lemma 3.6, $\widetilde{M} = \widetilde{M_I}$ is primary over $\widetilde{M_0}B$.

A model $\widetilde{M} \models \widehat{T}$ is minimal over $B\rho^{-1}(M_0)$ for a corresponding B, since M is minimal over BM_0. The classification of models follows once we note that, by the quantifier elimination, $\mathrm{tp}(B\widetilde{M_0})$ is determined by $\mathrm{tp}(\widetilde{M_0})$ and $\mathrm{tp}(B/M_0)$. □

4 Abelian varieties over number fields

Suppose $\mathbb{G}(\mathbb{C})$ is a complex abelian variety, or the product $A \times \mathbb{G}_m^n$ of an abelian variety and a (split) torus. Suppose $\mathbb{G}$ and its endomorphisms are defined over $k_0 \leq \mathbb{C}$. Let T be the theory of $\mathbb{G}(\mathbb{C})$ in the language consisting of a predicate for each k_0-Zariski-closed subset of $\mathbb{G}^n(\mathbb{C})$. This is a commutative divisible rigid group of finite Morley rank. Every connected subgroup of $\mathbb{G}^n(\mathbb{C})$ is the connected component of the kernel of an endomorphism (see below), so is over k_0.

Now let

$$\rho : T_0(\mathbb{G}(\mathbb{C})) \twoheadrightarrow \mathbb{G}(\mathbb{C})$$

be the Lie exponential map. We can view this as a structure $U_{\mathbb{G}}$ in the language of $\widehat{T}$ by defining $\rho_n := \rho({}^{x}/_{n})$, and interpreting $\widehat{H}$ as the analytic connected component of $\rho^{-1}(H(\mathbb{C}))$ containing 0, which is a $\mathbb{C}$-subspace of $T_0(\mathbb{G}(\mathbb{C}))$ of the same dimension as H.

That $U_{\mathbb{G}} \models \widehat{T}$ can be seen quite directly by analytic means - see section 5.3. In this section we work algebraically.

Lemma 4.1. *(i) Any connected algebraic subgroup $H \leq \mathbb{G}^n$ is the connected component of the kernel of an endomorphism $\eta \in \mathrm{End}(\mathbb{G}^n) \cong \mathrm{Mat}_{n,n}(\mathcal{O})$, and* *(ii)* $\widehat{H}(U_{\mathbb{G}})$ *is then the kernel of the action of* η *on* $(T_0(\mathbb{G}(\mathbb{C})))^n$.

Proof. (i) By Poincaré's complete reducibility theorem, there exists an algebraic subgroup H' such that the summation map $\Sigma : H \times H' \to G$ is an isogeny. So say $\theta : G \to H \times H'$ is an isogeny such that $\theta\Sigma = [m]$, and let $\pi_2 : H \times H' \to H'$ be the projection. Then $\pi_2\theta\Sigma(h, h') = mh'$, so $\ker(\pi_2\theta)^o = \Sigma(H \times H'[m])^o = (H + H'[m])^o = H$.

(ii) η takes values in $\ker(\rho)^n$ on $\widehat{H}(U_{\mathbb{G}})$, so by connectedness and continuity it is zero. Conversely, $\rho(\ker(\eta))$ is a divisible subgroup of $\ker(\eta)$, and hence is contained in $\ker(\eta)^o = H(\mathbb{C})$. So $\ker(\eta)$ is a subgroup of $\rho^{-1}(H(\mathbb{C}))$ containing $\widehat{H}(U_{\mathbb{G}})$; but $\ker(\eta)$ is a $\mathbb{C}$-subspace so is connected, so $\ker(\eta) = \widehat{H}(U_{\mathbb{G}})$.

□

Note that (i) can fail for $\mathbb{G}$ a semiabelian variety.

Proposition 4.2. *If* K *is an algebraically closed field extension of* k_0*, any surjective* $\mathcal{O} := \mathrm{End}(\mathbb{G})$*-module homomorphism* $\rho : V \twoheadrightarrow \mathbb{G}(K)$ *from a divisible torsion-free* $\mathcal{O}$*-module* V *with finitely generated kernel is a model of* $\widehat{T}$*, where* $\widehat{H}$ *is interpreted as the kernel of the action of* η *on* V^n *if* H *is the connected component of the kernel of* $\eta \in \mathrm{End}(\mathbb{G}^n) \cong \mathrm{Mat}_{n,n}(\mathcal{O})$.

Proof. We check the axioms $\widehat{T}'$ above. All but those considered below are immediate from the definitions.

For (A6): we have $H \leq \mathbb{G}^n$. By working in $\mathbb{G}^n$, we may assume $n = 1$. We wish to show $\rho_k(\widehat{H}) = H$. It is not hard to see that $\eta(\mathrm{Tor}(\mathbb{G})) = \mathrm{Tor}(\mathrm{im}\,\eta)$, and it follows that $\eta(\Lambda_0) = \mathrm{im}\,\eta \cap \Lambda_0$ where Λ_0 is the divisible hull of $\Lambda := \ker\rho$; so since Λ is finitely generated as an abelian group, $\mathrm{im}\,\eta \cap \Lambda$ is of finite index in $\eta(\Lambda)$. By the snake lemma, it follows that $\rho(\widehat{H})$ is of finite index in $\ker(\eta)$, so by divisibility of $\widehat{H}$, we have $\rho_k(\widehat{H}) = \ker(\eta)^o = H$.

$$\begin{tikzcd}
 & & \Lambda \arrow[r] \arrow[d] & \Lambda \cap \mathrm{im}\,\eta \arrow[r] \arrow[d] & \cdots \\
 & \widehat{H} \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "\rho"'] & \mathrm{im}\,\eta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\eta \arrow[r] \arrow[d] & \mathbb{G} \arrow[r] \arrow[d] & \mathrm{im}\,\eta & \\
\cdots \arrow[r] & \text{Finite} \arrow[r] & 0 & &
\end{tikzcd}$$

For (A9): by (A6), $\rho(\mathrm{pr}(\widehat{G})) = \mathrm{pr}(G) = (\ker\eta')^o$ say, and also $\rho(\ker\eta') = (\ker\eta')^o$; so $\mathrm{pr}(\widehat{G}) + \Lambda = \ker\eta' + \Lambda$, so by divisibility of both sides and finite generation of Λ, we have $\mathrm{pr}(\widehat{G}) = \ker\eta' = \widehat{\mathrm{pr}(G)}$. □

Corollary 4.2.1. $U_{\mathbb{G}} \models \widehat{T}$.

Note that $x \mapsto (\rho(x/n))_n$ is an embedding of $U(\mathbb{G})$ into $\widehat{\mathbb{G}}$, which, by the QE, is elementary.

Suppose now that k_0 is a number field. So $\mathbb{G}$ is an abelian variety over a number field, or a torus, or the product of one with the other. In this case we may appeal to Kummer theory to reduce consideration of the prime model to consideration of the kernel. This is essentially the same argument as in [Gav07, Lemma 4].

Lemma 4.3. *Let $\mathbb{G} = A \times \mathbb{G}_m^r$ be the product of an abelian variety and a (split) torus.*

Suppose that A is defined over a number field k_0, and moreover that every endomorphism of A is also defined over k_0.

Note that $\mathbb{G}$ is rigid.

Let $T := \mathrm{Th}(\mathbb{G}(\mathbb{C}))$ in the language with a predicate for each subvariety defined over k_0 of a cartesian power of $\mathbb{G}$.

Then any $\widetilde{M_0} \models \widehat{T}$ over the prime model $M_0 = \mathbb{G}(\bar{\mathbb{Q}})$ is primary over ker.

Proof. By Lemma 3.2, it suffices to show atomicity. Let $\widetilde{c} \in \widetilde{M_0}$.

Translating, we may assume that $\mathrm{grploc}(\widetilde{c}/\ker) = \widehat{H}$, some H a semi-abelian subvariety of $\mathbb{G}$.

By Lemma 2.3, it suffices to isolate

$$\mathrm{tp}(\widehat{c}/\widehat{\ker}) = \mathrm{tp}(\widehat{c}/\mathrm{Tor}(\mathbb{G})).$$

This is equivalent to the existence of an n such that

$$\mathrm{tp}(c_n/\mathrm{Tor}(\mathbb{G})) \cup \{\widetilde{c} \in \widehat{H}\} \models \mathrm{tp}(\widehat{c}/\mathrm{Tor}(\mathbb{G})),$$

which in turn is equivalent to the image of $Gal(\bar{\mathbb{Q}}/k_0(\mathrm{Tor}(\mathbb{G}))$ under the map

$$\sigma \mapsto \widehat{c} - \sigma\widehat{c}$$

being a finite index subgroup of T_H.

For $G = A$ an abelian variety, this is [Ber10, Theorem 5.2]; see also references cited there.

For $G = A \times \mathbb{G}_m^r$, we argue that the proof of [Ber10, Theorem 5.2] goes through. This extension is addressed in [Rib79, Section 2], and the arguments there directly handle the mod l claim for sufficiently large primes l. It remains to show the l-adic result for all primes l. Define $\mathbb{Q}_l$-vector spaces:

$$\begin{aligned} G_l &:= T_l(G) \otimes \mathbb{Q} \\ A_l &:= T_l(A) \otimes \mathbb{Q} \\ \mu_l &:= T_l(G_m) \otimes \mathbb{Q} \end{aligned}$$

and

$$\Gamma_l := Gal(k_0(\mathrm{Tor}_{l^\infty}(G))/k_0);$$

we need to see

(B_1') $\mathrm{End}_{\Gamma_l} G_l \cong \mathrm{End}(G) \otimes \mathbb{Q}_l$

(B_2') G_l is semi-simple as a $\mathbb{Q}_l(\Gamma_l)$-module

(B_3') $H^1(\Gamma_l, G_l) = 0$;

the result then follows as in the l-adic part of [Ber10, Theorem 5.2].

(B_1')-(B_3') are known to hold for an abelian variety by Faltings and Bogomolov-Serre. That they follow for the product of an abelian variety with a torus seems also to be well-known, but we could not find an explicit reference - so we indicate how the arguments of [Rib79, Section 2] adapt to this l-adic case.

First note that by the existence of the Weil pairing and a polarisation,

$$\mathrm{Tor}_{l^\infty}(G) \subseteq G(k_0(\mathrm{Tor}_{l^\infty}(A))),$$

so

$$k_0(\mathrm{Tor}_{l^\infty}(G)) = k_0(\mathrm{Tor}_{l^\infty}(A)),$$

so Γ_l is also the corresponding Galois group for A, and (B_1')-(B_3') hold for A with this same Galois group Γ_l.

(B_2') follows, since the product of semi-simple modules is semi-simple.

Now

$$\mathrm{Hom}_{\Gamma_l}(\mu_l, A_l) = 0;$$

indeed, by (B_1') and (B_2') for A_l, the image of such a homomorphism must be of even $\mathbb{Q}_l$-dimension, so must be a point - and the only k_0-rational point of A_l is 0.

Via the l-adic Weil pairing and a polarisation, elements of

$$\mathrm{Hom}_{\Gamma_l}(A_l, \mu_l)$$

also correspond to k_0-rational points of A_l, and again 0 is the only such.

As in the proof of [Rib79, Theorem 2.4], (B_1') follows.

For (B_3'): by Bogomolov, some $\sigma \in G_l$ acts on A_l as some $\theta \in \mathbb{Q}_l^* \setminus \{1, -1\}$, and hence via the Weil pairing σ acts as $\theta^2 \neq 1$ on μ_l; we conclude by Sah's lemma. □

Remark 4.1. This can fail when $\mathbb{G}$ is a semi-abelian variety over a number field, due to the existence of *deficient points* - see [JR87].

So by Lemma 4.3 and Theorem 3.11, we conclude:

Conclusion 4.4. *Let T be as in the previous lemma. Then a model $\widetilde{M}$ of $\widehat{T}$ is determined up to isomorphism by the isomorphism types of $M = \rho(\widetilde{M}) \models T$ and* ker*, the latter equipped with all structure induced from $\widehat{T}$.*

Corollary 4.4.1 (Categoricity)**.** *The model $U_{\mathbb{G}} \models \widehat{T}$ of Corollary 4.2.1 is the unique structure $\widetilde{M}$ in the language of $\widehat{T}$ satisfying:*

(I) $\widehat{T}'$

(II) $|\widetilde{M}| = 2^{\aleph_0}$

(III) $\ker^{\widetilde{M}} \cong \ker^{U_{\mathbb{G}}}$*, isomorphism of substructures in the language of $\widehat{T}$.*

Moreover, for any such $\widetilde{M}$, the isomorphism of (III) extends to an isomorphism of $\widetilde{M}$ with $U_{\mathbb{G}}$.

Proof. This follows from the above conclusion on noting that T is bi-interpretable with ACF_0 (with k_0 as parameters), and hence is uncountably categorical.

We indicate explicitly how to construct the isomorphism for the moreover clause: first by Lemma 4.3 extend the isomorphism to $\widetilde{M_0}$, then extend to $B\widetilde{M_0}$ where B is an arbitrary transcendence basis for the field over M_0, and then by Theorem 3.11 to $\widetilde{M}$. □

We spell out how to deduce the version of this stated in the introduction:

Theorem 4.5 (Theorem 1.1)**.** *Suppose $\rho' : T_0(\mathbb{G}(\mathbb{C})) \twoheadrightarrow \mathbb{G}(\mathbb{C})$ is another surjective $\mathcal{O}$-module homomorphism,* $\ker\rho' = \ker\rho$*, and $\rho'\restriction_{\langle\ker\rho'\rangle_{\mathbb{Q}}} = \rho\restriction_{\langle\ker\rho\rangle_{\mathbb{Q}}}$.*

Then there exists an $\mathcal{O}$-module automorphism $\sigma \in$ $\mathrm{Aut}_{\mathcal{O}}(T_0(\mathbb{G}(\mathbb{C}))/\ker\rho)$ *and a field automorphism $\tau \in$* $\mathrm{Aut}(\mathbb{C}/k_0)$ *of $\mathbb{C}$ fixing k_0 such that $\tau\rho'\sigma = \rho$.*

Proof. By Corollary 4.2.1, the structure $\widetilde{M'}$ corresponding to ρ' is a model of $\widehat{T}$. By the QE and the assumption on the kernels, (III) in Corollary 4.4.1 holds, so $\widetilde{M'} \cong U_{\mathbb{G}}$. Since the graph of addition and each $\eta \in \mathcal{O}$ are interpretations of appropriate $\widehat{H}$, this isomorphism induces an $\mathcal{O}$-module automorphism of $T_0(\mathbb{G}(\mathbb{C}))$, and we have σ and τ as required. □

Understanding the structure of ker involves an understanding of the action of Galois on the torsion, which in general is well-known to be a hard problem. But let us highlight a strengthening of Conclusion 4.4 in the case of the characteristic 0 multiplicative group:

Theorem 4.6. *Let $\mathbb{G} = \mathbb{G}_m(\mathbb{C})$. Then a model $\widetilde{M}$ of $\widehat{T}$ is determined up to isomorphism by the transcendence degree of the algebraically closed field K such that $\rho(\widetilde{M}) \cong \mathbb{G}_m(K)$, and the isomorphism type of* $\ker\rho$ *as an abstract group.*

Proof. This is immediate from Conclusion 4.4 once we see that the isomorphism type of ker as a structure in the language of $\widehat{T}$ is determined by its isomorphism type as an abstract group. But this follows easily from the quantifier elimination and the fact from cyclotomic theory that any group automorphism of the roots of unity is a Galois automorphism. □

Remark 4.2. In the case of an elliptic curve $\mathbb{G} = E$ there are only finitely many kernels with underlying group $\langle \mathbb{Z}^2; + \rangle$ ([Gav07], [Bay09, Theorem 4.3.2]) - and only one if we extend k_0 by an appropriate root of unity.

See also [Gav06, IV.6.3,IV.7.4] for some discussion of the higher dimensional situation.

Remark 4.3. The assumption that k_0 is a number field was used in Lemma 4.3. It is natural to ask whether this is essential. Does an appropriate version of Kummer theory go through for Abelian varieties over function fields? We are unaware of this question being fully addressed in the literature, but [Ber10, Theorem 5.4] goes some way toward it.

5 Other cases

In this section, we make some brief remarks on some other natural examples of Theorem 3.11.

5.1 Positive characteristic

We can not in general expect to improve on Theorem 3.11 in positive characteristic: if $\mathbb{G}$ is the multiplicative group of a characteristic $p > 0$ algebraically closed field, then the prime model is $G(\mathbb{F}_p^{\mathrm{alg}})$, which is also the torsion group of $\mathbb{G}$. In this case, we recover the main theorem, 2.2, of [BZ10].

5.2 Manin kernels

In the theory DCF of differentially closed fields of characteristic 0, the Kolchin closures of the torsion of semiabelian varieties, also known as *Manin kernels*, are commutative divisible rigid groups of finite Morley rank. Our classification theorems therefore apply to this case. By considering a local analytic trivialisation, a natural analytic model of $\widehat{T}$ for $\mathbb{G}$ a (non-isoconstant) Manin kernel can be given; this will be addressed in future work.

5.3 Meromorphic Groups

Let $\mathbb{G}$ be a connected meromorphic group in the sense of [PS03], i.e. a connected definable group in the structure $\mathcal{A}$ of compact complex spaces definable over $\emptyset$ (equivalently, over $\mathbb{C}$). By [PS03, Fact 2.10], $\mathbb{G}$ can be uniquely identified with a complex Lie group.

Considering $\mathbb{G}$ with its induced structure, it is a finite Morley rank group. Suppose $\mathbb{G}$ is commutative and rigid. By the classification in [PS03] and the fact that any commutative complex linear algebraic group is a product of copies of $\mathbb{G}_m$ and $\mathbb{G}_a$, there is a definable exact sequence of Lie groups

$$0 \to \mathbb{G}_m^n \to \mathbb{G} \to H \to 0$$

where H is a complex torus. It is also shown in [PS03] that $\mathbb{G}$ is definable in a Kähler space; the latter may be considered in a countable language by [Moo05], so we may consider the language of $\mathbb{G}$ to be the induced countable language. Let $T = \mathrm{Th}(\mathbb{G})$.

In particular, in the case that $\mathbb{G}$ is a complex semiabelian variety, we may take the language to be that induced from the field, as in Corollary 4.2.1 above.

Now let $U_{\mathbb{G}}$ be the analytic universal cover of the Lie group $\mathbb{G}$ as a structure in the language of $\widehat{T}$, where we interpret ρ as the Lie exponential $T_0(\mathbb{G}) \twoheadrightarrow \mathbb{G}$, with $\rho_n(x) = \rho(x/n)$, and interpret $\widehat{H}$ for $H \leq G^n$ as the $\mathbb{C}$-subspace $T_0(H)$ of $T_0(\mathbb{G}^n) = T_0(G)^n$.

Proposition 5.1. *$U_{\mathbb{G}} \models \widehat{T}$.*

Proof. We show that $U_{\mathbb{G}}$ satisfies the axioms $\widehat{T}'$ above. These are immediate from the definitions and basic properties of Lie groups. For (A9), note that the connected component in the model theoretic sense is the connected component of the identity in the analytic sense. □

So by Theorem 3.11, $U_{\mathbb{G}}$ is the unique kernel-preserving extension of its restriction to the prime model $\mathbb{G}_0$ of $\mathbb{G}$, which is a countable structure.

Question 5.0.1. Could the Kummer theory of Lemma 4.3 apply here? Concretely: is $\pi^{-1}(\mathbb{G}_0)$ atomic over Λ?

References

[Bay09] Martin Bays. *Categoricity Results for Exponential Maps of 1-Dimensional Algebraic Groups & Schanuel Conjectures for Powers and the CIT*. PhD thesis, Oxford University, 2009. URL: `http://www.maths.ox.ac.uk/~bays/dist/thesis`.

[BBP09] Franck Benoist, Elisabeth Bouscaren, and Anand Pillay. Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences. 2009. arXiv:0904.2083v1 [math.LO].

[Ber10] Daniel Bertrand. Galois descent in galois theories. 2010. arxiv:1007.3047v1 [math.NT].

[BGH11] Martin Bays, Misha Gavrilovich, and Martin Hils. Some definability results in abstract kummer theory. 2011. arXiv:1112.6012v1 [math.LO].

[BHH+13] Martin Bays, Bradd Hart, Tapani Hyttinen, Meeri Kesälä, and Jonathan Kirby. Quasiminimal structures and excellence. *Bulletins of the London Mathematical Society*, 2013. Accepted.

[Bue96] Steven Buechler. *Essential stability theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1996.

[BZ10] Martin Bays and Boris Zilber. Covers of multiplicative groups of algebraically closed fields in arbitrary characteristic. 2010. To appear in the Bull. London Math. Soc.; arXiv:0704.3561 [math.LO].

[Gav06] Misha Gavrilovich. *Model theory of the universal covering spaces of complex algebraic varieties*. PhD thesis, Oxford University, 2006.

[Gav07] Misha Gavrilovich. A remark on transitivity of galois action on the set of uniquely divisible abelian extensions of $E(\bar{\mathbb{Q}})$ by $\mathbb{Z}^2$. Preprint, March 16, 2007, K-theory Preprint Archives, `http://www.math.uiuc.edu/K-theory/0827/`, 2007.

[Har87] Bradd Hart. An exposition of OTOP. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Math.*, pages 107–126. Springer, Berlin, 1987.

[JR87] Olivier Jacquinot and Kenneth A. Ribet. Deficient points on extensions of abelian varieties by $\mathbf{G}_m$. *J. Number Theory*, 25(2):133–151, 1987.

[Lac73] Alistair H. Lachlan. On the number of countable models of a countable superstable theory. In *Logic, methodology, philosophy of science 4*, pages 45–56. 1973.

[Las85] Daniel Lascar. Les groupes ω-stables de rang fini. *Trans. Amer. Math. Soc.*, 292(2):451–462, 1985.

[Moo05] Rahim N. Moosa. The model theory of compact complex spaces. In *Logic Colloquium '01*, volume 20 of *Lect. Notes Log.*, pages 317–349. Assoc. Symbol. Logic, Urbana, IL, 2005.

[PS03] Anand Pillay and Thomas Scanlon. Meromorphic groups. *Trans. Amer. Math. Soc.*, 355(10):3843–3859, 2003.

[Rib79] Kenneth A. Ribet. Kummer theory on extensions of abelian varieties by tori. *Duke Math. J.*, 46(4):745–761, 1979.

[She90] S. Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1990.

[Zil02] B. Zilber. Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety. In *Model theory and applications*, volume 11 of *Quad. Mat.*, pages 427–458. Aracne, Rome, 2002.