Lecture notes - Applied Stability Theory (Math 414) Autumn 2003.

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1 Differential fields: first properties.

All the rings we deal with are assumed to be commutative, with a unit, and contain $\mathbf{Q}$.

Definition 1.1 *By a derivation on a ring R we mean a map $\partial : R \to R$ such that for all $a, b \in R$*
(i) $\partial(a+b) = \partial(a) + \partial(b)$, and
(ii) $\partial(ab) = \partial(a)b + a\partial(b)$

Sometimes the derivation is denoted by $a \to a'$.

A ring equipped with a derivation is called a differential ring. The notions, differential subring, differential ring extension and homomorphism between differential rings, are clear.

If (R, ∂) is a differential ring then the set of constants is by definition $\{r \in R : \partial(r) = 0\}$, and is easily seen to be a (differential) subring. ∂^k means the k-fold iteration of ∂ and ∂^0 is taken to be the identity mapping.

The basic examples of differential rings are rings (or fields) of functions: for example
(i) $\mathbf{C}(z)$ the field of rational functions over $\mathbf{C}$ in the single indeterminate z, with $\partial(f) = df/dz$,
(ii) $\mathbf{C}(z, e^z)$ with ∂ as in (i). (Note that this field IS closed under ∂.)

Exercise 1.2 *(i) Let (R, ∂) be a differential ring. Let $P(x_1, , , x_n)$ be a polynomial over R. Let P^∂ denote the polynomial over R obtained from P by*

applyng ∂ to the coefficients. Let $a_1,..,a_n \in R$. Show that $\partial(P(a_1,..,a_n)) = P^{\partial}(a_1,..,a_n) + \sum_{i=1,..,n}((\partial P/\partial x_i)(a))\partial a_i$.
(iii) Let (R,∂) be a differential ring, S a differential subring of S and A a subset of R. Then the differential subring of R generated by S and A is precisely the subring of R generated by S and $\{\partial^k(a) : a \in A, k \geq 0\}$.

Lemma 1.3 *(i) If (R,∂) is a differential ring and R is an integral domain, then there is a unique extension of ∂ to a derivation on the quotient field of R.*
(ii) If (K,∂) is a differential field, then ∂ has a unique extension to a derivation on the algebraic closure of K.
(ii) Suppose that (K,∂) is a differential field, $K < L$, and $a_1,..,a_n \in L$ are such that $a_1,..,a_{n-1}$ are algebraically independent over K, but that $P(a_1,..,a_n) = 0$ for some (nonzero) polynomial $P(x_1,..,x_n)$ over K. Then there is a unique extension of ∂ to a derivation ∂^* on $K(a_1,..,a_n)$ (the field generated by K together with $a_1,...,a_n$) such that $\partial(a_i) = a_{i+1}$ for $i = 1,..,n-1$.

Proof. (i) is left as an exercise. (ii) follows by iterating the special case of (iii) when $n = 1$.
(iii). We may assume that the polynomial $P(x_1,..,x_n)$ is irreducible over K, namely is not the product of two polynomials over K. We will be using the elementary algebraic fact that (under our assumptions on $a = (a_1,...,a_n)$), P divides every polynomial $Q(x_1,..,x_n)$ which vanishes on a. Note that it then follows that $\partial P/\partial x_n$ does not vanish on a. We will choose a value for $\partial(a_n)$ (which it will be forced to have by Exercise 1.2): namely put $a_{n+1} = -(P^{\partial}(a)\sum_{i=1,..n-1}((\partial P/\partial x_i)(a)a_{i+1})/(\partial P/\partial x_n)(a))$.

Now define ∂^* on $K[a]$ by:
$\partial^*(g(a)) = g^{\partial}(a)+\sum_{i=1,..,n}((\partial g/\partial x_i)(a))a_{i+1}$, for each polynomial $g(x_1,..,x_n)$ over K.

It is immediate that $\partial^*(gh(a)) = \partial^*(g(a))h(a) + g(a)\partial^*(h(a))$, and ∂^* agrees with ∂ on K. So all that has to be checked is that ∂^* is well-defined: namely if $g(a) = h(a)$ then $\partial^*(g(a)) = \partial^*(h(a)$. This is clearly equivalent to showing that
(*) $\partial^*(g(a)) = 0$ if $g(a) = 0$.
Now (*) is true for the case $g = P$ (by choice of a_{n+1}). If $g(a) = 0$ then as we noted earlier $g = P \cdot r$ for some polynomial r over K. But then $\partial^*(g(a)) = \partial^*(P(a))r(a) + P(a)\partial^*(r(a)) = 0$.

So ∂^* is a derivation on $K[a]$ extending ∂ and now use part (i) to extend to $K(a)$.

The above lemma will be used later to show that the theory of differential fields (of characteristic zero) has a model companion.

The language of differential rings is the language of rings $\{+, \cdot, -, 0, 1\}$ together with the unary operation symbol ∂. We sometimes call this language L_∂. In any case any differential ring is naturally an L_∂-structure. By DF_0 we mean the theory of differential fields (of characteristic zero), namely the axioms for fields of characteristic zero as well as the axioms for the derivation.

If $x = (x_1, .., x_n)$ is a sequence of variables, then $\partial^k(x)$ denotes $(\partial^k(x_1), .., \partial^k(x_n))$.

Exercise 1.4 *Let $\theta(x_1, .., x_n)$ be a quantifier-free formula of L_∂. Then modulo DF_0, $\theta(x)$ is equivalent to a finite Boolean combination of formulas of the form $P(x, \partial(x), .., \partial^k(x)) = 0$, where P is a polynomial with coefficients from $\mathbf{Z}$.*

We have seen in [2] the notion of the quantifier-free type of a over A in a structure M: it is just the set of quantifier-free formulas with parameters from A which are true in M. For the moment, if (K, ∂) is a differential field, then by a complete quantifier-free n-type over K we mean $qftp_F(a_1, .., a_n/K)$, where (F, ∂) is a differential field extension (not necessarily an elementary extension) of (K, ∂), and $a_1, .., a_n \in F$. (Note this is a notion belonging entirely to algebra, in spite of the "logical" notation.)

Lemma 1.5 *Suppose (K, ∂) is a countable differential field. Then there are only countably many complete quantifier-free n-types over K.*

Proof. It is enough to prove this for $n = 1$ (why??). This is the actual content of the discussion in Example 2.54 of [3]. But we will repeat the argument. So let $a \in F$ where F is a differential field extension of K. We call a differentially transcendental over K, if $\{a, \partial(a), .., \partial^k(a), ...\}$ is algebraically independent over K, namely for NO polynomial $P(x_1, x_2, ...)$ over K is $P(a, \partial(a), \partial^2(a), ...) = 0$. By Exercise 1.4, there is a unique quantifier-free type over K of a differential transcendental (over K) element.

So we will assume that a is NOT differentially transcendental over K. In this case:

Claim. Let n be least such that $(a, \partial(a), .., \partial^n(a))$ is algebraically dependent over K. Then $qftp(a/K)$ is determined by the quantifier-free type of $(a, \partial(a), .., \partial^n(a))$ over K in the language of rings.
Proof. Let $P(x_1, .., x_n)$ be an irreducible polynomial over K which vanishes on $(a, \partial(a), .., \partial^n(a))$. Exercise 1.2 gives a formula for $\partial^{n+1}(a)$ as $s_1(a, \partial(a), .., \partial^n(a))$, where $s_1(x_1, ...x_n)$ is a rational function over K depending only on P. We easily find rational functions $s_k(x_1, .., x_n)$ for all $k \geq 1$ (depending only on P) such that $\partial^{n+k}(a) = s_k(a, \partial(a), .., \partial^k(a))$. Now let L be another differential field extension of K and $b \in L$ such that $(b, \partial(b), .., \partial^n(b))$ has the ame quantifier-free type over K in the language of rings as $(a, \partial(a), .., \partial^n(a))$. So again $\partial^{n+k}(b) = s_k(b, \partial(b), .., \partial^n(b))$ for all k. It follows that for all k, $(a, \partial(a), ..., \partial^k(a))$ and $(b, \partial(b), .., \partial^k(b))$ have the same quantifier-free type over K in the language of rings. Thus (by 1.4 for example), a and b have the same quantifier-free type over K.

The claim is proved.

But there are only countably many complete quantifier-free finitary types over K in the language of rings. (Use the fact that ACF_0 has quantifier-elimination and is the model companion of the theory of fields of char. 0, so every such type is a complete finitary type over K in the sense of ACF_0, but ACF_0 is ω-stable.)

Let us make explicit an important observation implicit in the previous proof.

Remark 1.6 *Suppose that $k < F$ are differential fields, $a \in F$, $a, \partial(a), .., \partial^{n-1}(a)$ are algebraically independent over k (in the sense of fields) but $\partial^n(a)$ is algebraic over $k(a, .., \partial^{n-1}(a))$, of degree m say. Let $P(x_0, .., x_{n-1}, x_n)$ be a polynomial over k of degree m in x_n such that $P(a, \partial(a), .., \partial^n(a)) = 0$. Then $qftp(a/k)$ is determined by the formulas $\{P(x, \partial(x), .., \partial^n(x)) = 0\} \cup \{Q(x, \partial(x), .., \partial^{n-1}(x)) \neq O : Q(x_0, .., x_{n-1}) \in k[x_0, .., x_{n-1}]\}$.*

Corollary 1.7 *Any complete theory T (in L_∂) of differential fields which has quantifier-elimination is ω-stable.*

Proof. Let K be a countable model of T. Any complete 1-type over K (realized in an elementary extension) is by quantifier elimination, determined by the st of quantifier-free formulas in it. But the latter is clearly a complete quantifier-free type over K in the sense above. Thus there are only countably many complete 1-types over K, so T is ω-stable.

Proposition 1.8 *(i) DF_0 has a model companion DCF_0 (the theory of differentially closed fields).*
(ii) DCF_0 can be axiomatized by DF_0 together with the sentences " for all $d_1,..,d_k,e_1,..,e_{k'}$ there is c such that $P(c,\partial(c),..,\partial^n(c),d_1,..,d_k) = 0$ and $Q(c,\partial(c),..\partial^m(c),e_1,..,e_{k'}) \neq 0$", whenever $P(x_1,..,x_n,y_1,..,y_k)$ and $Q(x_1,..,x_m,z_1,..,z_{k'})$ are polynomials over $\mathbf{Z}$, $m < n$ (or $n = 0$ and Q is omitted) and x_n really appears in P.
(iii) DCF_0 is complete and has quantifier elimination.
(iv) DCF_0 is ω-stable.

Proof. We will prove that DCF_0 as axiomatized in (ii) is complete with quantifier elimination, and that moreover every differential field embeds in a model of DCF_0. By 2.39, 2.41 and 2.44 of [3] this will prove (i), (ii) and (iii). (iv) will then follow from (iii) and 1.6.

First we show that any differential field (K,∂) embeds in a model of DCF_0. Let $P(x_1,..,x_n)$ and $Q(x_1,..,x_m)$ (with $m < n$) be polynomials over K, such that x_n appears in P. Let $a_1,..,a_{n-1}$ be elements of some field extension L of K, which are algebraically independent over K. Let a_n be a solution of $P(a_1..,a_{n-1},x) = 0$ (in an algebraically closure of L), and let F be the field generated by K and $\{a_1,..,a_n\}$. By Lemma 1.3(iii) there is a derivation ∂^* on F extending ∂ such that $\partial^*(a_i) = a_{i+1}$ for $i = 1,..,n-1$. So $P(a_1,\partial(a_1,..,\partial(a_n)) = 0$ and $Q(a,\partial(a),.,\partial^m(a)) \neq 0$ for $m < n$.

So we have solved one instance of the axiom system in an extension of K. It easilly follows that K embeds in a model of DCF_0.

For completeness and quantifier-elimination, we will prove that for any two saturated models K_1,K_2 of DCF_0 the system of finite partial isomorphisms between K_1 and K_2 is nonempty and has the back-and-forth property (and apply 2.29 and 2.30 of [2]).

Both K_1 and K_2 contain the ring $\mathbf{Z}$ (with trivial derivation).

Supose a,b are n-tuples from K_1,K_2 with the same quantifier-free type. The map f taking a to b then extends to an isomorphism f between the differential fields k_1 and k_2 generated by a,b respectively. Let $c \in K$. If c is differentially transcendental over k_1 then (by saturation of L and the axioms) we can find $d \in K_2$ differentially transcendental over k_2 and (a,c) and (b,d) have the same quantifier-free types.

Otherwise let m be least such that $\{c,\partial(c),..,\partial^m(c)\}$ is algebraically dependent over k_1, and let $P(x_1,..,x_n)$ be irreducible such that $P(c,..,\partial^m(c)) =$

0. Then $f(P)$ is a polyomial over k_2. The axioms, together with saturation, yield some $d \in K_2$ such that $d, \partial(d), ..\partial^{m-1}(d)$ are algebraically independent over k_2, but $f(P)(d, \partial(d), .., \partial^m(d)) = 0$. As in the proof of Lemma 1.5, the quantifier-free type of d over k_2 is precisely the image of the quantifier-free type of c over k_1 under f. Thus (a, c) and (b, d) have the same quantifier-free types.

A model of DCF_0 is called a differentially closed field. ¿From now on we fix a large saturated model $(\mathcal{U}, +, -, \cdot, 0, 1, \partial)$ of DCF_0, in which we work. $k, F, ..$ denote (small) differential subfields of $\mathcal{U}$ and $a, b, ..$ usually denote finite tuples from $\mathcal{U}$. $\mathcal{C}$ denotes the field of constants of $\mathcal{U}$ and for any K, C_K denotes the field of constants of K, which note contains $\mathbf{Q}$. We let $\overline{K}$ denote the field-theoretic algebraic closure of K.

Exercise 1.9 *DF_0 has the amalgamation property. Namely if $F < K$ and $F < L$ are 2 extensions of differential fields, then K and L embed over F into some differential field.*

Hint. Use quantifier elimination and the fact that any (small) differential field embeds in $\mathcal{U}$ as well as homogeneity of $|calU$.

Remark 1.10 *(i) $\mathcal{U}$ is an algebraically closed field.*
(i) For any k, the (field-theoretic) algebraic closure of $\overline{C}_k$ is contained in $\mathcal{C}$

Proof. The case of the axioms when $n = 0$ yields that $\mathcal{U}$ is algebraically closed.
Now suppose that the element a is in $\overline{C}_k$. Let $P(x)$ be the minimal polynomial of a over C_k. Then $P^\partial = 0$, and $(dP/dx)(a) \neq 0$. By 1.2, $\partial(a) = 0$.

Remark 1.11 *For any n-tuples a, b, $tp(a/k) = tp(b/k)$ iff for all m $tp_{L_r}(a, \partial(a), .., \partial^m(a)/k) = tp_{L_r}(b, \partial(b), .., \partial^m(b)/k)$.*

Proof. By quantifier elimination and 1.4.

For any K we let $\hat{K}$ denote some copy in $\mathcal{U}$ of the prime model over K. $\hat{K}$ is called the differential closure of K. By a general fact about ω-stable theories, its isomorphism type over K is unique.

Lemma 1.12 *For any F, $C_{\hat{F}} = \overline{C}_F$.*

Proof. Let a be an element of $C_{\hat{F}}$, so its type over F is isolated. If $a \notin \overline{F}$ then by 1.4, $tp(a/k)$ is isolated by a formula of the form $x' = 0 \wedge \wedge_i P_i(x) \neq 0$, where the $P_i(x)$ are polynomials over F, but such a formula is realized in C_F as the latter is infinite. Thus $a \in \overline{F}$. Let $P(x)$ be its minimal poynomial over F. Assume P monic. By 1.2, $P^\partial(a) = 0$. But P^∂ has lower degree than P (why?), hence $P^\partial = 0$ and so all coefficients of P are constants. So $a \in \overline{C}_F$.

Recall that from Example 3.7 of [3], any definable subset of $\mathcal{C}^n$ which is L_∂ definable in $\mathcal{U}$ is definable in $(\mathcal{C}, +, \cdot)$. Hence $\mathcal{C}$ is strongly minimal as a definable set in $\mathcal{U}$.

Lemma 1.13 *For any set $A \subseteq \mathcal{U}$, the algebraic closure of A in the sense of the L_∂-structure $\mathcal{U}$ is precisely $\overline{k}$ where k is the differential field generated by A.*

Proof. We may assume $A = k$ is small. Suppose $a \notin \overline{k}$.

If a is differentially transcendental over a, then $tp(a/k)$ is determined by $P(a, \partial(a), .., \partial^n(a)) \neq 0$ for all n. By compactness and the axioms, for any $a_1, .., a_m$ we can find a realization b of $tp(a/k)$ such that $b \neq a_i$ for all i. But then $a \notin acl(k)$.

On the other hand if a is not differentially transcendental over k, then $tp(a/k)$ is determined by: $P(a, \partial(a), ..., \partial^n(a)) = 0$ and "$a, \partial(a), .., \partial^{n-1}(a)$ are algebraically independent over k", for some polynomial P over k (where $n \geq 1$). As above we find infinitely many solutions of $tp(a/k)$. So $a \notin acl(k)$.

Exercise 1.14 *For any $A \subset \mathcal{U}$, the definable closure of A in the sense of the L_∂-structure $\mathcal{U}$ is precisely the differential subfield generated by A.*

We now want to characterise independence (nonforking) in $\mathcal{U}$. We will make use of independence in ACF_0. Recall that an algebraically closed field as an L_r-structure is strongly minimal and algebraic closure in the model-theoretic sense equals field-theoretic algebraic closure.. Thus in a model of ACF_0, if a is a possibly infinite tuple, and $F < K$ are fields, $tp(a/K)$ does not fork over F if fr all finite subtuples b of a, $tr.deg(F(b)/F) = tr.deg(K(b)/K)$.

Exercise 1.15 *Suppose $F < K$ are differential subfields of $\mathcal{U}$ with F algebraically closed. Then for any differential field extension $F < L$ there is an isomorphic copy L_1 of L over F, with $L_1 < \mathcal{U}$ and K independent from L_1 in the sense of ACF.*

Proposition 1.16 *For any finite tuple a, and differential fields $F < K$,*
(*) $tp(a/K)$ does not fork over F if and only if
(**) $(a, \partial(a), .., \partial^n(a), ...)$ is independent from K over F in the sense of ACF.

Proof. We may assume that both F and K are algebraically closed (in any sense).
Claim I. Suppose that K is a model (of DCF_0). Then (**) implies (*).
Proof. By (**), for each n, $tp_{L_r}(a, \partial(a), ., \partial^n(a)/K)$ is definable over F. By 1.4 and QE it follows that $tp_{L_\partial}(a/K)$ is definable over F, so doesn't fork over F.

Claim II. Suppose that F is a model. Then (*) implies (**).
Proof. By 2.50 of [3], $tp(a/K)$ is finitely satisfiable in F. In particular $tp_{L_r}(a, \partial(a), .., \partial^n(a)/K)$ is finitely satisfiable in F for all n. Again by 2.50 cited above, we get (**).

Now we prove the Proposition for general $F < K$ (algebraically closed differential fields). Suppose first that (*) holds. Let $L > F$ be a model (of DCF_0) such that L is independent from $K < a >$ over F in the sense of ACF (by 1.15). So $(a, \partial(a), ...)$ is independent from L over F in the ACF sense. $K < a >$ is independent from L over F in the differential sense, by Claim I. Thus (by forking calculus) a is independent from K_1 over L in the differential sense. where K_1 is the (differential) field generated by K and L. By Claim II. $(a, \partial(a), ...)$ is independent from K_1 over L in the ACF sense. We conclude that $(a, \partial(a), ...)$ is independent from K_1 over F in the ACF sense, which yields (**).

Conversely, suppose (**) holds. Let $L > K$ be a model such that $(a, \partial(a), ...)$ is independent from L over K in the ACF sense. So $(a, \partial(a), ..)$ is independent from L over F in the ACF sense. By Claim I, a is independent from L over F in the differential sense, yielding (*).

Definition 1.17 *For A a single element in $\mathcal{U}$, define $ord(a/k)$ to be the transcendence degree of $k(a, \partial(a), .., ..)$ over k.*

Corollary 1.18 *(i) For a a single element, and $F < K$, $tp(a/K)$ forks over F if and only if $ord(a/K) < ord(a/F)$.*
(ii) For a a single element, $U(a/k) \leq ord(a/k)$

Lemma 1.19 *For a single element a, $RM(tp(a/k)) \leq ord(a/k)$.*

Proof. It is enough to prove this for complete 1-types over $\mathcal{U}$ (by taking nonforking extensions and using Corollary 1.18 (i)). So let $p(x)$ be such a global 1-type. We prove by induction on $n < \omega$, that $ord(p) \leq n$ implies $RM(p) \leq n$. This is OK for $n = 0$ (as then $p(x)$ is algebraic). Assume $ord(p) \leq n+1$. By induction we may assume $ord(p) = n+1$. Also by induction any formula of the form $Q(x, \partial(x), .., \partial^n(x)) = 0$ ($Q(x_0, .., x_n)$ a polynomial over $\mathcal{U}$) has Morley rank $\leq n$. By Remark 1.6 (and QE) p is determined by a single formula ($P(x, \partial(x), .., \partial^{n+1}(x)) = 0$ for suitable P) together with a collection of negations of formulas of Morley rank $\leq n$. It follows that $RM(p) \leq n+1$ (why?).

We now want to find definable subsets of $\mathcal{U}$ of arbitrarily large (finite) Morley rank.

Lemma 1.20 *For any $n \geq 1$, the formula $\partial^n(x) = 0$ has Morley rank n and Morley degree 1. Moreover the set defined by this formula is a subgroup of $(\mathcal{U}, +)$ and is an n-dimensional $\mathcal{C}$-vector space.*

Proof. Let us introduce some notation. Let V_n be the subset of $\mathcal{U}$ defined by $\partial^n(x) = 0$.
Claim I. V_n is a subgroup of $(\mathcal{U}, +)$, and for any c the solution set of $\partial^n(x) = c$ is an additive translate of V_n.
Proof. Clear. Note that $\partial^n(x) = c$ DOES have some solution, by the axioms.

Claim II. V_n is a $\mathcal{C}$-vector space.
Proof. If $c \in \mathcal{C}$ and $v \in \mathcal{U}$, then by induction, $\partial^i(cv) = c \cdot \partial^i(v)$ for all i. Hence if $v \in V_n$ then $\partial^n(cv) = 0$ and so $cv \in V_n$.

Claim III. $RM(V_n) \geq n$ for all n.
Proof. By induction on n. $V_1 = \mathcal{C}$ which we already know to be strongly minimal. Now consider V_{n+1}. The map ∂^n takes V_{n+1} onto $\mathcal{C}$ (using Claim I). By Claim I and induction, each fibre (defined by $\partial^n(x) = c$ for some $c \in \mathcal{C}$) has Morley rank $\geq n$. So we have partitioned V_{n+1} into infinitely many pairwise disjoint definable sets of Morley rank $\geq n$, so $RM(V_n) \geq n+1$.

By Lemma 1.19, $RM(V_n) \leq n$ for all n. It follows together with Claim III, that
Claim IV. $RM(V_n) = n$.

Now if V is a definable $\mathcal{C}$ vector space of dimension r, then (by choosing a basis for V), V is in definable bijection with $\mathcal{C}^r$, hence has Morley rank r and Morley degree 1 (as $\mathcal{C}$ is strongly minimal). So by Claim II, and Claim IV, V_n is an n-dimensional vector space over $\mathcal{C}$ so also has Morley degree 1. The Lemma is proved.

Note that for any F the type over F of an element a which is differentially transcendental over F is unique. Let us call this type $p_{F,1}(x)$. Note that if $F < K$ then $p_{K,1}(x)$ is the unique nonforking extension of $p_{F,1}(x)$ over K. In particular these types are stationary.

Corollary 1.21 *(i)* $x = x$ *has Morley rank* ω *and Morley degree* 1.
(ii) For any F *the unique type of Morley rank* ω *over* F *is precisely* $p_{F,1}(x)$.

Proof. By Lemma 1.20, $RM(x = x) \geq \omega$. On the other hand, by quantifier-elimination, and 1.19, for any definable subset X of $\mathcal{U}$, either X or its complement has finite Morley rank (why?). (i) follows.
(ii) If $RM(tp(a/F)) = \omega$ then, by 1.19, a is differentially transcendental over F.

Corollary 1.22 *For some (any)* F, $p_{F,1}(x)$ *has* U*-rank* ω.

Proof. By 1.21 (ii) and the fact that U-rank is $\leq$ Morley rank, we get $U(p_{F,1}(x)) \leq \omega$.

We may assume $F = \mathbf{Q}$. So $p_{\mathbf{Q},1} = tp(a/\emptyset)$ where a is differentially trancendental (over $\mathbf{Z}$ or equivalently over $\mathbf{Q}$). For each n let $a^{(n)} = \partial^n(a))$. Then a forks with $a^{(n)}$ over $\emptyset$ (by 1.18 (i)).
Claim I. $tp(a/a^{(n)})$ is isolated by the formula $\partial^n(x) = a^{(n)}$.
Proof. Note that $a^{(n)}$ is also differentially transcendental over $\emptyset$. So if c is any solution of $\partial^n(x) = a^{(n)}$, then c is differentially transcendental over $\emptyset$ so has the same complete tye as a. Hence $tp(c/a^{(n)}) = tp(a/a^{(n)})$.

Claim II. $U(tp(a/a^{(n)})) = n$.
Proof. By Claim I, and 1.20 (and Claim I there), $tp(a/a^{(n)})$ has Morley rank n. But the set of solutions of $\partial^n(x) = a^{(n)}$ is in definable bijection with $\mathcal{C}^n$,

and we know that Morley rank = U-rank = dim inside the strongly minimal set $\mathcal{C}$. It follows (why?) that $U(tp(a/a^{(n)})) = n$ too.

By Claim II, $tp(a/\emptyset)$ has forking extensions of arbitrarily large finite U-rank. Thus $U(tp(a/\emptyset) \geq \omega$ so by what we saw earlier, is precisely ω.

Corollary 1.23 *For a single element $a \in \mathcal{U}$, $ord(a/k)$ is finite iff $U(tp(a/k))$ is finite iff $RM(tp(a/k))$ is finite.*

Example 1.24 *Let k and c be such that $cc'' = c'$ and $ord(c/k) = 2$. Then $U(tp(c/K)) = 1$ (and also $RM(tp(c/k)) = 1$).*

Discussion. This is Corollary 5.13 of [1]. One has to show that $tp(c/k)$ has no extension of order 1. Suppose otherwise. So there is $K > k$ such that $ord(c/K) = 1$. Let $P(x,y)$ be an irreducible polynomial over K such that $P(c,c') = 0$, and note that P must have positive degree in y. Applying ∂ we see that
$P^{\partial}(c,c') + (\partial P/\partial x)(c,c')c' + (\partial P/\partial y)(c,c')c'/c = 0$ and thus
$c(P^{\partial}(c,c') + (\partial P/\partial x)(c,c')cc' + (\partial P/\partial y)(c,c')c' = 0$.
It follows that $P(x,y)$ divides the polynomial
$xP^{\partial}(x,y) + (\partial P/\partial x)(x,y)xy + (\partial P/\partial y)(x,y)y$. A computation shows this to be impossible.

Proposition 1.25 *DCF_0 has elimination of imaginaries.*

Proof. We will first show that for any stationary type $p(x)$ where x is a finite tuple of variables, $Cb(p)$ is interdefinable with a tuple of real elements of $\mathcal{U}$. We may assume that $p(x) \in S(K)$ where K is a saturated model (elementary substructure of $\mathcal{U}$). Let $p = tp(a/K)$. Now ACF_0 does have elimination of imaginaries, hence for each r, $tp_{L_r}(a, \partial(a), .., \partial^r(a)/K)$ has a canonical base say c_r, a finite tuple from K.
Claim. If f is an automorphism of the differential field K then $f(p) = p$ iff $f(c_r) = c_r$ for all r.
Proof. We may assume that f is the restriction to K of an automorphism f' of $\mathcal{U}$. Suppose $f(p) = p$ then f fixes $tp_{L_r}(a, .., \partial^r(a)/K)$ for all r, so as f is also a field automorphism, $f(c_r)$ for all r. Conversely if f fixes c_r for all r, then $tp_{L_r}(a, .., \partial^r(a)/K) = tp_{L_r}(f'(a), .., \partial^r(f'(a))/K)$ for all r, hence by 1.11, $tp(a/K) = tp(f(a)/K)$.

By the Claim, $Cb(p)$ is interdefinable with the sequence $(a_r)_r$ so clearly with a finite subtuple. On general grounds (Dominika's project last semester), it follows that for any imaginary e there is a real tuple c such that $e \in dcl(c)$ and $c \in acl(e)$. As we are working in a field, the finite set $\{c_1,..,c_n\}$ of realizations of $tp(c/e)$ is itself interdefinable with a real finite tuple d. So e and d are interdefinable.

Finally we discuss the "differential Zariski topology". A differential polynomial over K in indeterminates $x_1,.,x_n$ is simply an ordinary polynomial over K in indeterminates $x_1,.,x_n, \partial(x_1),..,\partial(x_n),..,\partial^r(x_1,..,\partial^r(x_n)$ for some r. (We may often write $x^{(m)}$ for $\partial^m(x)$.) Such a differential polynomial can be evaluated on any n-tuple from $\mathcal{U}$. By definition a subset X of $\mathcal{U}^n$ is *Kolchin closed* (over K), if X is the common zero set of some possibly infinite system of differential polynomials (over K). (In fact for any differential field k we can speak of a Kolchin closed subset of k^n.) It is rather easy to see that any finite union of Kolchin closed subsets of $\mathcal{U}^n$ is also Kolchin closed: Suppose $V_i \subseteq \mathcal{U}^n$ is the common zero set of the system $\mathcal{Q}_\rangle$, for $i = 1,..,n$. Let $\mathcal{Q} = \{P_1 \cdot ... \cdot P_n : P_i \in \mathcal{Q}_i\}$. Then $\cup_i V_i$ is the common zero set of $\mathcal{Q}$. On the other hand, by definition, the intersection of an arbitrary family of Kolhin closed sets is Kolchin closed. As both $\emptyset$ and $\mathcal{U}^n$ are Kolchin closed, we see that the Kolchin closed subsets of $\mathcal{U}^n$ are the closed sets for a topology on $\mathcal{U}^n$, the Kolchin or differential Zariski topology.

We now aim towards a proof of the following important result:

Theorem 1.26 *Let $V \subseteq \mathcal{U}^n$ be Kolchin closed. Then there is a* finite *set $P_1,..,P_m$ of differential polynomials over $\mathcal{U}$ such that $V = \{x \in \mathcal{U}^n : P_1(x) = P_2(x) = .. = P_m(x) = 0\}$.*

Before giving the proof, let us consider some consequences of the theorem.

Remark 1.27 *Assume Theorem 1.26 is true for n. Then*
(i) For any system S of differential polynomials in differential indeterminates $x_1,..,x_n$, there is a finite subset S' of S such that the zero set of S in $\mathcal{U}$ equals the zero set of S' in $\mathcal{U}$. (In fact this remains true for any differential field over which S is defined.)
(ii) There is NO infinite strict descending chain $V_1 \supset V_2 \supset V_3...$ of Kolchin closed subsets of $\mathcal{U}^n$.
(iii) Define $h(V)$ for V a Kolchin closed subset of $\mathcal{U}^n$, by $h(V) \geq \alpha + 1$ if

there is a proper Kolchin closed subset W of V such that $h(W) \geq \alpha$. (Also $h(V) \geq 0$ iff V is nonempty and for limit δ, $h(V) \geq \delta$ iff $h(V) \geq \alpha$ for all $\alpha < \delta$.) Then for any Kolchin closed $V \subseteq \mathcal{U}^n$, $h(V)$ is an ordinal. In particular $h(\mathcal{U}^n)$ is an ordinal.
(iv) Call a Kolchin closed set $V \subseteq \mathcal{U}^n$ irreducible, if V is NOT the union of two proper Kolchin closed subsets of V. Then any Kolchin closed set V can be written uniquely as an (irredundant) finite union of Kolchin closed sets $V_1, .., V_m$. The V_i are called the irreducible components of V.
(v) Suppose $V \subseteq \mathcal{U}^n$ is an irreducible Kolchin closed set defined over a small differential subfield K of $\mathcal{U}$. (So V is the zero set of a (finite) system of differential polynomials with coefficients from K.) Then there is $a \in \mathcal{U}^n$ such that $a \in V$ and $a \notin W$ for every Kolchin closed proper subset W of V which is defined over K. Moreover $tp(a/K)$ does not depend on the choice of a. We call a a generic point of V over K, and $tp(a/K)$ the generic type of V over K.

Proof. (i) We may assume that S is over a small differential subfield K of $\mathcal{U}$. Let $V \subseteq \mathcal{U}^n$ be the zero set of S. By 1.26, V is definable. By compactness V is defined as the zero set of a finite subset S' of S.
(ii) This is immediate. For suppose V_i is the zero set of the finite set S_i of differential polynomials. Then the zero set of $\cup\{S_i : i < \omega\}$ is by (i) the 0-set of $\cup_{i=1,..,m} S_i$ whereby $\cap_{i<\omega} V_i = V_m$.
(iii) Immediate from (ii).
(iv) This is proved by induction on $h(V)$. If V is already irreducible there is nothing to do. Otherwise $V = V_1 \cup V_2$ where V_i are proper Kolchin closed subsets of V. $h(V_i) < h(V)$ for $i = 1, .., 2$ so by induction each V_i can be uniquely written as an (irredundant) finite union of irreducible Kolchin closed sets. This gives an expression for V as a finite union of irreducible Kolchin closed subsets, and (after writing this in an irredundant fashion) we see easily it is unique (using the induction hypothesis for V_1 and V_2).
(v) Consider the set $\Sigma(x)$ of formulas $\{x \in V\} \cup \{x \notin W : W$ a proper Kolchin closed subset of V defined over $K\}$ over K. If Σ were inconsistent, then by compactness we could write V as a finite union of proper Kolchin closed subsets, contradicting irreducibility of V. Thus Σ is consistent, so there is $a \in \mathcal{U}^n$ as claimed. Note that for any differential polynomial $P(x_1, .., x_n)$ over K, $P(x) = 0 \in tp(a/K)$ iff $\models (\forall x)(x \in V \rightarrow P(x) = 0)$. By quantifier elimination, the latter information determines a complete type over K (which

we know to be consistent as it is realized by a).

Exercise 1.28 *(Under the same assumptions as Remark 1.27.) Suppose $V \subseteq \mathcal{U}^n$ is irreducible and defined over K. Let $p(x_1, ., x_n)$ be the generic type of V over K. Then p is stationary and for any $L > K$ the nonforking extension of p over L is precisely the generic type of V over L.*

Proof of Theorem 1.26. We will be making use of the Hilbert basis theorem (see the next section), which implies that in any field, the 0-set of a possibly infinite system of *polynomials* in $x_1, .., x_n$, is the 0-set of a finite subsystem.

The proof of 1.26 is by a global induction on n. Let us first do the $n = 1$ case (although this is also contained in our proof of the induction step). So let $V \subset \mathcal{U}$ be our Kolchin closed set. Without loss V is defined by a family of differential polyomials over a small differential subfield K of $\mathcal{U}$. Let $\mathcal{Q}$ be the family of all differential polynomials over K which vanish on V (so V is the 0-set of $\mathcal{Q}$). We proceed by induction on $(ord, deg)(\mathcal{Q})$, which is by definition the least (ord, deg) of a differential polynomial $Q(x)$ in $\mathcal{Q}$ (where $ord(Q)$ is greatest $x^{(m)}$ that appears in Q and $deg(Q)$ is the degree of Q in this $x^{(m)}$). (Here and subsequently we view Q as an ordinary polynomial over K in $x, x', ...$)

Suppose $(ord, deg)(\mathcal{Q}) = (m, d)$ witnessed by $Q(x)$. Then $\partial(Q(x)) \in \mathcal{Q}$. By 1.2 (ii),
(*) $\partial(Q(x)) = x^{(m+1)}s(x) + g(x)$
where $s(x)$ (called the separant of Q) is $\partial Q/\partial x^{(m)}$, and g is a differential polynomial over K of order at most m. Note that (ord, deg) of $s(x)$ is strictly less than that of $Q(x)$. So by induction the zero set V_1 of $\mathcal{Q} \cup \{s(x)\}$ is definable.

On the other hand, suppose that $(\partial Q)(a) = 0$, and $s(a) \neq 0$, then from (*) for each $i \geq 1$, we can write x^{m+i} as $s_i(x)$ where $s_i(x)$ is a quotient of differential polyomials in x over K of order at most m and the denominator of s_i is a power of $s(x)$. For each $P(x) \in \mathcal{Q}$, replace x^{m+i} for $i \geq 1$ by $s_i(x)$, and multiply through by the denominators to obtain a differential polyomial over K of order at most m. Let $\mathcal{Q}_2$ be the family of differential polynomials so obtained. Using the Hilbert basis theorem, it is easy to see that the zero set V_2 of $\mathcal{Q}_2$ is definable (why?) Moreover it is clear that if $s(a) \neq 0$ then $a \in V$ iff $a \in V_2$.

Thus V is the union of V_1 together with $\{a \in V_2 : s(a) \neq 0\}$, so is definable, hence by compactness, V is the 0-set of a finite set of differential polynomials.

Now we perform the induction step. We assume the theorem is true for n and prove it for $n+1$. So we work with differential indeterminates $x_1, .., x_n, x_{n+1}$. Let $V \subseteq \mathcal{U}^{n+1}$ be our Kolchin closed set. We assume V to be defined by a system of differential polynomials over a small differential field K, and let $\mathcal{Q}$ be all differential polynomials over K vanishing on V. Let $V|n$ be the Kolchin closure of $\{(a_1, .., a_n) \in \mathcal{U}^n : \exists a_{n+1}, (a_1, .., a_{n+1}) \in V\}$. So the induction hypothesis applies to $V|n$. Using Remark 1.27 we may also assume $V|n$ to be irreducible. Let $a = (a_1, .., a_n)$ be a generic point of $V|n$ over K. Note that there IS a_{n+1} such that $(a, a_{n+1}) \in V$ (why????). For $P(x_1, .., x_{n+1}) \in \mathcal{Q}$, let $P_a(x_{n+1})$ be the differential polynomial (over the differential ring $K[a]_\partial$ generated by a over K), obtained by substuting a_i for x_i for $i = 1, .., n$, and let $\mathcal{Q}_a$ be the set of all such $P_a(x_{n+1})$ for $P \in \mathcal{Q}$. Let (m, d) be the least (order,degree) (in x_{n+1}) of a differential polynomial in $\mathcal{Q}_a$ in which x_{n+1} appears. (Let us remark that if x_{n+1} does not appear in any polynomial in $\mathbf{Q}$ then V is precisely $V|n \times \mathcal{U}$.)

Our induction will be on $(h(V|n), (m, d))$. x denotes $(x_1, .., x_n)$

Suppose first that $(m, d) = (0, 1)$. So there is $Q(x, x_{n+1}) \in \mathcal{Q}$ of the form $x_{n+1}F(x) + G(x)$ (F, G differential polyomials in $x = (x_1, ., x_n)$) such that $F(a) \neq 0$. Then for $b \in \mathcal{U}^n$ such that $F(b) \neq 0$, clearly $(b, b_{n+1}) \in V$ iff $b \in V|n$ and $b_{n+1} = -G(b)/F(b)$. In particular

(**) $\{(b, b_{n+1}) \in V : F(b) \neq 0\}$ is definable.

On the other hand if V_2 is the 0-set of $\mathcal{Q} \cup \{F(x)\}$ then clearly $h(V_2|n) < h(V|n)$ so we can apply induction to see that V_2 is definable. Together with (**) this shows that V is definable, which is enough.

Now suppose that $(m, d) > (0, 1)$. Suppose this is witnessed by $P_a(x_{n+1}) \in \mathcal{Q}_a$. Then $\partial(P_a) = x_{n+1}^{(m+1)}(S_a(x_{n+1}) + G_a(x_{n+1})$, where $S_a(x_{n+1})$ has smaller (ord, deg) than $P_a(x_{n+1})$ and where $G_a(x_{n+1})$ has order at most m. The notation implies that $S_a(x_{n+1})$ comes from some $S(x, x_{n+1})$ by substituting a for x.

Claim I. Let V_1 be the 0-set of $\mathcal{Q} \cup \{S(x, x_{n+1})\}$. Then the induction hypothesis applies to V_1. In particular V_1 is definable.

Proof. Clearly $V_1 \subseteq V$, and so $V_1|n \subseteq V|n$. If $V_1|n$ is a proper subset of

$V|n$ then $h(V_1|n) < h(V|n)$ and the induction hypothesis applies. Otherwise $V_1|n = V|n$ and so (why?) there IS a_{n+1} such that $S_a(a_{n+1}) = 0$. As $(ord, deg)S_a(x_{n+1}) <$ the least (ord, deg) of any $Q_a(x_{n+1})$ in $\mathcal{Q}_a$ the induction hypothesis again applies.

Claim II. $V \cap \{(b, b_{n+1}) : S(b, b_{n+1}) \neq 0\}$ is definable.
Proof. Left to the reader !!

By Claims I and II, V is definable. This completes the proof of Theorem 1.26.

2 Algebraic geometry and algebraic groups

To pursue further the model theory of differential fields and the applications we have in mind, it will be convenient to introduce some language and basic notions of algebraic geometry.

¿From the model-theoretic point of view, where the category of definable sets in a structure is among the central objects of study, it is natural to think of algebraic varieties, groups,.. as special cases of definable sets (in an algebraically closed field). However the whole development of algebraic geometry, post-Weil, was in the opposite direction. First, there was an "intrinsic" definition of an algebraic variety, releasing it from any embedding in some ambient space. Second was the notion a scheme, in which rings played a central role and arithmetic issues were built into the general theory. I guess I will stick with the "naive" point of view here but I may try to point out the more general notions.

k will denote an algebraically closed field. Sometimes we may assume it to be of uncountable transcendence degree κ (and so also κ-saturated in the language L_r).

$k[x_1, .., x_n]$ denotes the polynomial ring over k in indeterminates $x_1, .., x_n$. One of the basic results (Hilbert's basis theorem) is that any ideal of $k[x_1, .., x_n]$ is finitely generated. (See Chapter VI, section 2 of Lang's Algebra.)

Definition 2.1 *An affine algebraic set V is the common zero set in k^n of some system $S = \{f_\lambda : \lambda \in \Lambda\}$ of polynomials in $k[x_1, .., x_n]$. We write $V = V(S)$.*

Lemma 2.2 *(With notation as in the definition above.)*
(i) $V(S) = V(I)$ where I is the ideal generated by S.
(ii) $V(S) = V(S')$ for some finite subset S' of S.
(iii) The affine algebraic subsets of k^n are the closed sets for a Noetherian topology on k^n, called the Zariski topology.

Proof. (i) is clear.
(ii). The ideal I generated by S is finitely generated, by the Hilbert basis theorem, thus generated by a finite subset S' of S. Clearly $V(S') = V(S)$.
(iii). By definition a Noetherian topology is a topology such that any intersection of closed sets is a finite subintersection. The algebraic subsets of k^n are closed under finite union (by considering products of polynomials) and the Noetherian condition comes from (ii). Clearly also both k^n and $\emptyset \subset k^n$ are algebraic subsets of k^n.

Remark 2.3 *(Hilbert's Nullstellensatz.)*
(i) If I is a proper ideal of $k[x_1, .., x_n]$, then $V(I) \neq \emptyset$.
(ii) If I is an ideal of $k[x_1, .., x_n]$, then the set of $f \in k[x_1, .., x_n]$ which vanish on $V(I)$ is precisely $\sqrt{I} =_{def} \{f : f^m \in I$ for some $m\}$.

Proof. (i) By the Hilbert basis theorem there is some (proper) prime ideal I' of $k[x_1, .., x_n]$ containing I (a maximal ideal for example). Then $R = k[x_1, .., x_n]/I'$ is an integral domain containing k. Let K be its field of fractions. Let $a = (a_1, .., a_n)$ be the image of $(x_1, .., x_n)$ in K. Then $f(a) = 0$ for f in some finite generating set S of I. As k is an existentially closed field, there is $a' \in k^n$ such that $f(a') = 0$ for $f \in S$. Hence $a \in V(I)$.
(ii) Clearly if $f \in \sqrt{I}$ then f is 0 on $V(I)$. Conversely, suppose f is 0 on $V(I)$. Let x_0 be a new variable, and let $J \subset k[x_0, x_1, .., x_n]$ be the ideal generated by I together with $1 - x_0 f(x_1, .., x_n)$. If J were a proper ideal of $k[x_0, .., x_n]$ then by part (i), $V(J) \neq \emptyset$, contradicting our assumptions. (There would be $a = (a_1, .., a_n) \in k^n$ such that $g(a) = 0$ for $g \in I$ but $f(a) \neq 0$.) Hence $J = k[x_0, .., x_n]$. So $1 \in J$ and it easily follows that $f^m \in I$ for some m: write $1 = h(x_0, .., x_n)(1 - x_0 f(x_1, .., x_n) + \sum_j g_j(x_0, .., x_n) f_j(x_1, .., x_n)$ with $f_j \in I$. Substitute $1/f$ for x_0 and multiply both sides by a suitable power of f.)

If X is an arbitrary subset of k^n, we let $I(X)$ denote the set of polynomials in $k[x_1, .., x_n]$ which vanish on X. Clearly $I(X)$ is an ideal. By 2.3 we have

Remark 2.4 *The map taking V to $I(V)$ sets up a bijection between (affine) algebraic sets $V \subset k^n$ and radical ideals of $k[x_1, .., x_n]$. Moreover $V(I(V)) = V$.*

We define the affine algebraic set $V \subseteq k^n$ to be irreducible if V can not be written as $V_1 \cup V_2$ for V_1, V_2 proper algebraic subsets of V. Note that by Lemma 2.2 (iii) any V decomposes (uniquely) into a finite union of irreducible algebraic sets.

Exercise 2.5 *Let $V \subseteq k^n$ be an algebraic set. Then V is irreducible if and only if $I(V)$ is a prime ideal of $k[x_1, .., x_n]$..*

Definition 2.6 *Let $V \subseteq k^n$ be an affine algebraic set.*
(i) By a regular function on V we mean a function from V to k given by a polynomial $f(x_1, .., x_n) \in k[x_1, .., x_n]$.
(ii) The coordinate ring $k[V]$ of V is the ring of all regular functions on V (with natural addition and multiplication).

Let us first remark that it is immediate that $k[V]$ is the ring $k[x_1, .., x_n]/I(V)$. For, if $f, g \in k[x_1, .., x_n]$ then f and g define the same (regular) function on V if and only if $f - g$ vanishes on V iff $f - g \in I(V)$. It follows from 2.5 that V is irreducible iff $k[V]$ is an integral domain. In the case that V is irreducible, we define $k(V)$ the field of rational functions on V to be the field of fractions of $k[V]$.

There is another natural candidate for the notion of a "regular" function on V, namely a "locally rational" function.

Definition 2.7 *Let V again be an algebraic subset of k^n. Let $f : V \to k$ by an (abstract) function. We call f locally rational if for every $a \in V$ there is a Zariski open neighbourhood U of a and a pair of polynomials $g_1, g_2 \in k[x_1, .., x_n]$ such that $g_2 \neq 0$ on U and $f = g_1/g_2$ on U.*

Lemma 2.8 *(With above notation.) The locally rational functions on V are precisely the regular (i.e. polynomial) functions on V.*

Proof. Clearly any regular function is locally rational.

Conversely, suppose that f is locally rational. Note that any covering of V by Zariski open subsets of V has, by 2.2 (iii), a finite subcover. Note also that any Zariski open subset of V is a finite union of sets defined by polyomial inequations $P(x_1,..,x_n) \neq 0$. It follows that we can find a finite set $Q_1,..,Q_m$ of polynomials, and for each $i = 1,..,m$ a pair f_i, g_i of polynomials, such that
(i) for each $a \in V$, $Q_i(a) \neq 0$ for some $i = 1,..,m$, and
(ii) whenever $a \in V$ and $Q_i(a) \neq 0$ then $g_i(a) \neq 0$ and $f(a) = f_i(a)/g_i(a)$.
Claim I. We may assume that $g_i = Q_i$ for $i = 1,..,m$.
Proof. Let $I = I(V)$. By 2.3 (ii), and (ii) above, Q_i is in the radical ideal generated by $I \cup \{g_i\}$. That is, for some positive integer r_i, $Q_i^r = h_i g_i + t_i$ for some $h_i \in k[x_1,..,x_n]$ and $t_i \in I$. Hence for $a \in V$ such that $Q_i(a) \neq 0$, $f_i/g_i = h_i f_i/Q_i^{r_i}$. Replace Q_i by $Q_i^{r_i}$ and f_i/g_i by $h_i f_i/Q_i^{r_i}$ for $i = 1,..,m$.

Claim II. There are polynomials $h_1,..,h_m$, such that for each $i = 1,..,m$, $\sum_{j=1,..,m} h_j f_j Q_j Q_i^2 = f_i Q_i$ on V.
Proof. First by 2.3 (i), I together with $\{Q_1^2,..,Q_m^2\}$ must generate the trivial ideal $k[x_1,..,x_n]$, hence there are $h_1,..,h_m$ such that
(*) $\sum_{j=1,..,m} h_j Q_j^2 = 1$ modulo I.
On the other hand, whenever $a \in V$ and both Q_i and Q_j are nonzero at a, then $f_i/Q_i f_j/Q_j$ at a. Thus (for $i, j \in \{1,..,m\}$),
(**) $(f_i Q_j - f_j Q_i) Q_i Q_j$ vanishes on V so is in I.
Now fix i. For each j, $f_j Q_j Q_i^2 = f_i Q_i Q_j^2$ mod I, by (**). Hence, working modulo I,
$\sum_{j=1,..,m} h_j f_j Q_j Q_i^2 = f_i Q_i \sum_{j=1,..,m} h_j Q_j^2$ which by (*) equals $f_i Q_i$. Thus yields Claim II.

By Claim II f is given on all of V by the polynomial $\sum_{j=1,..,m} h_j f_j Q_j$. So f is regular.

Definition 2.9 *An affine algebraic variety consists of a pair $(V, k[V])$ where $V \subset k^n$ is an affine algebraic set and $k[V]$ is its coordinate ring.*

Before continuing let us recall the *dimension* of an algebraic variety. It is convenient now to assume k to be reasonably saturated, even just ω-saturated. Let $V \subset k^n$ be an algebraic set (variety). The algebraic geometers say that V is defined over the subfield K of k if $I(V)$ is generated by polynomials over K, namely if $I(V)$ is the tensor product of $I_K(V)$ and k over K where $I_K(V)$ is the set of polynomials over K vanishing on V.

Fact 2.10 *V is defined over K (in the algebraic geometric sense) just if V is defined over the perfect closure of K in the structure $(k, +, \cdot)$ in the model-theoretic sense.*

In particular in characteristic zero the two notions coincide. In the following all model-theoretic notation will be with respect to k as an L_r-structure. Now suppose that V is a variety defined over a small subfield K of k.

Definition 2.11 *(i) $dim(V) = max\{tr.deg(K(a)/K) : a \in V\}$.*
(ii) By a generic point of V over K we mean $a \in V$ such that $tr.deg(K(a)/K) = dim(V)$.

Exercise 2.12 *Let $V \subseteq k^n$ be a variety, defined over $K < k$.*
(i) $dim(V) = RM(V)$.
(ii) If V is irreducible, then $dim(V) = tr.deg(k(V)/k)$.
(iii) If V is irreducible then $dM(V) = 1$.
(iv) If V is irreducible then $dim(V)$ equals the Krull dimension of V, namely the greatest m such that there exist a strict chain of irreducible varieties $V_0 \subset V_1 \subset ... \subset V_m = V$.
(v) If V is irreducible, then $a \in V$ is a generic point of V over K just if $a \notin W$ for all proper Zariski closed subsets W of V which are defined over K.

Let us now discuss morphisms.

Definition 2.13 *Let $V \subseteq k^n$, and $W \subseteq k^m$ be varieties.*
(i) A morphism from V to W is a polynomial map, namely a map $f : V \to W$ such that each of the coordinate maps $f_1, .., f_m$ is in $k[V]$.
(ii) By an isomorphism between V and W we mean a bijection f between V and W such that both f and F^{-1} are morphisms.
(iii) The morphism $f : V \to W$ is said to be defined over K if f can be represented by a sequence of polynomials with coefficients from K.

Note that if f is an abstract map from V to W then f induces a map $f^\sharp$ of k-algebras from the k-algebra of k-valued functions on W to the k-algebra of k-valued functions on V: for g a k-valued function on W, $f^\sharp(g)$ is the function on V whose value at $a \in V$ is $g(f(a))$.

Exercise 2.14 *(V, W affine varieties.) Let f be an (abstract) map from V to W. Then f is a morphism if $f^\sharp(k[W]) \subseteq k[V]$.*

Definition 2.15 *Let $V \subseteq k^n$ be an irreducible affine variety. Let $a = (a_1, .., a_n) \in V$. Then $\mathcal{M}_a$, the "maximal ideal of V at a", is the set of $f \in k[V]$ such that $f(a) = 0$. We may write $\mathcal{M}_{V,a}$ to emphasize the dependence on V.*

Lemma 2.16 *(With above notation.)*
(i) $\mathcal{M}_a$ is a maximal ideal of $k[V]$.
(ii) $\mathcal{M}_a$ is generated as an ideal by $\{(x_1 - a_1), .., (x_n - a_n)\}$ (where by abuse of notation x_i denotes the ith coordinate map on V).
(iii) Every maximal ideal of $k[V]$ is of the form $\mathcal{M}_a$ for some $a \in V$.

Proof. (i) The map which takes $f \in k[V]$ to $f(a) \in k$ is a surjective ring homomorphism whose kernel is precisely $\mathcal{M}_a$. So $k[V]/\mathcal{M}_a$ is a field, hence $\mathcal{M}_a$ is a maximal ideal.
(ii) Left to the reader.
(iii) Let I be a maximal ideal of $k[V]$. By 2.3 (i), there is $a \in V(I)$. Then $I \subseteq \mathcal{M}_a$, whereby $I = \mathcal{M}_a$ by maximality.

The maximal ideal of an irreducible variety V at a point $a \in V$ is often defined rather to be the (unique) maximal ideal of the local ring $\mathcal{O}_a$ of V at a. Here $\mathcal{O}_a$ is the subring of $k(V)$ consisting of (or rather represented by) those f/g where $f, g \in k[V]$ and $g(a) \neq 0$. This is often called the ring of functions on V which are regular at a. Any element of $\mathcal{O}_a$ can be evaluated at a, and the unique maximal ideal of $\mathcal{O}_a$ consists of those $h \in \mathcal{O}_a$ such that $h(a) = 0$. This of course coincides with the "localization" of $\mathcal{M}_a$ at itself in $k[V]$, that is $\{f/g \in k(V) : f(a) = 0, g(a) \neq 0\}$.

Exercise 2.17 *Let $V \subseteq k^n$ be irreducible, and $a \in V$. Let $f_1, g_1, f_2, g_2 \in k[V]$, with $g_i(a) \neq 0$ for $i = 1, 2$. Then $f_1/g_1 = f_2/g_2$ in $k(V)$ just if the function f_1/g_1 is defined and agrees with f_2/g_2 on some Zariski open neighbourhood of a.*

Because of the exercise, the local ring of V at a can be thought of as the ring of "germs of rational functions defined at a".

Recall that if I is an ideal of a ring R then for any n, I^n denotes the ideal of R generated by all products $r_1...r_n$ where $r_i \in I$ for $i = 1, .., n$.

Lemma 2.18 *Suppose that $f : V \to W$ is a morphism of irreducible affine varieties. Let $a \in V$ and $f(a) = b \in W$. Then*
(i) For any r, $f^\sharp((\mathcal{M}_{W,b})^r)) \subseteq (\mathcal{M}_{V,a})^r$.
(ii) Moreover, if $V \subseteq k^n$, $W = k^n$ and f is the natural embedding, then $f^\sharp((\mathcal{M}_{W,b})^r) = (\mathcal{M}_{V,a})^r$ for all r.

Proof. (i) is clear. For (ii), it is enough to prove it for $r = 1$. In this case $a = f(a)$. If $g \in k[V]$ then $g = h/I(V)$ for a polynomial $h \in k[x_1, .., x_n] = k[W]$, and $g(a) = 0$ iff $h(a) = 0$.

Lemma 2.19 *Let $V \subseteq k^n$ be an irreducible variety, and let $a \in V$. Then for any positive integer r, $\mathcal{M}_a/(\mathcal{M}_a)^r$ is a finite-dimensional k-vector space.*

Proof. By Lemma 2.10 it is enough to prove this when $V = k^n$. For convenience we restrict our attention to the case when $a = (0, .., 0)$. A polynomial f is in $\mathcal{M}_0$ iff it has no constant term. Moreover two such polynomials yield the same element in the quotient $\mathcal{M}_0/(\mathcal{M}_0)^r$ if they have the same terms of (total) degree $< r$. Thus the vector space in question is finite-dimensional.

Remark 2.20 *(With above notation.) (i) The k-vector space $\mathcal{M}_a/(\mathcal{M}_a)^2$ is called the (Zariski) cotangent space of V at a. Its dual is called the (Zariski) tangent space of V at a and will be discussed more later in this section.*
(ii) If $\mathcal{M}'$ say denotes the maximal ideal of the local ring of V at a, then there are canonical isomorphisms between $\mathcal{M}_a/(\mathcal{M}_a)^r$ and $\mathcal{M}'/(\mathcal{M}')^r$.

We wll need the following commutative algebraic result (which follows from Krull's Theorem, which in turn follows from the Artin-Rees lemma):

Fact 2.21 *Let R be a Noetherian domain (that is, an integral domain satisfying the ascending chain condition on ideals). Let I be an ideal of R with $I \neq R$. Then $\cap_n I^n = 0$.*

Sometimes the affine variety k^n is denoted by $\mathbf{A}^n$. Let $V \subseteq k^n$ be an irreducible variety and $a \in V$. Let $i_V : V \to \mathbf{A}^n$ be the canonical embedding. Then by 2.18(ii), for each r, $i_V^\sharp$ induces a surjective k-linear map from $\mathcal{M}_{\mathbf{A}^n,a}/(\mathcal{M}_{\mathbf{A}^n,a})^r$ to $\mathcal{M}_{V,a}/(\mathcal{M}_{V,a})^r$ which we denote $(i_V^\sharp)_r$ for now.

Lemma 2.22 *Let $V, W \subseteq k^n$ be irreducible affine varieties. Suppose $a \in V \cap W$. Then $V = W$ if and only if $ker((i_V^\sharp)_r) = ker((i_W^\sharp)_r)$ for all $r = 2, 3, ...$*

Proof. Clearly the left hand side implies the right hand side.

Suppose now that the RHS holds. We will show that $I(V) \subseteq I(W)$. By symmetry we conclude that $I(V) = I(W)$ hence $V = W$. So let $f \in k[x_1, .., x_n]$ be in $I(V)$. In particular $f(a) = 0$. So f is in the maximal ideal of $\mathbf{A}^n$ at a. As $f/I(V) \in k[V]$ equals 0, it follows that f (or rather its suitable quotient) is in the kernel of $(i_V^\sharp)_r$ for all r. By assumption, this implies f is in the kernel of $(i_W^\sharp)_r$ for all r. This means that $f/I(W)$ is in $(\mathcal{M}_{W,a})^r$ for all r. But $k[W]$ is a Noetherian domain, and $\mathcal{M}_{W,a}$ is a proper ideal of $k[W]$. So by Fact 2.21, $f/I(W) = 0$, that is $f \in I(W)$.

Let us now discuss a little more cotangent and tangent spaces. Let us fix an irreducible affine variety $V \subseteq k^n$, defined over $K < k$ say. Fix $a \in V$. As earlier $\mathcal{M}_{V,a}$ denotes the maximal ideal of V at a. We let $T(V)_a$ denote the tangent space to V at a.

Fact 2.23 *The k-dimension of $\mathcal{M}_a/(\mathcal{M}_a)^2$ is $\geq dim(V)$. In particular the dimension of $T(V)_a$ (as a k-vector space) is $\geq dim(V)$.*

Explanation. In some of the books this is done in a purely commutative algebraic way. The dimension of a ring is defined to be the sup of the lengths of strictly ascending chains of prime ideals of the ring. So note that $dim(k[V]) = dim(V)$ (why?). Next for R a finitely generated integral domain over a field k, $dim(R)$ is the same as $dim(R_\mathcal{M})$ where $\mathcal{M}$ is any maximal ideal of R and $R_\mathcal{M}$ denotes the localization of R at $\mathcal{M}$. Thus we see that $k[V]$ and the local ring $\mathcal{O}_a$ of V at a have the same dimension. Lastly, if A s a local Noetherian ring, then $dim(A)$ is finite and $\leq$ the $A/\mathcal{M}$-dimension of $\mathcal{M}/\mathcal{M}^2$ where $\mathcal{M}$ is the unique maximal ideal of A. Applying this to the local ring of V at a and using Remark 2.20(ii) we obtain the fact above.

Let us give a more explicit description of the cotangent and tangent spaces of V at a. We will start with the cotangent space. Note that any $f \in k[x_1, .., x_n]$ has a Taylor expansion at $a = (a_1, .., a_n)$, namely can be written as a polynomial in $(x_1 - a_1), .., (x_n - a_n)$. We will write $L_a(f)$ for the linear part of this expansion. Note that $f \in \mathcal{M}_{\mathbf{A}^n,a}$ iff the linear part of f has 0 constant term. Note also that $f = L_a(f) + g$ where $g \in (\mathcal{M}_{\mathbf{A}^n,a})^2$. $\mathcal{M}_{V,a}$ is precisely $\{f/I_V : f \in \mathcal{M}_{\mathbf{A}^n,a}\}$. Thus we can identify (the k-vector space) $\mathcal{M}_{V,a}/(\mathcal{M}_{V,a})^2$ with $k^n/\sim$ where $(c_1,..,c_n) \sim (d_1,..,d_n)$ if the linear polynomials $c_1(x_1-a_1)+..+c_n(x_n-a_n)$ and $d_1(x_1-a_1)+..+d_n(x_n-a_n)$ are equivalent modulo $I(V)+(\mathcal{M}_{V,a})^2 = I(V)+(\mathcal{M}_{\mathbf{A}^n,a})^2$.

Now the space of k-valued linear functions on k^n can be considered as k^n itself together with the action $(y_1,..,y_n)((c_1,..,c_n)) = y_1c_1+..+y_nc_n$. So we obtain $T(V)_a$ as a k-subspace of k^n (consisting of those $(y_1,..,y_n)$ such that $y_1c_1+..+y_nc_n = y_1d_1+...+y_nd_n$ whenenever $(c_1,.,c_n) \sim (d_1,..,d_n)$). Let us now give explicit equations for this subspace of k^n.

Fact 2.24 *(i) $(y_1,..,y_n) \in T(V)_a$ if and only if*
(*) $\sum_{i=1,..,n}((\partial P/\partial x_i)(a))y_i = 0$ for all $P(x_1,..,x_n) \in I(V)$.
(ii) In (i) it is enough to restrict our attention to those P in a fixed finite set of generators of $I(V)$. In particular, if V is defined over K, then $T(V)_a$ is defined over $K(a)$.

Proof. (i) First suppose (*) holds of $(y_1,..,y_n)$. Now if $(c_1,..,c_n) \sim (d_1,..,d_n)$ then $(c_1-d_1)(x_1-a_1)+..+(c_n-d_n)(x_n-a_n)+g = P \in I(V)$, for some $g(x_1,..,x_n) \in (\mathcal{M}_{\mathbf{A}^n,a})^2$. Hence by (*) $y_1(c_1-d_1)+..+y_n(c_n-d_n) = 0$. So $(y_1,..,y_n) \in T(V)_a$.
Conversely, suppose $(y_1,..,y_n) \in T(V)_a$, and let $P(x_1,..,x_n) \in I(V)$. Then $P(a) = 0$, so $P = L_a(P) + R$ where $L_a(f)$ has no constant term. Let $L_a(P) = c_1(x_1-a_1)+..+c_n(x_n-a_n)$. Then clearly
(I). $\sum_{i=1,..,n}((\partial P/\partial x_i)(a))y_i = \sum_{i=1,..,n} c_iy_i$.
On the other hand $(c_1,..,c_n) \sim (0,..,0)$. Hence
(II). $\sum_{i=1,..,n} c_iy_i = 0$.
By (I) and (II), we get that $(y_1,..,y_n)$ satisfies (*).
(ii) is clear.

Lemma 2.25 *Let a be a generic point of V over K. Then $dim(T(V)_a) = dim(V)$.*

Proof. By 2.23 we only have to prove $\leq$. Suppose that $dim(V) = r$. After reordering we may assume that $a_1,..,a_r$ are algebraically independent over K and each a_i for $i = r+1,..,n$ is algebraic over $K(a_1,..,a_r)$. For $i = r+1,..,n$, let s_i be the degree of a_i over $K(a_i,..,a_r)$. So (for each i) we can find a polynomial over K, $F_i(x_1,..,x_r,x_i)$ say, such that $F_i(a_1,..,a_r,a_i) = 0$ and the degree of F_i in x_i is s_i. Now as a is a generic point of V over K, $F_i(x_1,..,x_r,x_i) \in I(V)$ (Why??) So for any $(y_1,..,y_n) \in T(V)_a$, $(\sum_{j=1,..,r}(\partial F_i/\partial x_j)(a)y_j) + (\partial F_i/\partial x_i)(a)(y_i) = 0$. But $(\partial F_i/\partial x_i)(a) \neq 0$, hence y_i is in the k-linear span of $(y_1,..,y_r)$. As $(y_1,..,y_n) \in T(V)_a$ was arbitrary we obtain the result.

Definition 2.26 *$a \in V$ is said to be a nonsingular point of V if $dim(T_(V))_a) = dim(V)$. V is said to be smooth or nonsingular if all points on V are nonsingular.*

Corollary 2.27 *The set of nonsingular points of V is a (nonempty) Zariski open subset of V.*

Proof. Note from 2.24 that the tangent spaces of V at various points of V are uniformly definable. So it follows from 2.25 and quantifier-elimination that the set of nonsingular points of V CONTAINS a Zariski open subset of V. We leave it to the reader to conclude the proof.

Remark 2.28 *(i) Suppose $f(x)$ is a polynomial of one variable, and $a \in k$. Then the tangent space to the variety defined by $y = f(x)$ at $(a, f(a))$ is the line through the origin with slope $(df/dx)(a)$. After translating this line by $(a, f(a))$ we obtain the "usual" tangent line to the graph of $y = f(x)$ at $(a, f(a))$.*
(ii) More generally the tangent space to the variety V at a point a (as we defined it above) can be thought of as the totality of points on lines through a in k^n which are "tangent to" V at a (after translating the origin to a). Here "tangent to" means intersects at a with multiplicity ≥ 2, which can again be made precise.

We can put all the tangent spaces of V at various points together to form the "tangent bundle" of V.

Definition 2.29 *(Assume $V \subseteq k^n$ is an irreducible affine algebraic variety and that $I(V)$ is generated by $P_1,..,P_m$.) The tangent bundle $T(V)$ of V is the affine algebraic variety, contained in k^{2n} and defined by equations $P_j(x_1,..,x_n) = 0$ and $\sum_{i=1,..,n}((\partial P_j/\partial x_i)(x_1,..,x_n))y_i = 0$ for $j = 1,..,m$.*

So $T(V)$ is simply the set of pairs (a, v) where $a \in V$ and $v \in T(V)_a$.

Fact 2.30 *If $V \subset k^n$ is (irreducible and) smooth then $T(V)$ is irreducible and smooth.*

Definition 2.31 *Let V be an irreducible affine algebraic variety, and f a regular function on V, then df, the differential of f, is the regular function on $T(V)$ defined by $df(a, v) = \sum_{i=1,..,n}(\partial f/\partial x_i)(a)v_i$.*

Remark 2.32 *For each $a \in V$, the map taking $f \in k[V]$ to the function $df(a, -)$ defines an isomorphism between $\mathcal{M}_a/(\mathcal{M}_a)^2$ and the dual space to $T(V)_a$.*

Exercise 2.33 *Let $V \subset k^n$ and $W \subseteq k^m$ be irreducible affine varieties, and let $f : V \to W$ be a morphism. So $f = (f_1, .., f_m)$ with f_i regular functions on V. Define df (the differential of f) to be $(df_1, .., df_m)$. Show that df is a morphism from $T(V)$ to $T(W)$, and that for each $a \in V$, $df|T(V)_a$ is a linear map from $T(V)_a$ to $T(W)_{f(a)}$.*

Before discussing arbitrary (not necessarily affine) varieties we have to say a little more about rational functions.

Definition 2.34 *Let $V \subset k^n$ be an affine variety and U a nonempty Zariski-open subset of V.*
(i) A function $f : U \to k$ is said to be regular on U if it is locally rational, namely for every $a \in U$ there is a Zariski open neighbourhood $U' \subseteq U$ of a such that on U' f is given by a quotient of polynomials, or equivalently a quotient of functions in $k[V]$.
(ii) A map f from a Zariski open subset U of an affine $V \subseteq k^n$ to a Zariski open subset U' of an affine variety $W \subset k^m$ is said to be a morphism from U to U' if the coordinate functions $f_1, .., f_m$ are refular functions on U.

Note again that if f is a regular function on the open subset U of V then there is a finite covering of U by Zariski opens U_i such that for each i $f|U_i$ is given by a quotient of polynomials. Sometimes we call Zariski open subsets of affine varieties, quasi-affine varieties (although we should also equip them with a sheaf of regular functions..). Note that any Zariski open subset of V is defined by a finite disjunction of polynomial inequations (together with the equations defining V). By a *principal* open subset of V we mean a Zariski open subset of V defined by a single inequation $g(x_1, .., x_n \neq 0$.

Exercise 2.35 *Suppose U is a principal open subset of the affine variety $V \subset k^n$.*
(i) Then U is isomorphic to an affine variety.
(ii) Any regular function f on U can be represented by a quotient of polynomials.

Hint. Let U be defined by $g(x) \neq 0$ (plus $x \in V$). Consider the affine variety $W \subset k^{n+1}$ defined by $x \in V$ plus $g(x)y = 1$. Show that the the natural bijection between U and W is an isomorphism (in the sense of Definition 2.34. Also show that f gives rise to a locally rational function from V to k which is by 2.8 given by a polynomial.

Let us discuss the relationship between regular functions on Zariski open subsets of V and rational functions on V when V is affine and irreducible.

Lemma 2.36 *Let $V \subseteq k^n$ be an affine irreducible variety. Let X be the set of pairs (U, f) such that U is a nonempty Zariski open subset of V and f is a regular function on U. Define $(U, f) \sim (U', f')$ if there is some Zariski open $U'' \subseteq U \cap U'$ such that $f|U'' = f'|U''$. (We say that f and f' have the same germ.) Then*
(i) $\sim$ is an equivalence relation,
(ii) if $(U, f) \sim (U', f')$ then already $f|U \cap U' = f'|U \cap U'$.
(iii) The set of $\sim$-equivalence classes is precisely $k(V)$.

Proof. Note that any two nonempty open subsets of V intersect in a nonempty open subset. Thus (i) is clear.
(ii). It is enough to show that if $O \subseteq U \cap U'$ is a principal open subset of V then f agrees with f' on V. By 2.35 $f|0$ is given by P_1/Q_1 where P_1, Q_1 are polyomials and Q_1 is every nonzero on O. Likewise $f'|O$ is given by P_2/Q_2. Now by assumption there is nonempty Zariski open subset O' of O such that f and f' agree on O'. In particular $P_1Q_2 = P_2Q_1$ on O'. It follows (by irreducibility of V) then $P_1Q_2 = P_2Q_1$ on V, in particular on O. Thus $P_1/Q_1 = P_2/Q_2$ on O.
(iii). By 2.35, every (U, f) is $\sim$ to some (U', f') where f' is given by a quotient P/Q of polynomials, and U' is defined by $Q \neq 0$. That is each equivalence class of $\sim$ is "represented" by some quotient P/Q of polynomials. Clearly P/Q and P'/Q' give the same germ if $PQ' = P'Q$ on V iff $P/Q = P'/Q'$ in $k(V)$.

Definition 2.37 *By an (abstract) pre-variety (over k), we mean the following data:*
(i) a set V,
(ii) a covering $V = V_1 \cup .. \cup V_m$ of V.
(iii) for each $i = 1, .., m$, an affine variety $U_i \subseteq k^{n_i}$ and a bijection f_i between V_i and U_i such that for each i, j, $f_i(V_i \cap V_j)$ is a (possibly empty) Zariski open subset of U_i, and $f_j \circ f_i^{-1}|f_i(V_i \cap V_j)$ is an isomorphism between $f_i(V_i \cap V_j)$ and $f_j(V_i \cap V_j)$.
We say that this pre-variety is defined over K if all algebraic-geometric data (the U_i and the transition functions) are defined over K.

Remark 2.38 *It may be more convenient to call the above data on V an atlas on V, to define when two atlases are equivalent (compatible) and to say that the prevariety given above is defined over K if it has a compatible atlas which is defined over K.*

Example 2.39 *(i) An affine variety $V \subseteq k^n$ is a pre-variety (take the covering of V by itself),*
(ii) More generally a quasi-affine variety is a pre-variety. (Write U as a union of principal opens.)
(iii) Projective n-space over k, $\mathbf{P}^n(k)$ is the set of lines in k^{n+1} through the origin (that is 1-dimensional vector subspaces of k^{n+1}). We can think of projective n-space over k as the set of equivalence classes of $n+1$-tuples $(a_0, .., a_n)$ from k such that not all a_i are 0 and where $(a_0, .., a_n)$ is equivalent to $(b_0, .., b_n)$ if there is $\lambda \in k$ such that $b_i = \lambda a_i$ for $i = 0, .., n$. We write $[a_0 : .. : a_n]$ for the equivalence class of $(a_0, .., a_n)$. For each $i = 0, .., n$, let $V_i = \{[a_0 : .. : a_n] : a_i \neq 0\}$. Let $U_i = k^n = \mathbf{A}^n$ and let f_i be the bijection between V_i and k^n given by $f_i([a_0 : .. : a_n]) = (a_0/a_i, .., a_{i-1}/a_i, a_{i+1}/a_i, .., a_n/a_i)$. We leave it as an exercise to show that this gives $\mathbf{P}^n$ the structure of a pre-variety. (One has to check that the $f_i(V_i \cap V_j)$ are open subsets of k^n and that the transition functions are isomorphisms.)

Let us now (in a rather informal manner) adapt the Zariski topology and the notions of regular function, morphism etc. to pre-varieties. Given a pre-variety $(V, V_i, U_i, f_i)_i$ we say that a subset X of V is (Zariski) open if for each i, $f_i(V_i \cap X)$ is a Zariski open subset of the affine variety U_i. We leave it to the reader to check that the Zariski topology on V is Noetherian (DCC on closed sets). Also any open subset of V has naturally the structure of a pre-variety (why?) as does any closed subset subset of V. In particular any pre-variety is a union of finitely many irreducible closed subsets, its irreducible components.

By a regular function on the pre-variety V we mean a function $f : V \to k$, such that for each i, $f \circ f_i^{-1}$ is a regular function on the affine variety U_i. If V' is a Zariski open subset of V, then a map $f : V' \to k$ is regular if for each i, $f \circ f_i^{-1}$ is regular on the Zariski open subset $f_i(V' \cap V)$ of U_i, as defined earlier. We leave it to the reader to check that this coincides with the notion of a regular map on the pre-variety V'.

Remark 2.40 *Let V be a pre-variety. Then V equipped with its Zariski topology, and for each Zariski open $U \subseteq V$, the ring (in fact k-algebra) of regular functions on U, is a sheaf (of k-algebras). What this amounts to is that given open $U \subseteq V$ and a covering U_α of U by open sets, and s_α regular functions on U_α such that $s_\alpha = s_\beta$ on $U_\alpha \cap U_\beta$ then there is a regular function s on U whose restruction to each U_α is s_α.*

The notion of a morphism f between pre-varieties V and W is the obvious one. Assume V is given by data $(V_i, U_i, f_i)_i$ and W by data (W_j, O_j, g_j). Then f should be a map from V to W which is continuous in the Zariski topology, and moreover, for any i, j, $f|(V_i \cap f^{-1}(W_j)) : (V_i \cap f^{-1}(W_j)) \to W_j$, when read in the charts U_i and O_j is a morphism from an open subset of U_i to O_j. Note that this agrees with the notion of a morphism defined earlier between affine or quasi-affine varieties.

If V, W are pre-varieties, then $V \times W$ has a natural structure of a pre-variety (with covering by $V_i \times W_j$'s and bijections $f_i \times g_j$ between $V_i \times W_j$ and $U_i \times O_j$).

The pre-variety V is said to be a variety if it is *separated*, namely the diagonal $\{(x, x) : x \in V\}$ is Zariski closed in $V \times V$. (This is the analogue of Hausdorfness for topological spaces.) Note that affine and quasi-affine varieties are separated. An example of a pre-variety which is not a variety is the affine line with a double point: both charts are $\mathbf{A}^1 = k$, and the transition map identifies only the 2 copies of the line with $\{0\}$ removed.

By a projective variety we mean a variety which is a Zariski closed subset of some $\mathbf{P}^n(k)$ (with the induced structure of a variety). It is a fact that a projective variety, as a subset of $\mathbf{P}^n(k)$ can be defined by a finite system of homogeneous equations.

Definition 2.41 *A variety X is said to be complete if for any other variety, the projection map $\pi : X \times Y \to Y$ is closed, namely takes Zariski-closed sets to Zariski-closed sets.*

Note that any closed subvariety of a complete variety is complete.

Fact 2.42 *Any projective variety is complete.*

Lemma 2.43 *If X is an irreducible complete variety then any regular function on X is constant.*

Proof. Suppose $f : X \to k$ is a regular function. We may assume that f is not identically 0. Then $Z = \{(x, y) \in X \times k : f(x)y = 1\}$ is a nonempty Zariski closed in $X \times k$ (why?). So $\pi(Z)$ is nonempty and closed in k (and clearly irreducible). As $0 \notin \pi(Z)$, this forces $\pi(Z)$ to be a singleton.

Corollary 2.44 *(i) If X is a complete variety, and Y is an irreducible closed subvariety which is affine (that is, isomorphic to an affine variety), then Y is a singleton.*
(ii) Any morphism from an irreducible complete variety to an affine variety is constant.

Proof. (i) Now Y is also complete. If Y were affine and infinite then some coordiate function would give a noncontant regular map on Y, contradicting 2.43.
(ii) Similar.

Lemma 2.45 *Suppose X is a complete irreducible variety, Y, Z are irreducible varieties and $f : X \times Y \to Z$ is a morphism such that for some $y_0 \in Y$, the map $f(-, y_0) : X \to Z$ is constant. Then for all $y \in Y$, the map $f(-, y) : X \to Z$ is constant.*

Proof. Let $z_0 \in Z$ be the common value of $f(x, y_0)$ for $x \in X$. Let U be an affine open neighbourhood of z_0 in Z. Then $V = \{(x, y) \in X \times Y : f(x, y) \notin U\}$ is closed in $X \times Y$. So the projection V_1 of V on Y is closed in Y. As $y_0 \notin V_1$, the complement of V_1 in Y, O say, is open and nonempty. For any $y_1 \in O$, $f(-, y_1)$ is a morphism from X into the affine variety U, hence by 2.44 (ii), is constant. Now pick $x_0, x_1 \in X$. Then $\{y \in Y : f(x_0, y) = f(x_1, y)\}$

is clearly closed in Y and by what we just saw, contains a nonempty open subset of Y. As Y is irreducible, this set is all of Y which is what we want.

Among the important things we want to point out that is for an arbitrary irreducible variety X, the tangent bundle $T(X)$ of X makes sense, and is another variety. Suppose X is covered by affine varieties V_i. So here we mean that the V_i are open subsets of X, each identified (by a given bijection) with a (neccesarily irreducible affine variety). For each i, j, we have an isomorphism $f_{i,j}$ say between an open subset U_i say of V_i and U_j of V_j. We have the affine varieties $T(V_i)$ and $T(V_j)$ defined earlier. Let π_i be the canonical surjection from $T(V_i)$ to V_i, and likewise for π_j. Then $\pi_i^{-1}(V_i)$ and $\pi_j^{-1}(V_j)$ are open subsets of $T(V_i), T(V_j)$ respectively, and we leave it to the reader to check that $df_{i,j}$ is an isomorphism between these two quasi-affine varieties. This gives us the required transition functions between open subsets of the various $T(V_i)$ and yields a variety $T(X)$ together with a surjective morphism $\pi : T(X) \to X$.

The tangent space $T(V)_a$ to V at a is just $\pi^{-1}(a)$. Note on the other hand that this can be obtained wthout constructing the full tangent bundle. Simply consider an affine neighbourhood V_1 of a in V and define the tangent space at a to be $T(V_1)_a$. The point $a \in V$ (V irreducible) is said to be nonsingular on V if $dim(V) = dim(T_V)_a$. V is smooth or nonsingular if all points on it are nonsingular.

Let us now discuss rational functions. Let V be an irreducible variety. Consider regular functions from (nonempty) Zariski open subsets of V to k. As in the affine case, call two such functions $(U_1, f_1), (U_2, f_2)$ equivalent if f_1 agrees with f_2 on some nonempty Zariski open of $U_1 \cap U_2$. We will call the set of equivalence classes the field of rational functions $k(V)$ of V. Note that if V' is one of the affine charts on V then $k(V) = k(V')$ (why?)

Let W be another irreducible variety. We will say that V and W are birationally isomorphic if there is an isomorphism f between nonempty Zariski open subsets U, U' of V, W respectively. One of the main aims of algebraic geometry is the classification of algebraic varieties up to birational isomorphism.

Fact 2.46 *V and W are birationally isomorphic if $k(V)$ is isomorphic to $k(W)$ over k (as fields).*

It is also important to see that all these objects, varieties, morphisms,.. are *definable* in the algebraically closed field k. On the face of it, they live in $(k, +, \cdot)^{eq}$. For example if the variety is given by data $(V_i, U_i, f_i)_i$, then we can identify V set-theoretically with the disjoint union of the affine varieties U_i quotiented out by a definable equivalence relation. By elimination of imaginaries this latter definable set in k^{eq} is in definable bijection with some definable subset X of some k^n. If the original variety V was defined over K, X will be too. Likewise if W is another variety with corresponding definable set Y then a morphism from V to W gives rise to a definable map from X to Y. With this notation we have:

Exercise 2.47 *If V has dimension m then X has Morley rank m.*

On the other hand:

Fact 2.48 *(i) Let X be a definable subset of k^n. Let $\bar{X}$ be the Zariski closure of X. Then X contains a Zariski-open, Zariski-dense subset of $\bar{X}$. Moreover $RM(\bar{X} \setminus X) < RM(X)$.*
(ii) (characteristic zero.) Let $X \subset k^n$, $Y \subset k^m$ and let f be a definable function from X to Y. Then there is $U \subset X$ such that U is Zariski-open and Zariski-dense in $\bar{X}$ and $f|U$ is a morphism from U to $\bar{Y}$.

Proof. (i) is by quantifier-elimination in algebraically closed fields. (ii) can be dediced from (i) and the fact that definable functions are "piecewise rational".

In any case we will often identify a variety with the corresponding definable set and we will simply talk about points of the variety as if they were tuples from k. If V is a variety defined over K, and $K < L < k$ then $V(L)$ denotes those points of V all of whose coordinates are in L. With this convention we have the following, which should be seen as the analogue of 2.48 when k^n is replaced by an arbitrary algebraic variety"

Exercise 2.49 *(i) Let V be a variety, and X a definable subset of V. Then X is a Boolean combination of closed sets, X contains a Zariski-dense Zariski-open subset of its Zariski closure $\bar{X}$ in V, and $RM(\bar{X} \setminus X) < RM(X) = RM(\bar{X})$,*
(ii) (char. 0.) Let W be another variety, and $f : V \to W$ a definable function. Then there is a Zariski-open Zariski-dense subset U of V such that $f|U$ is a morphism from U to W.

Definition 2.50 *(i) By an algebraic group we mean an variety X equipped with a pair of morphisms $mult : X \times X \to X$, and $inv : X \to X$ and a distinguished point $id \in X$ such that $(X, mult)$ is a group, inv is group inversion and id is the identity element of the the group. We will say that the algebraic group is defined over K if all this data is defined over K.*
(ii) By an algebraic subgroup of an algebraic group G we mean a subgroup H of G which is also a Zariski closed subset of G.
(iii) By a morphism of algebraic groups we mean a morphism of varieties which is also a group homomorphism.

Remark 2.51 *Any algebraic group G is smooth.*

Proof. For any $a, b \in G$ there is an isomorphism (of algebraic varieties) taking a to b (left multiplication by ba^{-1}). So as some point on G is smooth, every point is.

Example 2.52 *The general linear group over k, $GL(n, k)$ is an algebraic group. On the face of it the underlying set of $GL(n, k)$ is the principal open subset of k^{n^2} defined by the vanishing of the determinant. In any case, multiplication is given by a polynomial map and inversion by an everywhere defined rational map. On the other hand, we can view $GL(n, k)$ as the affine variety $\{x, y) \in k^{n+1} : det(x)y = 1\}$, and multiplication and inversion are now given by polynomial functions.*

Definition 2.53 *By a linear algebraic group, we mean a closed subgroup of some $GL(n, k)$.*

At the opposite extreme we have abelian varieties.

Definition 2.54 *An abelian variety is a complete irreducible algebraic group.*

Remark 2.55 *Any abelian variety is a commutative group.*

Proof. Write the group operation on G multiplicatively, and let e denote the identity. The map $f : G \times G \to G$ given by $f(x, y) = xyx^{-1}y^{-1}$ is a morphism and $f(-, e)$ is constant. Hence by 2.45, for any $y \in G$, $f(-, y)$ is constant,so (by taking $x = e$) as constant value e. So G is commutative.

Abelian varieties are closely related to algebraic curves: any (smooth, projective) algebraic curve C which is not birational to $\mathbf{A}^1$ embeds as a closed subvariety of some abelian variety $J(C)$ which it actually generates under addition. Elliptic curves are precisely 1-dimensional abelian varieties.

Example 2.56 *Consider the projective variety E defined by $y^2z = x^3 + axz^2 + bz^3$ where the polynomial $x^3 + ax + b$ has distinct zeroes. This can be thought of as the affine curve defined by $y^2 = x^3 + ax + b$ (embedded in $P^{\mathbf{2}}$ by $(x, y) \to [x, y, 1]$) together with the point at infinity $[0, 1, 0]$. Let O denote the point at infinity. Define a binary operation on E as follows: for $P, Q \in E$, let L be the line through P and Q (tangent to P if $P = Q$), and let R be the third point on the intersection of L with E. Let L' be the line through O and R. Then $P \oplus Q$ is the third point on the intersection of L' with E. Then $(C, \oplus)$ is an abelian variety of dimension 1.*

Elliptic curves give among the easiest examples of definable families of essentially mutually nonisomorphic varieties. For example, the elliptic curves E_λ given by $y^2 = x(x-1)(x-\lambda)$ as λ varies include (up to isomorphism) all elliptic curves, and E_λ is isomorphic to $E_{\lambda'}$ just if $j(\lambda) = j(\lambda')$ where $j(\lambda) = (\lambda^2 - \lambda + 1)^3/\lambda^2(\lambda - 1)^2$.

Note that an algebraic group G is in particular a group definable in k. We will see that there is a close relationship between the algebraic-geometric and definability properties of G.

Lemma 2.57 *Let G be an algebraic group, and H a definable subgroup of G. Then H is closed in G, hence an algebraic subgroup.*

Proof. Let X be the Zariski-closure of H in X. Then X is a (definable) subgroup of G (Why??) By Exercise 2.49, $RM(X \setminus H) < RM(X)$, hence by 4.27 of [3], $X = H \cdot H = H$.

Lemma 2.58 *(characteristic zero.) Let G and H be algebraic groups. Then any definable homomorphism f from G to H is a morphism.*

Proof. By 2.49 (ii), there is a nonempty open subset U of G such that $f|U$ is a morphism. Fix $a \in G$, then the translate aU of U by a is also open, and $f|aU$ is also a morphism. (For $u \in a.U$, $f(u) = f(a).f(a^{-1}u)$ and $a^{-1}u \in U$.) Hence f is locally rational so a morphism.

Lemma 2.59 *Let G be an algebraic group. Then the connected component of G in the model-theoretic sense coincides with the irreducible component of the variety G which contains the identity.*

Proof. Let G^0 be the model-theoretic connected component of G (smallest definable subgroup of finite index). By the previous lemma, G^0 is closed (as are each of its finitely many translates). So all we have to see is that G^0 is irreducible. It is rather easy to see that G^0 acts (by left multiplication say) transitively and definably on its set of irreducible components. So the set of $g \in G^0$ which leave each irreducible component invariant is a definable subgroup of G^0 of finite index, hence equals G^0. So there is only one irreducible component G^0 itself.

By the above lemma, the translates of G^0 coincide with the connected components of G in the Zariski topology, hence the expression "connected component".

A basic result connecting the definable and geometric categories is:

Proposition 2.60 *Any group definable in $(k, +, \cdot)$ is definably isomorphic to an algebraic group.*

Let us sketch a proof of this, in the characteristic zero case, making crucial use of a result of Weil.

Let G be our definable group. There is no harm in assuming G to be connected in the model-theoretic sense. Assume $G \subset k^n$. Let K be an algebraically closed subfield of k such that G is K-definable, and we assume k to be $|K|^+$-saturated. G has a unique generic type $p(x)$ over K. (p is the unique type over K of maximal Morley rank $= m$ say of G, and p is stationary.) Let $f : G \times G \to G$ be the group operation. Let $\bar{G}$ be the Zariski-closure of G (also defined over K). By 2.48 there is $U \subseteq G \times G$ such that U is Zariski-open and Zariski-dense in $\bar{G} \times \bar{G}$ and $f : U \to \bar{G}$ is a morphism. We can choose U to be K-definable. Now $\bar{G}$ may have several irreducible components. But one of the irreducible components has p as its "generic type". Call this irreducible component X. Replace U by $\{u \in U : f(u) \in X\} \cap (X \times X)$. Then with this new U we have that $U \subset G \times G$ is Zariski-open (and dense) in $X \times X$, and $f|U : U \to X$ is a morphism. Note that p is the generic type of X ($a \in X$ is generic over K iff $tp(a/K) = p$.) Note also that if a, b realize p

and are independent over K then $(a,b) \in U$, in fact (a,b) is a generic point of (the irreducible quasi-affine variety) U over K. Summarizing we have: an irreducible affine variety X defined over K, and a morphism f from a Zariski-open subset U of $X \times X$ (defined over K) to X such that
(i) if (a,b) is generic in $X \times X$ over K, and $c = f(a,b)$ then $K(a,b) = K(a,c) = K(b,c)$ and
(ii) if (a,b,c) is a generic of $X \times X \times X$ over K, then $f(f(a,b),c) = f(a,f(b,c))$.
¿From this data, Weil's theorem produces a connected algebraic group $(H,\cdot)$ defined over K and a birational isomorphism h between X and H, such that for generic $(a,b) \in X \times X$ over K, $h(f(a,b)) = h(a) \cdot h(b)$. In particular h is a (definable, partial) map from G to H such that $h(a)$ is defined whenever a realizes p, and for for K-independent realizations a,b of p, $h(ab) = h(a) \cdot h(b)$. For any $c \in G$, choose a,b realizing p such that $c = ab$. Define $h'(c) = h(a) \cdot h(b)$. Then h' is a definable isomorphism of G with H.

Among the consequences of Proposition 2.60 is a rather straightforward proof that the quotient of an algebraic group by a normal algebraic subgroup is also an algebraic group. Namely, suppose G is an algebraic group, and N a normal algebraic subgroup. Them there is another algebraic group H and a surjective homomorphism $h : G \to H$ of algebraic groups, whose kernel is precisely N. For G/N is a definable group, hence by 2.60 definably isomorphic to an algebraic group H. We obtain a definable surjective homomorphism $f : G \to H$ whose kernel is N. By 2.58, f is a morphism.

Let us complete this section with some remarks on the structure of algebraic groups. We will assume k has characteristic zero, for simplicity. In general we call an algebraic linear (abelian variety), if it is isomorphic to a linear algebraic group (abelian variety).

Fact 2.61 *(i) (Chevalley) Suppose G is a connected algebraic group defined over $K < k$. Then there is a unique maximal normal connected linear algebraic subgroup L of G (defined over K too), and G/L is an abelian variety.*
(ii) Let G be a commutative connected linear algebraic group, defined over K. Then G has unique connected algebraic subgroups N, T, also defined over K, such that $G = N \cdot T$, $N \cap T = \{1\}$, N is a "vector group", namely N is isomorphic to $(k,+)^d$ for some d, and T is a "algebraic torus", namely T is

isomorphic to $(k^*, \cdot)^s$ *for some* s.

Definition 2.62 *By a semiabelian variety we mean a connected algebraic group* G *which is an extension of an abelian variety by an algebraic torus. (That is there is* $T < G$ *an algebraic torus, and* G/T *is an abelian variety.)*

Often we write commutative algebraic groups in additive notation.

Fact 2.63 *Let* A *be a semiabelian variety. Then* A *is divisible and for any* n *the* n*-torsion subgroup of* $A = \{x \in A : nx = 0\}$ *is finite. Moreover the torsion subgroup of* A *is Zariski-dense in* A.

In the special case where A *is an abelian variety of dimension* d*, then for each prime* p*, the group of* p*-torsion elements of* A *is* $(\mathbf{Z}/p\mathbf{Z})^{2d}$*. Assuming* k *to be the field of complex numbers, then* A *has the structure of a real Lie group, and as such is isomorphic to* $(S^1)^{2d}$ *where* S^1 *is the circle group.*

Corollary 2.64 *Semiabelian varieties satisfy the following rigidity property: assume* A *to be a semi-abelian variety defined over* K*. Then any connected algebraic subgroup of* A *is defined over* $acl(K)$.

Proof. Let $B < A$ be a connected algebraic subgroup of the semi-abelian variety A. Then the torsion of B is contained in the torsion of A so by 2.63, contained in $B(acl(K))$. As B is also a semiabelian variety, the torsion of B is Zariski-dense in B. So any automorphism of k fixing $acl(K)$ pointwise leaves B invariant. Hence B is defined over $acl(K)$.

We have discussed stabilizers of types in the context of ω-stable groups. This of course applies to algebraic groups too.

Exercise 2.65 *Let* G *be an algebraic group and* X *an irreducible closed subset of* G*, all defined over* K*. Let* $p(x) \in S(K)$ *be the generic type of* X *(that is* $p(x)$ *says* $x \in X$ *and* $x \notin Y$ *for* Y *any proper Zariski-closed subset of* X *defined over* K*). Then* $Stab(p)$ *coincides with the "set-theoretic" stabilizer* $\{g \in G : g \cdot X = X\}$ *of* X *in* G.

Finally we consider tangent bundles of algebraic groups.

Fact 2.66 *Suppose G is a connected algebraic group, defined over K. Let $f : G \times G \to G$ denote the group operation $(g, h) \to gh = f(g, h)$. Let $T(G)$ be the tangent bundle of G (an irreducible algebraic variety also defined over K). Define the following binary operation on $T(G) : (g, u) \cdot (h, v) = (gh, df_{(g,h)}(u, v))$. Then*
(i) $(T(G), \cdot)$ is an algebraic group, defined over K, and the canonical surjection $\pi : T(G) \to G$ is a homomorphism of algebraic groups.
(ii) For $g \in G$, write $\lambda^g : G \to G$ and $\rho^g : G \to G$ for left and right multiplication respectively by g. Then $(g, u) \cdot (h, v) = (gh, d(\lambda^g)_h(v) + d(\rho^h)_g(u))$.

3 Finite Morley rank sets and groups in differentially closed fields

We will make use of the algebraic geometric language and results from the previous section to shed some more light on the structure of sets of finite Morley rank definable in differentially closed fields. As in section 1, we fix a saturated differentially closed field $\mathcal{U}$. We have both the language of rings L_r and the language of differential rings L_∂. By definable we usually mean L_∂-definable. On the other hand $\mathcal{U}$ is also an algebraically closed field of characteristic zero, and so taking k from the last section to be $\mathcal{U}$, we have algebraic varieties, morphisms,.. at our disposal too.

The first aim is to give a canonical form for stationary types of finite Morley rank (finite order).

Definition 3.1 *Let $X \subseteq \mathcal{U}^n$ be an irreducible affine algebraic variety defined over the (small, differential) subfield K of $\mathcal{U}$. By $\tau(X)$ we mean the subset of $\mathcal{U}^{2n}$ defined by:*
(i) $x \in X$,
(ii) $(\partial P/\partial x_i)(x)v_i + P^\partial(x) = 0$, for all $P(x) \in I_K(X)$.

Lemma 3.2 *(i) $\tau(X)$ is an affine algebraic variety defined over K, and comes equipped with a surjective morphism π (over K) to X.*
(ii) Let $a \in X$. Then $(a, \partial(a)) \in \tau(X)$.
(iii) Let $\pi_0 : T(X) \to X$ be the tangent bundle of X. Then for each $a \in X$, $T(X)_a$ acts strictly transitively on $\tau(X)_a$ by addition. In fact these actions fit together to give a morphism $T(X) \times_X \tau(X) \to \tau(X)$ which sends

$((a,u),(a,v))$ to $(a,u+v)$.
(iv) Let Y be another irreducible variety defined over K and $f : X \to Y$ a morphism defined over K. For $a \in X$, define $\tau(f)_a$ to be $df_a + f^{\partial}(a)$. Namely, assuming f is given by the sequence $(f_1,..,f_m)$ of regular functions on X, $\tau(f)_a(u) = ((df_1)_a(u) + f_1^{\partial}(a),..,(df_m)_a(u) + f_m^{\partial}(a))$ for $(a,u) \in \tau(X)$. Then the map taking (a,u) to $(f(a),\tau(f)_a(u))$ is a morphism from X to Y which we sometimes just call $\tau(f)$.

Proof. Clear.

By virtue of (iv) above, we can define $\tau(X)$ for any irreducible (abstract) variety. We just piece together the $\tau(V_i)$ for open affine parts of X, and use isomorphisms given by (iv) above to define the transition maps.

Lemma 3.3 *Let V be an irreducible variety (over $\mathcal{U}$), W an irreducible variety of $\tau(V)$ such that $\pi|W : W \to V$ is dominant ($\pi(W)$ is Zariski-dense in V, equivalently contains a nonempty Zariski open of V). Let U be a nonempty Zariski open subset of V. THEN there is $a \in V$ such that $(a,\partial(a)) \in W$.*

The above lemma can be considered to be give an alternative system of axiom for DCF_0.

Remark 3.4 *(i) By an algebraic D-variety (over the differential subfield K of $\mathcal{U}$) we mean an (irreducible) algebraic variety X defined over K together with a morphism $s : X \to \tau(X)$ (of algebraic varieties, defined over K) which is a section of $\pi : \tau(X) \to X$ (that is $\pi \circ s$ is the identity on X.*
(ii) Let (X,s) be an algebraic D-variety defined over s (affine if you wish). Let $(X,s)^{\sharp} = \{a \in X : s(a) = (a,\partial(a))\}$. Then $(X,s)^{\sharp}$ is a (L_{∂}-) definable Zariski-dense subset of X.
(iii) Let (X,s) be an algebraic D-variety defined over K. By a generic point of $(X,s)^{\sharp}$ over K we mean $a \in (X,s)^{\sharp}$ such that a is generic in X over K in the sense of algebraic geometry (or ACF). Then there exists a generic point of $(X,s)^{\sharp}$ over K and moreover the type of such a point over K is unique.

Proof. (ii) Note that $\{s(a) : a \in X\}$ is closed and irreducible and projects onto X. Hence by the previous lemma for any open subset U of X there is $a \in U$ such that $s(a) = (a,\partial(a))$ which is precisely what we want.
(iii) By saturation of $\mathcal{U}$ and (ii) there is some generic point a of $(X,s)^{\sharp}$ over

K. Note that $tp_{L_r}(a/K) = p_0$ is uniquely determined as X is irreducible. Now s is a polyomial map. So $p_0(x) \cup \{\partial(x) = s(x)\}$ determines a unique L_∂-type over K.

Our general point will be that any definable set of finite Morley rank is essentially of the form $(X, s)^\sharp$ (for (X, s) some algebraic D-variety.

For a a finite tuple, we will say $tp(a/K)$ has finite order if $tr.deg(K(a, \partial(a), \partial^2(a), ..)/K)$ is finite, and moreover we define the order of $tp(a/K)$ to be this transcendence degree.

Lemma 3.5 *Let $K < \mathcal{U}$ be algebraically closed. Let a be a finite tuple such that $tp(a/K)$ has finite order. Then there is some (irreducible) algebraic D-variety (X, s) defined over K, and some generic point b of $(X, s)^\sharp$ over K such that a is interdefinable with b over K. Moreover $dim(X) = order(tp(a/K))$.*

Proof. Replacing the tuple a by the tuple $(a, \partial(a), .., \partial^r(a))$ for some r we may assume that the tuple $\partial(a)$ is (field-theoretically) algebraic over $K(a)$. Suppose $a = (a_1, .., a_n)$. By 1.2, for each $i = 1, ., , n$, $\partial^2(a_i) \in K(a, \partial(a))$ say, $\partial^2(a_i) = F_i(a, \partial(a))/G_i(a, \partial(a))$ where F_i, G_i are polynomials over K. Let b be the finite tuple $(a, \partial(a), (G_1(a, \partial(a)))^{-1}, .., (G_n(a, \partial(a)))^{-1})$. Then it is not hard to see that there is a polynomial s over K such that $\partial(b) = s(b)$.

Let X be the irreducible affine variety over K whose generic point is b. By 3.2 (ii) $(b, s(b)) \in \tau(X)$. So for all $c \in X$, $(c, s(c)) \in \tau(X)$ (why?). Thus (X, s) is an algebraic D-variety over K, and b is a generic point of $(X, s)^\sharp$ over K. Clearly b is interdefinable with a over K. Moreover $dim(X) = tr.deg(K(b)/K) = tr.deg(K(b, \partial(b), ...)/K) = tr.deg(K(a, \partial(a), ..)/K) = order(tp(a/K))$.

Corollary 3.6 *Let a be a finite tuple, and $K < \mathcal{U}$. Then the following are equivalent:*
(i) $tp(a/K)$ has finite order.
(ii) $tp(a/K)$ has finite Morley rank.
(iii) $tp(a/K)$ has finite U-rank.

Proof. Suppose $a = (a_1, .., a_n)$. Notice that
(a) if $tp(a/K)$ has finite U-rank then so does $tp(a_i/K)$ for each i.
(b) $tp(a/K)$ has finite order iff $tp(a_i/K)$ has finite order for each $i = 1, .., n$.

As also $U(tp(a/K)) \leq RM(tp(a/K)$, we see by Corollary 1.18 (ii) that (ii) implies (iii) implies (i). So all we need to prove is that (i) implies (ii). In fact we will prove that $RM(tp(a/K)) \leq order(tp(a/K))$. We may assume K is algebraically closed and even saturated. By Lemma 3.5, we may assume a is a generic point of $(X,s)^\sharp$ over K for some algebraic D-variety defined over K. Suppose $dim(X)=m$. Then $ord(tp(a/K))=m$. Note that $tp(a/K)$ is determined (axiomatized) by "$x \in X$" $\wedge\, \partial(x)=s(x) \cup \{\neg($"$x \in Y$" $\wedge\, \partial(x)=s(x)) : Y$ a proper irreducible subvariety of X defined over $K\}$. By induction hypothesis, each of the formulas "$x \in Y$" $\wedge\, \partial(x)=s(x)$ has Morley rank $< m$. Hence $tp(a/K)$ is axiomatized by a single formula together with a collection of negations of formulas of Morley rank $< m$. It follows that $RM(tp(a/K)) \leq m$.

We now aim towards understanding canonical bases of types of finite Morley rank. Specifically we will prove:

Theorem 3.7 *Suppose $K < L$ are (small) algebraically closed (differential) subfields of $\mathcal{U}$. Suppose that $tp(a/K)$ has finite Morley rank. Let $c = Cb(tp(a/L))$. Then $tp(c/K,a)$ is internal to $\mathcal{C}$. Namely there is some tuple d such that d is independent from c over $K \cup \{a\}$ and $c \in dcl(K \cup \{a,d,e\})$ for e some tuple from $\mathcal{C}$.*

Proof. We may assume, by 3.5, that $a \in \mathcal{U}^n$ is the generic point over K of $(X,s)^\sharp$ for some (affine) algebraic D-variety (X,s) defined over K. Let Y be the irreducible subvariety over L whose generic point over L (in the sense of algebraic geometry) is a. Note that $(Y, s|Y)$ is an algebraic D-variety (why?), hence a is a generic point of $(Y,s|Y)^\sharp$ over L. Let q be the nonforking extension of $tp(a/L)$ over $\mathcal{U}$. Then $order(q) = order(tp(a/L)) = dim(Y) = m$. $q(x)$ is determined by the data $\partial(x) = s(x)$ and $x \in Y$ and $x \notin Z$ for every irreducible variety $Z \subset \mathcal{U}^n$ of dimension $< m$.

Claim I. Let f be an automorphism of $\mathcal{U}$ fixing K-pointwise. Then $f(q)=q$ if $f(Y)=Y$.

Proof. Note that $\partial(x)=s(x) \in f(q)$ as f fixes K pointwise. Also $order(f(q)) = m$. So if $f(Y)=Y$, then $order(f(q))=m$ and $x \in Y \wedge \partial(x)=s(x) \in f(q)$ hence $f(q)=q$. On the other hand if $f(q)=q$ then $x \in (Y \cap f(Y)) \in q(x)$, hence $f(Y) \cap Y$ has dimension m, so $f(Y)=Y$ (as Y is irreducible).

So by Claim I, $Cb(tp(a/L))$ is interdefinable over K with the canonical parameter for Y. We now look back at Lemma 2.22. We have for each r a surjective linear map $(i_Y^\sharp)_r$ from $\mathcal{M}_{\mathbf{A}^n,a}/(\mathcal{M}_{\mathbf{A}^n,a})^r$ to $\mathcal{M}_{Y,a}/(\mathcal{M}_{Y,a})^r$, and Y

is determined by the sequence $(Ker(i_Y^\sharp)_r) : r = 2, 3, ..)$. We will leave it as an exercise to strengthen Lemma 2.22 as follows (where $\mathbf{A}^n$ is replaced by the variety X).
(*) Let V_r denote the vector space $\mathcal{M}_{X,a}/(\mathcal{M}_{X,a})^{r+1}$ and W_r the vector space $\mathcal{M}_{Y,a}/(\mathcal{M}_{Y,a})^{r+1}$, and f_r the canonical linear map from V_r to W_r. Then f_r is surjective (this uses the fact that a is a generic point of X), and Y is determined by the sequence $(Ker(f_r) : r = 1, 2, ..)$.

Now we leave it to the reader to see how V_{r+1} is naturally a definable $\mathcal{U}$-vector space, defined over $K(a)$ (in L_r), likewise W_{r+1} is definable over $L(a)$ and the map f_r definable over $L(a)$. In particular $Ker(f_r)$ is a definable subspace of V_{r+1}, so has a canonical parameter. So from Claim I, (*), and compactness we see:
Claim II. $Cb(tp(a/L))$ is interdefinable over $K \cup \{a\}$ with the canonical parameter of $Ker(f_r)$ for large enough r.

In order to prove the Theorem 3.7 from Claim II, we will show that V_r and W_r are naturally equipped with the structure of ∂-modules, that f_r is a ∂-module map, and thus $Ker(f_r)$ is a ∂-submodule of V_r. The theory of ∂-modules (linear differential equations), will yield that the type of the canonical parameter of $Ker(f_r)$ over $K \cup \{a\}$ is internal to $\mathcal{C}$.

So let us first discuss ∂-modules.

Definition 3.8 *Let (K, ∂) be a differential field. By a ∂-module over K we mean a finite-dimensional K-vector space V together with an additive homomorphism $D_V : V \to V$ satisfying*
$D_V(\lambda v) = \partial(\lambda)v + \lambda D_V(v)$ for all $\lambda \in K$ and $v \in V$.

Remark 3.9 *Let (K, ∂) be a differential field.*
(i) Let A be an $n \times n$ matrix over K, and $V = K^n$. Then $(V, \partial - A)$ is a ∂-module over K (where we think of elements of V as column vectors, with ∂ acting coordinatewise and A acting by left multiplication).
(ii) Conversely, if (V, D_V) is a ∂-module over K such that $dim_K(V) = n$, then with respect to any basis of V over K, (V, D_V) becomes $(K^n, \partial - A)$ for some $n \times n$-matrix A over K.

Proof. Exercise.

One of the basic results on linear differential equations is:

Lemma 3.10 *Let (K, ∂) be a differentially closed field, and (V, D_V) a ∂-module over K.*
(i) Then there is a basis $e_1, .., e_n$ of V over K, consisting of solutions of $D_V(v) = 0$. Moreover
(ii) The solution set V^∂ of $D_V = 0$ is precisely the C_K-subspace of V spanned by $e_1, .., e_n$.

Proof. (i) By Remark 3.9, we may assume that $V = K^n$ and $D_V = \partial - A$ for some $n \times n$ matrix over K. Consider the variety $X = K^{n^2} = \mathbf{A}^{n^2}$ which we identify with the set of $n \times n$ matrices over K. Then $\tau(X) = T(X) = K^{2n^2}$. Moreover left matrix multiplication by A defines a section of $\tau(X) \to X$. By Lemma 3.3 (although this is a bit of overkill), for any Zariski open subset O of X, there is $x \in O$ such that $\partial(x) = Ax$. In particular if we take O as defined by $det(x) \neq 0$, then we find a nonsingular $n \times n$ matrix U over K such that $\partial U = AU$. If $u_1, ..., u_n$ are the columns of U then $\partial(u_i) = Au_i$ for each $i = 1, , .n$, so the u_i are solutions of $D_V = 0$. Clearly $\{u_1, ., u_n\}$ is also a basis of the K-vector space K^n. We have proved (i).
(ii) Let $e_1, .., e_n$ be the basis of V over K given by (i). Let $v \in V$. So $v = c_1e_1 + ... + c_ne_n$ for some $c_1, .., c_n \in K$. Then $D_V(v) = \partial(c_1)e_1 + .. + \partial(c_n)e_n$. Hence $D_V(v) = 0$ iff $c_i \in C_K$ for $i = 1, .., n$. This yields (ii).

If (V, D_V) is a ∂-module over a differential field (K, ∂), then there is a natural notion of a ∂-submodule of V: it should be just a K-vector subspace W of V such that $D_V|W : W \to W$. There is also a natural notion of a ∂-module map between ∂-modules and note that the kernel of such a map will be a ∂-submodule.

Lemma 3.11 *Let (K, ∂) be a differentially closed field. Give $V = K^n$ the trivial ∂-module structure $D_V = \partial$. Let W be a ∂-submodule of K^n. Then W is defined over C_K.*

Proof. By 3.10, W has a K-basis $v_1, .., v_s$ such that $\partial(v_i) = 0$ for $i = 1, .., s$, namely that each $v_i \in (C_K)^n$. So clearly W is defined over C_K.

We now return to the proof of Theorem 3.7. We follow the notation in the discussion following its statement. So $a \in \mathcal{U}^n$ is a generic point over K of the irreducible affine algebraic variety X (defined over K), a is also a generic point over L of the irreducible subvariety Y of X (defined over L), and $\partial(a) =$

$s(a) = (s_1(a), .., s_n(a))$ where s is a polynomial function defined over K. Also recall the definitions of V_{r+1} and W_{r+1} (higher-dimensional cotangent spaces of X, Y repectively at a). Let $\mathcal{U}[X]$ be the coordinate ring of X. We can use s to equip this coordinate ring with the structure of a differential ring (extending the derivation on $\mathcal{U}$). Namely, for $f(x_1, .., x_n) \in \mathcal{U}[X]$, define $\partial'(f) = f^{\partial}(x) + \sum_{i=1,..,n}(\partial f/\partial x_i)(x)s_i(x)$. (Why is this a derivation?)

Lemma 3.12 *$\mathcal{M}_{X,a}$ is a differential ideal of $(\mathcal{U}[X], \partial')$ as is every power of it.*

Proof. Let $f \in \mathcal{U}[X]$ be zero at a. Then $\partial(f(a)) = 0$. But by 1.2, the definition above of ∂' and the fact that $\partial(a) = s(a)$, we see that $\partial(f(a))$ is precisely $(\partial'(f))(a)$. Hence $\mathcal{M}_{X,a}$ is closed under ∂'. It is easily follows that each $(\mathcal{M}_{X,a})^{r+1}$ is too.

¿From Lemma 3.12, ∂' equips each $\mathcal{U}$-vector space V_r with a ∂-module structure D_{V_r} say.

Now we can do exactly the same thing with Y in place of X to equip each $\mathcal{U}$-vector space W_r with a ∂-module structure D_{W_r}. We leave it to the reader to prove:

Lemma 3.13 *$f_r : V_r \to W_r$ is a ∂-module map.*

Recall that c was the canonical base of $tp(a/L)$. By Lemma 3.10, for each r we can choose a basis d_r for V_r over $\mathcal{U}$ consisting of solutions of $D_{V_r} = 0$, and we can clearly choose d_r to be independent (in DCF_0) of c over $K(a)$.

Now fix r as in Claim II. With respect to the basis d_r, (V_r, D_{V_r}) identifies with $(\mathcal{U}^n, \partial)$. By 3.11 and 3.13, $Ker(f_r)$ is defined over d_r together with some tuple e of constants (as well as $K(a)$). So the canonical parameter of $Ker(f_r)$ is in $dcl(K, a, d_r, e)$. By Claim II, this shows that $tp(c/K, a)$ is internal to $\mathcal{C}$. Theorem 3.7 is proved.

As $\mathcal{C}$ is a strongly minimal field definable in $\mathcal{U}$, we obtain from Theorem 3.7, a positive answer to Conjecture ZC3 from [3] for DCF_0.

Let us explain how this yields the appropriate version of (ZC1) in our context:

Corollary 3.14 *Let $p(x) \in S(A)$ be a stationary type of U-rank 1 in $\mathcal{U}$, the saturated model of DCF_0. Suppose $p(x)$ is nonmodular. Then p is nonorthogonal to the strongly minimal set $\mathcal{C}$, namely to the unique nonalgebraic stationary type $p_0(x)$ over $\emptyset$ containing the formula $x \in \mathcal{C}$.*

Proof. Suppose $p(x)$ is non locally modular. By Exercise 5.7 and Theorem 3.35 of [3], there are realizations a, b of p and some $B \supset A$ such that if $c = Cb(stp(a, b/B)$, then $c \notin acl(A, a, b)$. By Theorem 3.7, $stp(c/A, a, b)$ is internal to $\mathcal{C}$. In particular, for some $D \supseteq (A \cup \{a, b\})$ independent of c over A, a, b there is e realizing $p_0|D$ such that c and e are interalgebraic over D. As $tp(c/D)$ is internal to p it follows that p is nonorthogonal to p_0. (Why?)

Theorem 3.7 is rather more powerful than Corollary 3.14. Here is another consequence which cannot be formally obtained from 3.7 (as far as I know).

Corollary 3.15 *Let G be a group of finite Morley rank, definable in $\mathcal{U}$ over A. Let $p(x) = tp(g/A)$ be stationary, where $g \in G$. Suppose that $Stab(p)$ is trivial (the identity). Then p is internal to $\mathcal{C}$.*

Proof. Let $h \in G$ be generic in G over $A \cup \{g\}$. By Lemma 4.19 of [3], g is interdefinable with $d = Cb(stp(h/A \cup \{g.h\})$ over $A \cup \{h\}$. By Theorem 3.7, $stp(d/A \cup \{h\})$ is internal to $\mathcal{C}$. Thus $tp(g/A \cup \{h\})$ is internal to $\mathcal{C}$. As g is independent from h over A, p is internal to $\mathcal{C}$.

Corollary 3.16 *Let G be a connected group of finite Morley rank, definable in $\mathcal{U}$ over A. Suppose that $p(x) = tp(g/A)$ is stationary and has trivial stabilizer. Let X be the set of realizations of $p(x)$ and suppose that $g^{-1} \cdot X$ generates G. Then there is an algebraic group H defined over $\mathcal{C}$ such that G is definably isomorphic to $H(\mathcal{C})$.*

Proof. By Proposition 5.17 of [3], for some $n < \omega$, every element of G is of the form $h_1 \cdot h_2 .. \cdot h_n$ for $h_i \in g^{-1} \cdot X$, so of the form $g^{-1} \cdot a_1 .... \cdot g^{-1} \cdot a_n$ where the a_i realize p. By Corollary 3.15, p is internal to $\mathcal{C}$. By Lemma 3.41 of [3], there is small $B \supset A$ (which we may suppose contains g) such that every realization of p is in the definable closure of B together with some elements of $\mathcal{C}$. It follows that every element of G is in the definable closure of B together with some elements of $\mathcal{C}$. By compactness there is some B-definable partial function f such that every element of G is of the form $f(c)$ for some tuple c from $\mathcal{C}$.

Let Y be the set of tuples c from $\mathcal{C}$ such that $f(c)$ is defined and in G, and let E be the equivalence relation on Y defined by $f(c_1) = f(c_2)$. Then both Y and E are definable in the structure $(\mathcal{C}, +, \cdot)$ (by Example 3.7 of [3]). So we have a definable bijection f between G and Y/E. This transports the group structure on G to a group structure on Y/E, definable in $\mathcal{U}$. As above, this group structure is definable in $(\mathcal{C}, +, \cdot)$. By Proposition 2.60 (and elimination of imaginaries in algebraically closed fields), Y/E together with its group structure is definably isomorphic to a (connected) algebraic group in $\mathcal{C}$. This latter group can be considered as $H(\mathcal{C})$ for some algebraic group in $\mathcal{U}$ which is defined over $\mathcal{C}$. Putting everything together we have a definable (in L_∂) isomorphism of G with $H(\mathcal{C})$.

To complete these notes we will show how Corollary 3.16 yields a rather fast proof of the Mordell-Lang conjecture for function fields in characteristic zero.

We will first need a few more remarks/facts about definable groups in $\mathcal{U}$. Definability means as usual L_∂-definability unless we say otherwise.

Lemma 3.17 *Any definable subgroup of $(\mathcal{U}, +)^d$ is a $\mathcal{C}$-vector space.*

Proof. Let G be a definable subgroup of $\mathcal{U}^d$. Then $\{c \in \mathcal{C} : c \cdot G \subset G\}$ is a (definable) subgroup of $(\mathcal{C}, +)$ which is infinite (as it contains $\mathbf{Z}$) and thus, by strong minimality of $\mathcal{C}$ must be all of $\mathcal{C}$.

The next fact is due to Buium. For convenience we state it without proof (although it is not difficult, modulo the existence of "universal" extensions of abelian varieties).

Fact 3.18 *Let A be a semiabelian variety defined over $\mathcal{U}$. Then there is some (L_∂)-definable homomorphism μ from A to $(\mathcal{U}^d, +)$ (for some d) such that $Ker(\mu)$ has finite Morley rank and is connected.*

Corollary 3.19 *Suppose A is a semiabelian variety defined over $\mathcal{U}$. Let Γ be a finitely generated subgroup of A. Then there is a connected definable subgroup G of A such that G has finite Morley rank, and contains Γ.*

Proof. Let μ be as is the fact above. Let $\Gamma' = \mu(\Gamma)$. The Γ' is a finitely generated subgroup of $Im(\mu)$. By Lemma 3.17, $\mu(A)$ is a $\mathcal{C}$-vector space. The $\mathcal{C}$-subspace V generated by Γ' will then be finite-dimensional so definable and of finite Morley rank (why?). $\mu^{-1}(V)$ will then be a definable subgroup of A of finite Morley rank. (Why?)

We will prove:

Theorem 3.20 *Let $k < K$ be a algebraically closed fields of characteristic zero. Let A be a semiabelian variety over K (identified as usual with $A(K)$), and Γ a finitely generated subgroup of A. Let X be an irreducible subvariety of A (defined over K) such that $\Gamma \cap X$ is Zariski-dense in X. Then there are a semiabelian variety B_0 defined over k and a subvariety X_0 of B_0 also defined over k, and a homomorphism f of algebraic groups from some algebraic subgroup B of A onto B_0, such that (after possibly translating X), $X = f^{-1}(X_0)$.*

Corollary 3.21 *Suppose $k < K$ are algebraically closed fields, and A is an abelin variety defined over K which has k-trace 0. (This means that no (nonzero) abelian subvariety of A is isomorphic to an abelian variety defined over k, and implies that no abelian subvariety of A has some (nonzero) homomorphic image defined over k). Let Γ be a finitely generated subgroup of A and X a subvariety of A (defined over K too). Then $X \cap \Gamma$ is a finite union of translates of subgroups of A. In particular, if X is a one-dimensional irreducible subvariety of A and X is NOT a translate of an algebraic subgroup of A, then $X \cap \Gamma$ is finite.*

Proof of Theorem 3.20. Let $S = Stab_A(X) = \{a \in A : X + a = X\}$. Then S is an algebraic subgroup of A, and X is a union of translates of S. Replacing A by A/S, X by X/S and Γ by Γ/S, we may assume that S is trivial.

We may also assume that $K = \mathcal{U}$ and $k = \mathcal{C}$. (The theory $(ACF_0)_P$ of proper pairs of algebraically closed fields of characteristic zero, in the language of rings with an additional predicate P for the bottom field, is complete. But $(\mathcal{U}, \mathcal{C})$ is a saturated model of this theory whereby (K, k) elementarily embeds in $(\mathcal{U}, \mathcal{C})$.)

So we now talk about definability in L_∂. By 3.19, let G be a connected definable subgroup of A of finite Morley rank, which contains Γ. So $X \cap G$ is Zariski-dense in X. Now $X \cap G$ is a definable set. Let W be a definable subset of $X \cap G$ of least (RM, dM) such that W is Zariski-dense in X. Then W has Morley degree 1. (why?) Assume $RM(W) = m$. Assume also W is defined over the small algebraically closed differential field K_0. Let $p(x)$ be the unique stationary type over K_0 containing "$x \in W$" and of Morley rank m.

Claim I. $Stab_G(p)$ is trivial.

Proof. Note that $g \in Stab(p)$ iff $RM((g+W) \cap W) = m$. But by choice of W, if $RM((g+W) \cap W) = m$ then X is the Zariski closure of $(g+W) \cap W$ so $g + X = X$, so $g \in Stab_A(X)$.

Claim II. The set of realizations of $p(x)$ is a Zariski-dense subset of X.

Proof. Left to the reader.

We will now apply Corollary 3.16. Let Z denote the set of realizations of $p(x)$. Let $g \in Z$. Let G_1 be the connected definable subgroup of G generated by $Z - g$. Then G_1 is definably isomorphic to $H(\mathcal{C})$ for some (connected) algebraic group H defined over $\mathcal{C}$. Note that H is a commutative (as $H(\mathcal{C})$ is). Let $h : H(\mathcal{C}) \to G_1$ be the definable isomorphism. Without loss h is defined over K_0.

Claim III. h extends to a homomorphism h_1 of algebraic groups from H into A.

*Proof.*By our understanding of definable closure in DCF_0, for any $a \in H(\mathcal{C})$, $h(a)$ is in the differential field generated by K_0 and a which is precisely $K_0(a)$ as a is a tuple of constants. We can find $a \in H(\mathcal{C})$ which is a generic point of H over K_0 in the sense of ACF (why??). So $h(a) = f(a)$ for some K_0-rational function f. For each generic point b of H over K_0 (in the sense of ACF) define $h_1(b) = f(b)$, and if $d \in H$ is a product of generic (over K_0) elements b_1, b_2 define $h_1(d) = f(b_1) \cdot f(b)_2$. Check that h_1 is a homomorphism (of algebraic groups) from H into A and that $h_1|H(\mathcal{C})$ is precisely h.

Claim IV. h_1 is an isomorphism between H and its image, say B.

Proof. Let $B = h_1(H)$. Then B is a semiabelian subvariety of A. Also H is a connected commutative algebraic group, defined over $\mathcal{C}$. We will apply some of the structure theorem for commutative algebraic groups (Fact 2.61). Note that the maximal algebraic "vector subgroup" N of H must be in $Ker(h_1)$ (as the image of a vector group under a homomorphism of algebraic groups is also a vector group, and as B is a semi-abelian variety, B contains no vector subgroup.) But N is defined over $\mathcal{C}$, so if nonzero has nonzero points in $\mathcal{C}$, but $h_1|H(\mathcal{C})$ is bijective. Hence $N = 0$. Thus H is a semiabelian variety. The connected component of $Ker(h_1)$ is, by Corollary 2.64, defined over $\mathcal{C}$ so as above must be trivial. So $Ker(h_1)$ is finite, so contained in the torsion of H, so again contained in $H(\mathcal{C})$. As before, $Ker(h_1)$ is trivial.

Now $Z - g$ is Zariski-dense in $X - g$ which is thus contained in B. Let $Y = h_1^{-1}(X - g)$ an irreducible subvariety of H. Note that $h_1^{-1}(Z - g)$ is a subset of $H(\mathcal{C})$ which is Zariski-dense in Y. Hence Y is defined over $\mathcal{C}$.

So putting everything together B is a semiabelian subvariety of A which contains a translae $X - g$ of X and there is an isomorphism between B and a semiabelian variety H defined over $\mathcal{C}$ which takes $X - g$ to a variety also defined over $\mathcal{C}$. This completes the proof of Theorem 3.20.

References

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