A note on generically stable measures and fsg groups

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Abstract

We prove (Proposition 2.1) that if μ is a generically stable measure in an NIP theory, and $\mu(\phi(x, b)) = 0$ for all b then for some n, $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge .. \wedge \phi(x_n, y))) = 0$. As a consequence we show (Proposition 3.2) that if G is a definable group with fsg in an NIP theory, and X is a definable subset of G then X is generic if and only if every translate of X does not fork over $\emptyset$, precisely as in stable groups, answering positively Problem 5.5 from [3].

1 Introduction and preliminaries

This short paper is a contribution to the generalization of stability theory and stable group theory to NIP theories, and also provides another example where we need to resort to measures to prove statements (about definable sets and/or types) which do not explicitly mention measures. The observations in the current paper can and will be used in the future to sharpen existing results around measure and NIP theories (and this is why we wanted to record the

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observations here). Included in these sharpenings will be: (i) replacing average types by generically stable types in a characterization of strong dependence in terms of measure and weight in [6], and (ii) showing the existence of "external generic types" (in the sense of Newelski [5]), over any model, for fsg groups in NIP theories, improving on Lemma 4.14 and related results from [5].

If $p(x) \in S(A)$ is a stationary type in a stable theory and $\phi(x,b)$ any formula, then we know that $\phi(x,b) \in p|\mathfrak{C}$ if and only if $\models \bigwedge_{i=1,,.n} \phi(a_i,b)$ for some independent realizations $a_1,..,a_n$ of p (for some n depending on $\phi(x,y)$). Hence $\phi(x,b) \notin p|\mathfrak{C}$ for all b implies that (and is clearly implied by) the inconsistency of $\bigwedge_{i=1,..,n} \phi(a_i,y)$ for some (any) independent set $a_1,..,a_n$ of realizations of p. This also holds for generically stable types in NIP theories (as well as for generically stable types in arbitrary theories, with definition as in [7]). In [6], an analogous result was proved for "average measures" in strongly dependent theories. Here we prove it (Proposition 2.1) for generically stable measures in arbitrary NIP theories, as well as giving a generalization (Remark 2.2).

The fsg condition on a definable group G is a kind of "definable compactness" assumption, and in fact means precisely this in o-minimal theories and suitable theories of valued fields (and of course stable groups are fsg). Genericity of a definable subset X of G means that finitely many translates of X cover G. Proposition 2.1 is used to show that for X a definable subset of an fsg group G, X is generic if and only if every translate of X does not fork over $\emptyset$. This is a somewhat striking extension of stable group theory to the NIP environment.

We work with an NIP theory T and inside some monster model $\mathfrak{C}$. If A is any set of parameters, let $L_x(A)$ denote the Boolean algebra of A-definable sets in the variable x. A *Keisler measure* over A is a finitely additive probability measure on $L_x(A)$. Equivalently, it is a regular Borel probability measure on the compact space $S_x(A)$. We will denote by $\mathfrak{M}_x(A)$ the space of Keisler measures over A in the variable x. We might omit x when it is not needed or when it is included in the notation of the measure itself (*e.g.* μ_x). If X is a sort, or more generally definable set, we may also use notation such $L_X(A)$, $S_X(A)$, $\mathfrak{M}_X(A)$, where for example $S_X(A)$ denote the complete types over A which contain the formula defining X (or which "concentrate on X").

Definition 1.1. A type $p \in S_x(A)$ is *weakly random* for μ_x if $\mu(\phi(x)) > 0$ for any $\phi(x) \in L(A)$ such that $p \vdash \phi(x)$. A point b is weakly random for μ over A if $\mathrm{tp}(b/A)$ is weakly random for μ.

We briefly recall some definitions and properties of Keisler measures, referring the reader to [4] for more details.

If $\mu \in \mathfrak{M}_x(\mathfrak{C})$ is a global measure and M a small model, we say that μ is M-invariant if $\mu(\phi(x,a)\triangle\phi(x,a')) = 0$ for every formula $\phi(x,y)$ and $a, a' \in \mathfrak{C}$ having the same type over M. Such a measure admits a Borel defining scheme over M: For every formula $\phi(x,y)$, the value $\mu(\phi(x,b))$ depends only on $\mathrm{tp}(b/M)$ and for any Borel $B \subset [0,1]$, the set $\{p \in S_y(M) : \mu(\phi(x,b)) \in B$ for some $b \models p\}$ is a Borel subset of $S_y(M)$.

Let $\mu_x \in \mathfrak{M}(\mathfrak{C})$ be M-invariant. If $\lambda_y \in \mathfrak{M}(\mathfrak{C})$ is any measure, then we can define the *invariant extension* of μ_x over λ_y, denoted $\mu_x \otimes \lambda_y$. It is a measure in the two variables x, y defined in the following way. Let $\phi(x,y) \in L(\mathfrak{C})$. Take a small model N containing M and the parameters of ϕ. Define $\mu_x \otimes \lambda_y(\phi(x,y)) = \int f(p) d\lambda_y$, the integral ranging over $S_y(N)$ where $f(p) = \mu_x(\phi(x,b))$ for $b \in \mathfrak{C}$, $b \models p$ (this function is Borel by Borel definability). It is easy to check that this does not depend on the choice of N.

If λ_y is also invariant, we can also form the product $\lambda_y \otimes \mu_x$. In general it will not be the case that $\lambda_y \otimes \mu_x = \mu_x \otimes \lambda_y$.

If μ_x is a global M-invariant measure, we define by induction: $\mu^{(n)}_{x_1 \ldots x_n}$ by $\mu^{(1)}_{x_1} = \mu_{x_1}$ and $\mu^{n+1}_{x_1 \ldots x_{n+1}} = \mu_{x_{n+1}} \otimes \mu^{(n)}_{x_1 \ldots x_n}$. We let $\mu^{(\omega)}_{x_1 x_2 \ldots}$ be the union and call it the *Morley sequence* of μ_x.

Special cases of M-invariant measures include definable and finitely satisfiable measures. A global measure μ_x is *definable* over M if it is M-invariant and for every formula $\phi(x,y)$ and open interval $I \subset [0,1]$ the set $\{p \in S_y(M) : \mu(\phi(x,b)) \in I$ for some $b \models p\}$ is open in $S_y(M)$. The measure μ is *finitely satisfiable* in M if $\mu(\phi(x,b)) > 0$ implies that $\phi(x,b)$ is satisfied in M. Equivalently, any weakly random type for μ is finitely satisfiable in M.

Lemma 1.2. *Let $\mu \in \mathfrak{M}_x(\mathfrak{C})$ be definable over M, and $p(x) \in S_x(\mathfrak{C})$ be weakly random for μ. Let $\phi(x_1, .., x_n)$ be a formula over $\mathfrak{C}$. Suppose that $\phi(x_1, .., x_n) \in p^{(n)}$. Then $\mu^{(n)}(\phi(x_1, .., x_n)) > 0$.*

Proof. We will carry out the proof in the case where μ is definable (over M), which is anyway the case we need. Note that $p^{(m)}$ is M-invariant for all m. The proof of the lemma is by induction on n. For $n = 1$ it is just the definition of weakly random. Assume true for n and we prove for $n+1$. So suppose $\phi(x_1, .., x_n, x_{n+1}) \in p^{(n+1)}$. This means that for $(a_1, .., a_n)$ realizing $p^{(n)}|M$, $\phi(a_1, .., a_n, x) \in p$. So as p is weakly random for μ, $\mu(\phi(a_1, .., a_n, x)) = r > 0$. So as μ is M-invariant, $tp(a'_1, .., a'_n/M) = tp(a_1, .., a_n/M)$ implies

$\mu(\phi(a'_1, .., a'_n, x)) = r$ and thus also $r - \epsilon < \mu(\phi(a'_1, .., a'_n, x))$ for any small positive ϵ. By definability of μ and compactness there is a formula $\psi(x_1, .., x_n) \in tp(a_1, .., a_n/A)$ such that $\models \psi(a'_1, .., a'_n)$ implies $0 < r - \epsilon < \mu(\phi(a'_1, .., a'_n, x))$. By induction hypothesis, $\mu^{(n)}(\psi(x_1, .., x_n)) > 0$. So by definition of $\mu^{(n+1)}$ we have that $\mu^{(n+1)}(\phi(x_1, .., x_n, x_{n+1})) > 0$ as required. $\square$

A measure $\mu_{x_1,...,x_n}$ is *symmetric* if for any permutation σ of $\{1, ..., n\}$ and any formula $\phi(x_1, ..., x_n)$, we have $\mu(\phi(x_1, ..., x_n)) = \mu(\phi(x_{\sigma.1}, ..., x_{\sigma.n}))$. A special case of a symmetric measure is given by powers of a generically stable measure as we recall now. The following is Theorem 3.2 of [4]:

Fact 1.3. *Let μ_x be a global M-invariant measure. Then the following are equivalent:*

1. *μ_x is both definable and finitely satisfiable (necessarily over M),*
2. *$\mu^{(n)}_{x_1,...,x_n}|_M$ is symmetric for all $n < \omega$,*
3. *for any global M-invariant Keisler measure λ_y, $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$,*
4. *μ commutes with itself: $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$.*

If μ_x satisfies one of those properties, we say it is generically stable.

If $\mu \in \mathfrak{M}_x(A)$ and D is a definable set such that $\mu(D) > 0$, we can consider the *localisation* of μ at D which is a Keisler measure μ_D over A defined by $\mu_D(X) = \mu(X \cap D)/\mu(X)$ for any definable set X.

We will use the notation $Fr(\theta(x), x_1, ..., x_n)$ to mean

$$\frac{1}{n}|\{i \in \{1, ..., n\} :\models \theta(x_i)\}|.$$

The following is a special case of Lemma 3.4. of [4].

Proposition 1.4. *Let $\phi(x, y)$ be a formula over M and fix $r \in (0, 1)$ and $\epsilon > 0$. Then there is n such that for any symmetric measure $\mu_{x_1,...,x_{2n}}$, we have*

$$\mu_{x_1,...,x_{2n}}(\exists y(|Fr(\phi(x, y), x_1, ..., x_n) - Fr(\phi(x, y), x_{n+1}, ..., x_{2n})| > r)) \leq \epsilon.$$

2 Main result

Proposition 2.1. *Let μ_x be a global generically stable measure. Let $\phi(x,y)$ be any formula in $L(\mathfrak{C})$. Suppose that $\mu(\phi(x,b)) = 0$ for all $b \in \mathfrak{C}$. Then there is n such that $\mu^{(n)}(\exists y(\phi(x_1,y) \wedge ... \wedge \phi(x_n,y))) = 0$.*

Moreover, n depends only on $\phi(x,y)$ and not on μ.

Proof. Let μ_x be a global generically stable measure and M a small model over which $\phi(x,y)$ is defined and such that μ_x is M-invariant. Assume that $\mu(\phi(x,b)) = 0$ for all $b \in \mathfrak{C}$. For any k, define

$$W_k = \{(x_1,...,x_n) : \exists y(\wedge_{i=1..k}\phi(x_i,y))\}.$$

This is a definable set. We want to show that $\mu^{(n)}(W_n) = 0$ for n big enough. Assume for a contradiction that this is not the case.

Let n be given by Proposition 1.4 for $r = 1/2$ and $\epsilon = 1/2$. Consider the measure $\lambda_{x_1,...,x_{2n}}$ over M defined as being equal to $\mu^{(2n)}$ localised on the set W_{2n} (by our assumption, this is well defined). As the measure $\mu^{(2n)}$ is symmetric and the set W_{2n} is symmetric in the $2n$ variables, the measure λ is symmetric. Let $\chi(x_1,...,x_{2n})$ be the formula "$(x_1,...,x_{2n}) \in W_{2n} \wedge \forall y(|Fr(\phi(x,y),x_1,...,x_n) - Fr(\phi(x,y),x_{n+1},...,x_{2n})| \leq 1/2)$". By definition of n, we have $\lambda(\exists y(|Fr(\phi(x,y),x_1,....,x_n) - Fr(\phi(x,y),x_{n+1},...,x_{2n})| > 1/2)) \leq 1/2$. Therefore $\mu^{(2n)}(\chi(x_1,...,x_{2n})) > 0$.

As μ is M-invariant, we can write

$$\mu^{(2n)}(\chi(x_1,...,x_{2n})) = \int_{q \in S_{x_1,...,x_n}(M)} \mu^{(n)}(\chi(q,x_{n+1},...,x_{2n}))d\mu^{(n)},$$

where $\mu^{(n)}(\chi(q,x_{n+1},...,x_{2n}))$ stands for $\mu^{(n)}(\chi(a_1,...,a_n,x_{n+1},...,x_{2n}))$ for some (any) realization $(a_1,...,a_n)$ of q. As $\mu^{(2n)}(\chi(x_1,...,x_{2n})) > 0$, there is $q \in S_{x_1,...,x_n}$ such that
(*) $\mu^{(n)}(\chi(q,x_{n+1},...,x_{2n})) > 0$.
Fix some $(a_1,...,a_n) \models q$. By (*), we have $(a_1,...,a_n) \in W_n$. So let $b \in \mathfrak{C}$ such that $\models \bigwedge_{i=1...n}\phi(a_i,b)$. Again by (*), we can find some $(a_{n+1},...,a_{2n})$ weakly random for $\mu^{(n)}$ over $Ma_1...a_nb$ and such that
(**) $\models \chi(a_1,...,a_n,a_{n+1},...,a_{2n})$.
In particular, for $j = n+1,...,2n$, a_j is weakly random for μ over Mb hence $\models \neg\phi(a_j,b)$. But then $|Fr(\phi(x,b);a_1,...,a_n) - Fr(\phi(x,b);a_{n+1},...,a_{2n})| = 1$. This contradicts (**). □

Remark 2.2. The proof above adapts to showing the following generalization: Let μ_x be a global generically stable measure, $\phi(x,y)$ a formula in $L(\mathfrak{C})$. Let $\Sigma(x)$ be the partial type (over the parameters in ϕ together with a small model over which μ is definable) defining $\{b : \mu(\phi(x,b)) = 0\}$. Then for some n: $\mu^{(n)}(\exists y(\Sigma(y) \wedge \phi(x_1,y) \wedge .. \wedge \phi(x_n,y))) = 0$.

3 Generics in fsg groups

Let G be a definable group, without loss defined over $\emptyset$. We call a definable subset X of G left (right) generic if finitely many left (right) translates of X cover G, and a type $p(x) \in S_G(A)$ is left (right) generic if every formula in p is. We originally defined ([2]) G to have "finitely satisfiable generics", or to be fsg, if there is some global complete type $p(x) \in S_G(\mathfrak{C})$ of G every left G-translate of which is finitely satisfiable in some fixed small model M.

The following summarizes the situation, where the reader is referred to Proposition 4.2 of [2] for (i) and Theorem 7.7 of [3] and Theorem 4.3 of [4] for (ii), (iii), and (iv).

Fact 3.1. *Suppose G is an fsg group. Then*
(i) A definable subset X of G is left generic iff it is right generic, and the family of nongeneric definable sets is a (proper) ideal of the Boolean algebra of definable subsets of G,
(ii) There is a left G-invariant Keisler measure $\mu \in \mathfrak{M}_G(\mathfrak{C})$ which is generically stable,
(iii) Moreover μ from (ii) is the unique left G-invariant global Keisler measure on G as well as the unique right G-invariant global Keisler measure on G,
(iv) Moreover μ from (ii) is generic *in the sense that for any definable set X, $\mu(X) > 0$ iff X is generic.*

Remember that a definable set X (or rather a formula $\phi(x,b)$ defining it) forks over a set A if $\phi(x,b)$ implies a finite disjunction of formulas $\psi(x,c)$ each of which divide over A, and $\psi(x,c)$ is said to divide over A if for some A-indiscernible sequence $(c_i : i < \omega)$ with $c_0 = c$, $\{\phi(x,c_i) : i < \omega\}$ is inconsistent.

Proposition 3.2. *Suppose G is fsg and $X \subseteq G$ a definable set. Then X is generic if and only if for all $g \in X$, $g \cdot X$ does not fork over $\emptyset$ (if and only if for all $g \in G$, $X \cdot g$ does not fork over $\emptyset$).*

Proof. Left to right: It suffices to prove that any generic definable set X does not fork over $\emptyset$, and as the set of nongenerics forms an ideal it is enough to prove that any generic definable set does not divide over $\emptyset$. This is carried out in (the proof of) Proposition 5.12 of [3].

Right to left: Assume that X is nongeneric. We will prove that for some $g \in G$, $g \cdot X$ divides over $\emptyset$ (so also forks over $\emptyset$).

Let μ_x be the generically stable G-invariant global Keisler measure given by Fact 3.1. Let M_0 be a small model such that μ does not fork over M_0 (namely, as μ is generic, every generic formula does not fork over M_0) and X is definable over M_0. Let $\phi(x,y)$ denote the formula defining $\{(x,y) \in G \times G : y \in x \cdot X\}$. So ϕ has additional (suppressed) parameters from M_0. Note that for $b \in G$, $\phi(x,b)$ defines the set $b \cdot X^{-1}$. As X is nongeneric, so is X^{-1} so also $b \cdot X^{-1}$ for all $b \in G$. Hence, as μ is generic, $\mu(\phi(x,b)) = 0$ for all b. By Proposition 2.1, for some n $\mu^{(n)}(\exists y(\phi(x_1,y) \wedge .. \wedge \phi(x_n,y))) = 0$. Let p be any weakly random type for μ (which in this case amounts to a global generic type, which note is M_0-invariant). So by Lemma 1.2 the formula $\exists y(\phi(x_1,y) \wedge .. \wedge \phi(x_n,y))) \notin p^{(n)}$. Let $(a_1,..,a_n)$ realize $p^{(n)}|M_0$. Then $(a_1,..,a_n)$ extends to an M_0-indiscernible sequence $(a_i : i = 1,2,....)$, a Morley sequence in p over M_0, and $\models \neg\exists y(\phi(a_1,y) \wedge ... \wedge \phi(a_n,y))$. So in particular $\{\phi(a_i,y) : i = 1,2,...\}$ is inconsistent. Hence the formula $\phi(a_i,y)$ divides over M_0, so also divides over $\emptyset$. But $\phi(a_1,y)$ defines the set $a_1 \cdot X$, so $a_1 \cdot X$ divides over $\emptyset$ as required. $\square$

Recall that we called a global type $p(x)$ of a $\emptyset$-definable group G, left f-generic if every left G-translate of p does not fork over $\emptyset$.

We conclude the following (answering positively Problem 5.5 from [3] as well as strengthening Lemma 4.14 of [1]):

Corollary 3.3. *Suppose G is fsg and $p(x) \in S_G(\mathfrak{C})$. Then the following are equivalent:*
(i) p is generic,
(ii) p is left (right) f-generic,
(iii) (Left or right) $Stab(p)$ has bounded index in G (where left $Stab(p) = \{g \in G : g \cdot p = p\}$).

Proof. The equivalence of (i) and (ii) is given by Proposition 3.2 and the definitions. We know from [2], Corollary 4.3, that if p is generic then $Stab(p)$ is precisely G^{00}. Now suppose that p is nongeneric. Hence there is a definable set $X \in p$ such that X is nongeneric. Let M be a small model over which X is defined. Note that the fsg property is invariant under naming parameters. Hence G is fsg in $Th(\mathfrak{C}, m)_{m \in M}$. By Proposition 3.2 (as well as what is proved in "Right to left" there), for some $g \in G$, $g \cdot X$ divides over M. As X is defined over M this means that there is an M-indiscernible sequence $(g_\alpha : \alpha < \bar{\kappa})$ (where $\bar{\kappa}$ is the cardinality of the monster model) and some n such that $g_{\alpha_1} \cdot X \cap ... \cap g_{\alpha_n} \cdot X = \emptyset$ whenever $\alpha_1 < ... < \alpha_n$. This clearly implies that among $\{g_\alpha \cdot p : \alpha < \bar{\kappa}\}$, there are $\bar{\kappa}$ many types, whereby $Stab(p)$ has unbounded index. $\square$

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