# Lecture notes on strongly minimal sets (and fields) with a generic automorphism

Anand Pillay

October 5, 2005

## 1 Introduction

These lecture notes develop the theory of strongly minimal sets with a generic automorphism. They are strongly influenced by, and in some sense an exposition of the papers [1] and [2] which look at the case of algebraically closed fields with an automorphism. The deepest parts of these latter papers are concerned with dichotomy theorems (a type of $SU$-rank 1 is locally modular or nonorthogonal to a fixed field) and have diophantine-geometric implications. These theorems can be formulated in the general context, but we don't know how to prove them. On the other hand a large amount of the general model theory of fields with an automorphism is valid in the strongly minimal context, and this is the point of view we take (influenced by [3] from which some of what we do here comes). We will try to get to the second paper [2] which proves the dichotomy theorem in all characteristics by appealing to a non first order version of the Zariski geometry theorem [5]. We also exhibit the basic model of strongly minimal sets with a generic automorphism as coming directly from the theory of existentially closed structures.

## 2 Existentially closed models and model companions

**Definition 2.1** *Let* $\mathbf{K}$ *be a class of structures in a language* $L$*.* $M \in \mathbf{K}$ *is existentially closed (e.c) in* $\mathbf{K}$ *if whenever* $M \subseteq N$*, then any existential* *sentence over $M$ true in $N$ is true in $M$; equivalently, any quantifier free formula over $M$ which has a solution in $N$ has a solution in $M$. If $T$ is a (first order) theory, then by an e.c. model of $T$ we mean a structure which is e.c. in the class of models of $T$.*

**Fact 2.2** *If $T$ is $\forall\exists$ axiomatizable, then for any any $M \in \mathbf{K}$ there is $N \supseteq M$ in $\mathbf{K}$ such that $N$ is e.c. for $K$ and $N$ is at most $M + L$.*

**Remark 2.3** *Suppose $T$ is $\forall\exists$ axiomatizable. Then $M$ is an e.c. model of $T$ iff $M$ is an e.c. model of $T_\forall$.*

*Proof.* Any model of $T_\forall$ is a substructure of a model of $T$.

We now fix $T$ to be a $\forall\exists$ theory in $L$.

**Lemma 2.4** *Let $M$ be an e.c. model of $T$ and let $a$ be a finite tuple from $M$. Then $etp_M(a)$ ( = the set of existential formulas of $L$ true of $a$ in $M$) is maximal among existential types realised in models of $T$.*

*Proof.* Suppose $N \models T$ and $b$ is a tuple in $N$ whose existential type contains $etp_M(a)$. Add constants for elements of $M$ and $N$ identifying only $a$ and $b$ with constants $c$. Consider $\Sigma = T \cup D(M) \cup D(N)$ in this language. Let $\phi(c,m) \in D(M)$ and $\psi(c,n) \in D(N)$. So $\exists y(\phi(x,y)$ is true of $a$ in $M$ so true of $b \in N$, whereby $N \models \phi(b,n')$ for some $n' \in N$. So $N \models \phi(b,n') \wedge \psi(b,n)$. This shows that $\Sigma$ is consistent. Thus we have embeddings $f, g$ of $M, N$ respectively into a model $N'$ of $T$ which we may assume to be e.c., with $f(a) = g(b)$. Then $etp_M(a) = etp_{N'}(f(a))$ (as $M$ is e.c.). On the other hand $etp_N(b) \subseteq etp_{N'}(g(b))$. So we get equality throughout.

**Definition 2.5** *(i) We say that $T$ has a model companion if the class of e.c. models of $T$ is an elementary class (the set of models of a theory $T_{e.c.}$), in which case we call $T_{e.c.}$ the model companion of $T$.*
*(ii) A theory $T'$ is said to be model-complete if any pair $M \subseteq N$ of models of $T$ is an elementary pair. This is well-known to be equivalent to saying that any formula $\phi(x)$ is equivalent (mod T') to a universal (extistential) formula.*

**Lemma 2.6** *If $T$ has a model companion,then $T_{e.c.}$ is model-complete. (Conversely, suppose $T'$ is a model-complete theory such that $(T')_\forall = T_\forall$. Then $T'$ is the model companion of $T$.)*

*Proof.* First we show that any existential formula is equivalent modulo $T_{e.c.}$ to a universal formula: Let $\phi(x)$ be an existential formula. Let $P$ be the set of existential types realised in e.c. models of $T$ which do not contain $\phi$. By Lemma 2.4 and compactness, each $p \in P$ contains a formula $\psi_p(x)$ say, inconsistent with $\phi(x)$ (modulo $T$). Let $\Psi(x)$ the the set of negations of these $\psi_p$ ($p \in P$). It is then clear that the statement $\forall x(\phi(x) \leftrightarrow \bigwedge \Psi(x))$ holds in all e.c. models of $T$, namely in all models of $T_{e.c.}$. By compactness $T \models \phi(x) \leftrightarrow \chi(x)$ where $\chi$ is some finite conjunction of members of $\Psi(x)$. $\chi$ is a universal formula.
An inductive argument (consider formulas in prenex form) shows that every formula is equivalent (mod $T_{e.c.}$) to a universal (and existential) formula, so $T_{e.c}$ is model-complete.
Now let $T'$ satisfy the second sentence in the lemma. Let $M$ be an *e.c.* model of $T$. So $M$ has an extension to a model $N$ of $T'$. $T'$ being model-complete is inductive so $\forall\exists$ axiomatizable, so $M$ is a model of $T'$. On the other hand, suppose $M$ is a model of $T'$. Let $N$ be a model of $T$ containing $M$, let $a \in M$ and suppose $N \models \phi(a)$ for some existential formula $\phi(x)$. Let $N \subseteq N' \models T'$. Then $M$ is an elementary substructure of $N'$, but also $N' \models \phi(a)$. So $M \models \phi(a)$. By Remark 2.3, this is enough.

**Definition 2.7** *Let $M$ be a substructure a model of $T$ (namely a model of $T_\forall$. We call $M$ an amalgamation base (for $T$) if whenever $f : M \to N_1$, $g : M \to N_2$ are embeddings into models of $T$, then there are embeddings $h, k$ of $N_1, N_2$ respectively into a model $N'$ of $T$ such that $h.f = k.g$.*

**Remark 2.8** *The above definition would be equivalent if we required only that $N_1, N_2, N'$ be models of $T_\forall$.*

**Lemma 2.9** *Let $M$ be an amalgamation base for $T$. Suppose $T$ has a model companion. Let $f, g$ be embeddings of $M$ in models $N_1, N_2$ of $T_{e.c}$. Let $m$ enumerate $M$. Then $f(m)$ and $g(m)$ have the same type in $N_1, N_2$ respectively.*

*Proof.* As $M$ is an amalgamation base we can find embeddings $h, k$ of $N_1, N_2$ into a model $N'$ of $T_{e.c.}$ with $h.f = k.g$. Let $m_1$=$f(m)$, $m_2 = g(m)$ and $m'$ = $h(m_1)$ (= $k(m_2)$. So, as $T_{e.c}$ is model-complete, $tp_{N_1}(m_1) = tp_{N'}(m') = tp_{N_2}(m_2)$.

**Corollary 2.10** *Suppose every model of $T_\forall$ is an amalgamation base, and $T$ has a model companion. Then $T_{e.c.}$ has quantifier-elimination. (In fact this is an if and only if.)*

**Remark 2.11** *Any e.c. model of $T$ is an amalgamation base.*

*Proof.* Like the proof of Lemma 2.4.

**Definition 2.12** *A model $M$ of $T_\forall$ is said to be a strong amalgamation base if Definition 1 holds with the additional requirement that $h(N_1) \cap k(N_2) = h.f(M)$.*

**Lemma 2.13** *Assume that $T$ has a model companion. Let $M \models T_\forall$ be a strong amalgamation base for $T$. Assume that $M \subseteq N$ where $N$ is e.c. Then $M$ is algebraically closed in $N$ in the model-theoretic sense. ($acl_N(M) = M$.)*

*Proof.* From the definition, we can find an e.c. model $N'$ of $T$ containing $M$ and $N_i$ for $i < \omega$ where each $(M, N_i)$ is isomorphic (via $f_i$ say which is identity on $M$) to $(M, N)$, and the $N_i$ are pairwise disjoint over $M$. We may assume that $N_0 = N$, and note that $N_0$ is an elementary substructure of $N'$. Moreover by Lemma 2.9, each $N_i$ has the same type over $M$ in the structure $N'$. in particular, for each $b \in N \setminus M$, $tp_N(b/M)$ has infinitely many realisations in $N'$, so is nonalgebraic.

# 3 ACFA and strongly minimal theories with a generic automorphism

$ACF$ is the (incomplete) theory of algebraically closed fields in the language of rings. It's completions are given by fixing the characteristic. It is convenient to define $L^-$ to be the language of rings, and $L$ to be $L^-$ together with a unary function symbol $\sigma$. We will, whenever possible, be working with a (possibly) incomplete theory $T$ with QE (in place of $ACF$) and in this case we again let $L^-$ denote the language of $T$ and $L$ this language augmented by $\sigma$. So unless we say otherwise $T$ denotes a theory with QE. (Allowing $T$ to be incomplete is not really a big deal, and we can equally well work with complete $T$ and some completion of $ACF$.)

**Fact 3.1** *ACF has QE, is strongly minimal has the definable multiplity property (DMP) and has elimination of imaginaries.*

We give some explanations. QE is classical. An incomplete theory is called strongly minimal, if all models of $T$ are infinite, and each definable (with parameters) subset of each model of $T$ is finite or cofinite (so this is equivalent to each completion of $T$ being strongly minimal). If $M$ is a (saturated) model of a strongly minimal theory, then any definable set $X \subseteq M^n$ has a well-defined Morley rank and degree (natural numbers). The Morley rank of $X$ is defined inductively by $RM(X) \geq 0$ if $X$ is nonempty, and $RM(X) \geq k+1$ if there is a pairwise disjoint family $(X_i)_{i\in\omega}$ of definable sets, each a subset of $X$, and each of Morley rank $\geq k$. If $RM(X) = k$, then there is some maximal $d$ such that $X$ can be partitioned into $d$ definable sets each of Morley rank $k$, and this $d$ is called the Morley degree or multiplicity of $X$. $RM(tp(a/A))$ is the min. of the Morley rank for formulas in $tp(a/A)$, and similarly for Morley degree. We will say $a$ is independent from $B$ over $A$, $(B \supseteq A)$, if $RM(tp(a/A)) = RM(tp(a/B))$. If $a = (a_1, .., a_n)$, this is equivalent to saying that $dim(a_1, .., a_n/A) = dim(a_1, .., a_n/B)$ (explain). Independence is symmetric, transitive, with the extension property (explain). Moreover $mlt(tp(a/A)) = 1$ just if $tp(a/A)$ is "stationary", namely whenever $B \supset A$ and $a_1, a_2$ realise $tp(a/A)$ such that each is independent from $B$ over $A$, then $tp(a_1/B) = tp(a_2/B)$. If $\phi(x, a)$ has Morley rank $k$, and $B$ contains $a$ then a generic solution of $\phi(x, a)$ over $B$ is by definition some $c$ satisfying $\phi(x, a)$ such that $RM(tp(c/B)) = k$. If in addition $\phi(x, a)$ has multiplicity 1 then such a type is unique. Any infinite algebraically closed subset of a model of (strongly minimal) $T$ will be an elementary submodel (why?) and any type over such a set will be stationary (Finite Equivalence Relation Theorem, but in the strongly minimal case it should be easier.)

**Fact 3.2** *(Strongly minimal $T$.) Let $M$ be a model of $T$, and $a, b$ tuples, and $A \subset M$. Then $RM(tp(a, b/A) = RM(tp(a/Ab) + RM(tp(b/A)$.*

The following is well-known:

**Exercise 3.3** *Suppose $T$ is strongly minimal. Then for every formula $\phi(x, y)$ and $k$ there is $\psi(y)$ such that for any model $M$ of $T$, and $b$ in $M$, $RM(\phi(x, b)) = k$ iff $M \models \psi(b)$. Moreover there a bound on the multiplities of such $\phi(x, b)$.*

The DMP is concerned with defining multiplicities too. We will give a rather restricted definition:

**Definition 3.4** *(Strongly minimal) $T$ has the DMP, if whenever $M \models T$, and $tp(b/A)$ is stationary with $RM = k$ and (multiplicity 1) then there is a formula $\phi(x,a)$ in $tp(b/A)$ such for any $a'$ in a model $M'$ of $T$, if $\phi(x,a')$ is consistent then it has $RM = k$ and multiplicity 1.*

**Lemma 3.5** *(Strongly minimal) $T$ has the DMP iff the definition above holds for the case $k=1$.*

*Proof.* We will assume to make life easy that in every model of $T$ $acl(\emptyset)$ is infinite (so every algebraically closed set is an elementary substructure). We prove the DMP by induction on $k$. It is trivial for $k=0$ and true by hypothesis for $k=1$. Let $k>1$, and let $tp(c/A)$ be stationary of Morley rank $k$. There is no harm in letting $A$ be algebraically closed. $c$ is a finite tuple not contained in $A$ so there is an element, say $c_0$ of $c$ not in $A$. By Fact 3.2 above, $RM(tp(c/Ac_0) = k-1$. Let $c'$ be a finite tuple containing $c_0$ and contained in $acl(A,c_0)$ such that $tp(c/Ac')$ is stationary (so of multiplicity 1). Note that $tp(c'/A)$ is stationary of $RM$ 1. By induction hypothesis, we can find suitable formulas $\phi(x,c',a) \in tp(c/Ac')$ and $\psi(y,a) \in tp(c'/A)$. Then check that the formula $\exists y(\phi(x,y,a) \wedge \psi(y,a))$ works for $tp(c/A)$. ????

Let's now prove that $ACF$ has the DMP. Let $K$ be an algebraically closed field, $k$ a (say algebraically closed) subfield and $\bar{b} = (b, b_1, .., b_n)$ in $K$ such that $RM(\bar{b}/k) = 1$, that is $\bar{b}$ is a generic point of an absolutely irreducible curve defined over $k$. We assume $b \notin k$ and the $b_i \in acl(k,b)$. By the primitive element theorem, there is a single element $c$ such that $dcl(k,b,b_1,..,b_n) = dcl(k,b,c)$. We may thus consider $tp(bc/k)$ in place of $tp(\bar{b}/k)$. As $k$ is algebraically closed there is an absolutely irreducible polynomial $f(X,Y) \in k[X,Y]$ such that $f(b,c)=0$. The absolute irreducibility of $f$ can be expressed in a first order way in terms of the coefficients (one need only consider polynomial factors of $f$ of a bounded degree). So we have the required formula.

**Example 3.6** *There is a strongly minimal theory without the DMP.*

*Proof.* Let $V$ be a vector space over $\mathbf{Q}$, and let $a \in V$, and let $D = V \times \{0,1\}$ equipped with the projection $\pi : D \to V$ and the function $f : D \to D$

defined as $f(v,i) = (v+a,i)$. We get a strongly minimal structure. For any $v \in V$,$\{(x,y) \in D \times D : \pi(y) = \pi(x) + v\}$ is a set of Morley rank 1 which has multiplicity 1 iff $v$ is not equal to an integral multiple of $a$.

What about elimination of imaginaries: A complete theory $T$ is said to have elimination of imaginaries, if (in a saturated model $M$ of $T$ say) for any $L(T)$-definable equivalence relation $E$ on $n$-tuples, and $a \in M^n$, there is some finite tuple $c$ from $M$ which is "interdefinable" with $a/E$. What this amounts to is that there should be an $L(T)$ formula $\phi(x)$ true of $a$, and some $L(T)$ formula $\psi(y)$ ($y$ a tuple of variables), and another formula $\chi(x,y) \in L(T)$, such that $T$ proves that $\chi$ defines a bijection between $\phi/E$ and $\psi$. If $T$ has at least two (?) constant symbols, then one can show that $T$ has EI iff for any 0-definable $E$ on $n$-space, there is a formula $\chi(x,y)$ inducing a 1-1 mapping from $M^n/E$ into $M^k$ for some $k$. We are here defining an incomplete theory to have EI if every completion does, but I suppose this gives some uniformity across the completions. EI for $ACF$ is due to Poizat and also has a proof in [7] coming from a weaker result valid in all strongly minimal theories with $acl(\emptyset)$ infinite.

**Definition 3.7** *$ACF_\sigma$ is $ACF \cup \{\sigma$ is an automorphism$\}$. This is a $\forall\exists$ axiomatizable theory in $L$. (Similarly if $T$ is a theory with QE in $L^-$ then $T_\sigma = T \cup \{\sigma$ is an automorphism$\}$, also $\forall\exists$.)*

**Lemma 3.8** *A model $(M,\sigma)$ of $T_\sigma$ is existentially closed (for $T_\sigma$) if and only if, whenever $N$ is an elementary extension of $M$, $\psi(x,y)$ is an $L^-$-formula over $M$, and $N \models \psi(b,c)$ for some tuples $b,c$ from $N$ such that $tp(c/M) = \sigma(tp(b/M))$, then $M \models \psi(a,\sigma(a))$ for some tuple $a$ from $M$.*

*Proof.* Left to right direction: Assume $(M,\sigma)$ is am e.c. model of $T_\sigma$. We may assume $\psi$ is quantifier-free. Also, by replacing $N$ by an elementary extension, we may assume the $\sigma'(b) = c$ for some automorpgism $\sigma'$ of $N$ extending $\sigma$. So we easily find suitable $a \in M$.
Right to left: Let us assume that $(M,\sigma)$ satisfies the right hand condition. Let $\phi(x)$ be a quantifier-free formula of $L$ with parameters from $M$ which is satisfied by a tuple $c$ say in some $(N,\sigma') \models T_\sigma$ extending $(M,\sigma)$. Note that $N$ is an elementary extension of $M$. We may assume that $\phi(x)$ has the form $\chi(x,\sigma(x),..,\sigma^k(x))$ where $\chi$ is a quantifier-free $L^-$-formula with parameters from $M$. Let $\psi(x_0,..,x_{k-1},y_0,y_1,..,y_{k-1})$ be the following $L^-$-formula over

$M$: $\chi(x_0, x_1, .., x_{k-1}, y_{k-1}) \wedge x_1 = y_0 \wedge x_2 = y_1 \wedge ... \wedge x_{k-1} = y_{k-2}$. Let $c' = (c, \sigma'(c), .., \sigma'^{k-1}(c))$. Then note that $(c', \sigma'(c'))$ satisfies $\psi$ in $N$. By hypothesis there is a tuple $a'$ from $M$ such that $(a', \sigma(a'))$ satisfies $\psi$ in $M$. If $a'$ is of the form $(a_0, .., a_{k-1})$ then it is clear that $a_0$ satisfies the original formula $\phi(x)$ in $(M, \sigma)$.

**Proposition 3.9** *(T strongly minimal with DMP). $T_\sigma$ has a model companion, which we call $TA$.*

*Proof.* Consider the following conditions on a model $(M, \sigma)$ of $T_\sigma$: Let $\phi_1(x), \phi_2(x), \psi(x, y)$ be $L^-$-formulas over $M$ such that
(i) $\phi_1, \phi_2$ have Morley rank $m$ and multiplicity 1, and $\psi$ has Morley rank $m + r$ and multiplicity 1, and $M \models \psi(x, y) \rightarrow \phi_1(x) \wedge \phi_2(y)$.
(ii) for any $b$ satisying $\phi_1(x)$, $\psi(b, y)$ has $RM$ $r$, and for any $c$ satisfying $\phi_2(y)$, $\psi(x, c)$ has $RM$ $r$.
(iii) $\sigma(\phi_1(x) = \phi_2(x)$ up to a formula of Morley rank $< m$.
*Then* there is $a \in M$ such that $M \models \psi(a, \sigma(a))$.
Owing to $T$ having the DMP, these conditions are expressed by a set of sentences of $L$. (One has to quantify over the parameters in the formulas.) We will show that $(M, \sigma)$ satisfies these conditions just if it satisfies the right hand side of Lemma 3.8. Suppose first that $(M, \sigma)$ satisfies these conditions. Let $N$ be an elementary extension of $M$ and suppose $N \models \psi(b, c)$ where $tp(c/M) = \sigma(tp(b/M))$. Let $RM(tp(b/M)) = m$ and $RM(tp(b, c/M)) = m + r$. So $RM(tp(b/Mc)) = RM(tp(c/Mb)) = r$. We may find (strengthening $\psi$) formulas $\phi_1(x) \in tp(b/M)$ and $\phi_2(y) \in tp(c/M)$ such that (i), (ii), (iii) above hold. So we find suitable $a \in M$ as required.
Conversely suppose $M$ satisfies the RHS of Lemma 3.8. Let $\phi_i$ and $\psi$ satisfy (i), (ii) and (iii) above. Let $(b, c)$ be a generic solution of $\psi(x, y)$ over $M$ in some elementary extension $N$ of $M$. So $RM(tp(b, c/M)) = m + r$. But (by (i) and (ii)), $RM(tp(b/M) \leq m$, $RM(tp(c/M)) \leq m$ and $RM(tp(b/Mc)) \leq r$ and $RM(tp(c/Mb)) \leq r$. By Fact 3.2, we have equality throughout. By (iii) above, $tp(c/M) = \sigma(tp(b/M))$. So by the RHS of 3.8, we find $a \in M$ with $M \models \psi(a, \sigma(a))$.
Thus by Lemma 3.8, the first order conditions above axiomatize the class of e.c. models of $T_\sigma$, so $T_\sigma$ has a model companion.

**Corollary 3.10** *$ACF_\sigma$ has a model companion (ACFA).*

**Remark 3.11** *In the case $T = ACF$ the axioms (conditions in the proof above) for $ACFA$ are usually expressed as: whenever $V$ is an irreducible variety over $K$, $W$ is an irreducible subvariety of $V \times \sigma(V)$ over $K$ projecting generically onto $V$ and $\sigma(V)$ and $U$ is a nonempty Zariski open subset of $W$ defined over $K$, then there is $a \in L$ such that $(a, \sigma(a)) \in U$.*

**Problem 3.12** *(T strongly minimal.) Is it the case that $T_\sigma$ has a model companion if and only if $T$ has the DMP?*

From now on $T$ will be assumed to be a strongly minimal theory with the DMP such that any algebraically closed subset of a model of $T$ is infinite, or equivalently is an elementary substructure (e.g. $ACF$).

**Fact 3.13** *Any model of $T_\sigma$ is a strong amalgamation base for $T_\sigma$.*

*Proof.* Let the model $(M, \sigma)$ of $T_\sigma$ have extensions $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$, both models of $T_\sigma$. We may assume that $M_1$ and $M_2$ are independent over $M$ inside a larger model $N$ of $T$. Then $M_1 \cap M_2 = M$ (why?). Moreover by stationarity of types over models, $\sigma_1 \cup \sigma_2$ is an elementary map so we may assume it extends to an automorphism of $N$.

**Corollary 3.14** *(i) For any model $(M, \sigma)$ of $T_\sigma$, $TA \cup D((M, \sigma)$ is complete. (Namely $qftp((M, \sigma)$ determines its complete type in any model of $TA$).*
*(ii) The completions of $TA$ are classified by the isomorphism types of $(acl^-)(\emptyset), \sigma)$.*
*(iii) Let $(M, \sigma)$ be model of $TA$ and let $A$ be a substructure which is closed under $\sigma$ and $\sigma^{-1}$ and is algebraically closed for $L^-$. Then $A$ is algebraically closed in $(M, \sigma)$.*

*Proof.* By Lemmas 2.9, 2.13, and Fact 3.13.

**Remark 3.15** *So if $(M, \sigma)$ is a model of $TA$ and $A$ is a subset, then the algebraically closure of $A$ is obtained by first closing under $\sigma$ and its inverse, and then taking the algebraic closure in the sense of $L^-$.*

**Definition 3.16** *Let $\Sigma$ be the collection of $L$-formulas of the form $\exists y(\theta(x, \sigma(x), .., \sigma^n(x), y, \sigma(y), .., \sigma^m(y)))$ where $\theta$ is an $L^-$-formula, and, in $TA$, $\theta$ implies that $(y, \sigma(y), .., \sigma^m(y))$ is $L^-$-algebraic over $(x, \sigma(x), .., \sigma^n(x))$.*

**Lemma 3.17** *(Weak quantifier elimination, or strong model-completeness.) In $TA$ any formula $\phi(x)$ is equivalent to a finite disjunction of formulas in $\Sigma$. (In the case of $ACFA$ we can strengthen the condition on $\Sigma$ by requiring that $y$ be a single variable.)*

*Proof.* It is enough (why?) to prove that if $a$, $b$ are tuples in models $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ respectively of $TA$ and $\Sigma - tp(a) \subseteq \Sigma - tp(b)$ (in the respective models) then $a$ and $b$ have the same type (in the respective models). First, as $a$ and $b$ have the same quantifier-free types, there is an $L^-$ elementary map $f$ taking $(\sigma_1^i(a) : i \in \mathbf{Z})$ to $(\sigma_2^i(b) : i \in \mathbf{Z})$. A Konig's Lemma argument, together with our hypothesis shows that $f$ extends to an $L$-isomorphism $f'$ from $acl(a)$ into $acl(b)$. $f'$ has to be surjective (why?). So we have an $L$-isomorphism between $acl(a)$ and $acl(b)$ taking $a$ to $b$. By Corollary 3.14 (i), $a$ and $b$ have the same types (in their respective models).
The $ACFA$ version follows from the primitive element theorem.

**Lemma 3.18** *Suppose $(M, \sigma)$ is a model of $TA$. Then so is $(M, \sigma^k)$ for any $k \geq 1$.*

*Proof.* We only have to prove that $(M, \sigma^k)$ is also existentially closed. For this, all we need to show is that if $(N, \tau)$ is an extension of $(M, \sigma^k)$ then there is an elementary extension $N'$ of $N$ and an extension $\sigma'$ of $\sigma$ to $N'$ such that $(\sigma')^k$ agrees with $\tau$ on $N$. Let $p(x) = tp^-(N/M)$. For $i = 2, .., k-1$ let $N_i$ realise $\sigma^i(p(x))$, such that $\{N, N_1, N_2, .., N_{k-1}\}$ is $M$-independent (in sense of $T$). Then $\sigma(tp^-((N, N_1, N_2, .., N_{k-1}/M))) = tp^-((N_1, N_2, .., N_{k-1}, \tau(N))/M)$, so $\sigma$ extends to an automorphism $\sigma'$ of a larger model $N'$ of $T$ such that $\sigma'(N, N_1, N_2, .., N_{k-1}) = (N_1, N_2, .., N_{k-1}, \tau(N))$. It follows that $(\sigma')^k$ agrees with $\tau$ on $N$.

**Definition 3.19** *Let $(M, \sigma)$ be a model of $TA$. Then by a fixed set, we mean any subset of $M$ of the form $\{a \in M : f(a) = \sigma^k(a)$, where $f$ is a $\emptyset$-definable (in $L^-$) automorphism of $M$ and $k > 0$. In the case where $f = id$ and $k = 1$ we call it* the *fixed set.*

**Remark 3.20** *In the ACFA case, the only possibilities for $f$ are the identity in characteristic* 0 *and powers (possibly negative) of the Frobenius in positive characteristic. In any case we talk about fixed fields in place of fixed sets.*

We want to develop properties of fixed sets in the general strongly minimal set context. As Galois groups play a role we need an additional assumption on $T$, a little weaker than elimination of imaginaries. We should remark that our assumption that $acl(\emptyset)$ is infinite in all models of $T$ implies that $T$ has weak elimination of imaginaries: for any $e \in M^{eq}$ there is some real tuple $c$ such that $c \in acl(e)$ and $e \in dcl(c)$. There is another property worth mentioning, geometric elimination of imaginaries: any imaginary $e$ is interalgebraic with some real tuple.

**Definition 3.21** *The (strongly minimal) structure $M$ has elimination of Galois imaginaries (EGI), if whenever $e \in M^{eq}$ and $e \in acl(c)$ for some real tuple $c$ from $M$, then there is a real tuple $e'$ such that $dcl(e, c) = dcl(e', c)$. $T$ is said to have the property if all its models (completions) do.*

**Remark 3.22** *Infinite sets (with no structure) and vector spaces are examples of strongly minimal sets which have elimination of Galois imaginaries but not elimination of imaginaries.*

**Lemma 3.23** *($M$ has EGI.) Let $M \models T$ and $A \subset M$ a definably closed subset. Let $G = Aut(acl(A)/A)$ (the group of elementary permutations of $acl(A)$ fixing $A$), considered as a profinite group. Then there is the usual Galois correspondence between closed subgroups of $G$ and definably closed subsets of $acl(A)$ containing $A$. We also call $G$ the (absolute) Galois group of $A$, $Gal(A)$.*

*Proof.* Exercise.

**Definition 3.24** *A substructure $A$ of $M$ is $PAC$ if it is definably closed in $M$ and every formula $\phi(x)$ over $A$ which has multiplicity $1$ has a solution in $A$.*

**Remark 3.25** *A pseudofinite field is by definition an infinite model of the theory of finite fields. Ax proved that $F$ is a pseudofinite field iff it is perfect, PAC (in its field-theoretic algebraic closure) and $Gal(F)$ is the profinite completion of $\mathbf{Z}$ (equivalently $Gal(F)$ has a unique open subgroup of index $n$ for all $n$).*

**Proposition 3.26** *Let $(M, \sigma)$ be a model of $TA$. Let $A$ be any fixed set. Then $A$ is PAC (in $M$). If $T$ has EGI then $Gal(A)$ is procyclic (has at most one open subgroup of index $n$ for each $n$). If moreover $T$ has EI then $Gal(A)$ is the profinite completion of* $\mathbf{Z}$. *In particular in the ACFA case any fixed field is pseudofinite.*

*Proof.* Assume $A$ is defined as the fixed set of $f.\sigma^k$. By Lemma 3.18, we may assume $k = 1$. It is clear first that $A$ is definably closed (in $M$). Let $\phi(x)$ be a multiplicity 1 $L^-$ formula over $A$. Let $c$ be a generic solution of $\phi(x)$ in some elementary extension $N$ of $M$, and $p(x) = tp^-(c/M)$. Note that $f.\sigma(p) = p$ and so clearly we can extend $\sigma$ to an automorphism $\sigma'$ of $N$ such that $f.\sigma'(c) = c$. Now use existential closure of $(M, \sigma)$ to find $d$ in $M$ satisfying $\phi(x)$ such that $f.\sigma(d) = d$. So $d \in A$.
Now we look at the Galois group business. Let $G$ be the Galois group of $A$. We first show that $G$ has at most one open subgroup of index $n$ for each $n$. It is enough to show that $< f.\sigma >$ is dense in $G$. By Lemma 3.2, the closure of $< f.\sigma >$ in $G$ is the set of elements of $G$ fixing the fixed set of $< f.\sigma >$. As the latter is precisely $A$, we are finished.
Finally we show, assuming that $T$ has EI, that $Gal(A)$ has at least one open subgroup of index $n$ for each $n$. Let $N$ be an elementary extension of $M$ containing algebraically independent (over $M$) elements (1-tuples) $c_1, .., c_n$, and let $\sigma'$ be an automorphism of $N$ extending $\sigma$ such that $f.\sigma'(c_i) = c_{i+1}$ for $i = 1, .., n-1$ and $f.\sigma'(c_n) = c_1$. As $(M, \sigma)$ is e.c., we find distinct such elements $d_1, .., d_n$ in $M$ (with $f.\sigma$ in place of $f.\sigma'$). Let $e$ be the imaginary element $\{d_1, .., d_n\}$. Clearly $e$ is fixed by $\sigma$. By EI, $e \in dcl(A)$ and thus $d_1, .., d_n \in acl(A)$ and moreover $B = dcl(A, d_1, .., d_n)$ is a finite Galois extension of $A$. By the second paragraph $Aut(B/A)$ is generated by $f.\sigma$. As $(f.\sigma)^n$ is the identity on $B$ (and $n$ is smallest such), $Aut(B/A)$ has cardinality exactly $n$ and we finish.

**Corollary 3.27** *$ACFA$ is unstable.*

*Proof.* A basic result in stable group theory says that if $F$ is an infinite field definable in a model of a stable theory, then neither the additive nor multiplicative groups of $F$ have any definable subgroups of finite index. However, if $F$ is a pseudofinite field, then for $n \neq char(F)$, the $nth$ powers form a (definable) subgroup of $F^*$ of index $n$.

**Exercise 3.28** *Show that if $F$ is a pseudofinite field of characteristic $\neq n$, and $R$ is the set of $n$th powers of the multiplicative group of $F$, then the additive translates of $R$ form an independent family of sets (any finite Boolean combination is consistent). Thus $ACFA$ has the independence property.*

**Remark 3.29** *We will study in a later section the general question of when $TA$ is unstable. This will be related to the geometry of algebraic closure in models of $T$. (We should remark that as $TA$ will be "simple", it will be unstable iff it has the independence property.) For example we'll see that if $T$ is "not locally modular" then $TA$ is unstable. In particular, using a result from [8], it follows that if $T$ has $EI$ then $TA$ is unstable. It would be nice to see a direct proof of this last result.*

# 4 Independence theorem, simplicity and consequences.

The assumptions on the strongly minimal theory $T$ remain in place. Let us fix $(\bar{M}, \sigma)$ a very saturated model of $TA$ in which we will work. $acl(-)$ denotes algebraic closure in this structure (called $acl_\sigma(-)$ by Zoe). ($acl^-(-)$ denotes algebraic closure in the $L^-$ -structure $\bar{M}$.) $A, B, C...$ denote small subsets of $\bar{M}$

**Definition 4.1** *Let $A \subseteq B$, $A \subseteq C$. We will say that $B$ is independent from $C$ over $A$ if $acl(B)$ is independent from $acl(C)$ over $acl(A)$ in the sense of the strongly minimal structure $\bar{M}$ (equivalently $\{\sigma^i(b) : i \in \mathbf{Z}, b \in B\}$ is independent from $\{\sigma^i(c) : i \in \mathbf{Z}, c \in C\}$ over $\{\sigma^i(a) : i \in \mathbf{Z}, a \in A\}$ in the sense of $M$). If $b$ is a finite tuple we say that $b$ is independent from $C$ over $A$ ($A \subseteq C$ again) if $A \cup \{b\}$ is.*

As a matter of notation, for $A$ a set, we let $cl_\sigma(A)$ denote $\{\sigma^i(a) : i \in \mathbf{Z}, a \in A\}$. Note that $acl(A)$ is precisely $acl^-(cl_\sigma(A))$.

**Lemma 4.2** *Let $A \subseteq C \subseteq D$, and let $b$ be a finite tuple.*
*(i) $b$ is independent from $acl(A)$ over $A$.*
*(ii) There is a countable subset $A_0$ of $A$ such that $b$ is independent from $A$ over $A_0$.*
*(iii) $b$ is independent from $C$ over $A$ iff $C$ (or equivalently every finite tuple*

*from $C$) is independent from $A \cup \{b\}$ over $A$.*
*(iv) $b$ is independent from $D$ over $A$ iff $b$ is independent from $D$ over $C$ and $b$ is independent from $C$ over $A$.*
*(v) there is $b'$ such that $tp(b'/A) = tp(b/A)$ and $b'$ is independent from $C$ over $A$.*
*(vi) if $b \in acl(C)$ and $b$ is independent from $C$ over $A$ then $b \in acl(A)$.*

*Proof.* (i) - (iv) follow from the definition and the analogous properties for independence in the strongly minimal structure $\bar{M}$.
(v) needs a couple of words. We may assume that $A$ and $C$ are algebraically closed, and we may replace $b$ by $acl(A \cup \{b\})$ (although it is now a tuple of infinite length). By the corresponding property for strongly minimal sets we may find $b'$ in $\bar{M}$, such that $tp^-(b'/A) = tp^-(b/A)$ and $b'$ is independent from $C$ over $A$ in the sense of $M$. Let $\tau$ be the image of $\sigma$ under the $L^-$-elementary map taking $b$ to $b'$. By the stationarity of types over models in strongly minimal sets, $\tau$ and $\sigma$ are compatible, in that they have a common extension to an automorphism $\sigma'$ of $acl^-(C, b')$. By 3.14, $T_\sigma \cup D((C, \sigma))$ is complete, whereby (by saturation) there is an embedding of $(acl^-(C, b'), \sigma')$ into $(\bar{M}, \sigma)$ over $(C, \sigma)$. Let $b''$ be the image of $b$. Thus $qftp(b''/A) = qftp(b/A)$ (under the canonical map taking the tuple $b$ to the tuple $b''$). By 3.14 again $tp(b''/A) = tp(b/A)$ and note that $b''$ is independent from $C$ over $A$.

**Proposition 4.3** *(Independence Theorem over algebraically closed sets.) Let $A, B, C$ be algebraically closed, with $A \subseteq B$, $A \subseteq C$ and $B$ independent fom $C$ over $A$. Let $d_1$, $d_2$ be such that $tp(d_1/A) = tp(d_2/A)$, $d_1$ is independent from $B$ over $A$ and $d_2$ is independent from $C$ over $A$. Then there is $d$ such that $tp(d/B) = tp(d_1/B)$, $tp(d/C) = tp(d_2/C)$ and $d$ is independent from $B \cup C$ over $A$.*

*Proof.* We may assume, by Lemma 4.2 (v) that $d_1$ is independent from $B \cup C$ over $B$ (and thus over $A$). Let $D_1 = acl(A, d_1)$. Note that $acl^-(B, D_1) = acl(B, D_1)$ and similarly for $C$ in place of $B$.
*Claim.* Let $e \in \bar{M}^{eq}$. Then $e \in dcl^-(acl^-(B, C), acl^-(B, D_1)) \cap acl^-(C, D_1)$ iff $e \in dcl^-(C, D_1)$.
*Proof of Claim.* Right to left is clear. Left to right: Suppose $e = f(g, h)$ where $g \in acl^-(B, C)$, $h \in acl^-(B, D_1)$, $e \in acl^-(C, D_1)$, and $f$ is an $L^-$-definable function (over $\emptyset$). So there are tuples $b \in B, c \in C$ and $d' \in$

$D_1$ and $L^-$-formulas (over $A$), and $L^-$ formulas (over $A$) $\phi_i(x_i, y_i, z_i)$ for $i = 1, 2$, each implying that $z_i$ is $L^-$-algebraic over $A, x_i, y_i$ such that $\models \phi_1(b, c, g) \wedge \phi_2(b, d', h)$. Note that $acl^-(C, D_1)$ is $L^-$-independent from $B$ over $A$. Thus (as $A$ is an elementary substructure of $\bar{M}$, $tp^-(e, c, d'/B)$ is finitely satisfisfiable in $A$ (why?). Thus we can find a tuple $a \in A$, such that $\models \exists g', h'(e = f(g', h') \wedge \phi_1(a, c, g') \wedge \phi_2(a, d', h))$.
Thus $e \in dcl^-(C, D_1)$, proving the claim.

It follows from the claim that
(*) $Aut^-(acl^-(C, D_1)/(C \cup D_1)) =$
$Aut^-(acl^-(C, D_1)/acl^-(B, C) \cup acl^-(B, D_1))$ (exercise).
Let $D_2 = acl(A, d_2)$. As $tp(d_2/A) = tp(d_1/A)$ there is an $A$-elementary map $f$ taking $D_2$ to $D_1$ (and taking $d_2$ to $d_1$). Note that (as $C$ is independent from each of $D_2$, $D_1$ over $A$), $f$ is also $C$-elementary. $f$ extends to an $L^-$-elementary map (over $C$) $f'$ taking $acl(C, D_2)$ to $acl(C, D_1)$ and let $\tau$ be the image of $\sigma$ under $f'$. (So $\tau$ agrees with $\sigma$ on $dcl^-(C, D_1)$.) By (*), the restriction of $\sigma$ to $acl(B, C) \cup acl(B, D_1)$ together with $\tau$ on $acl(C, D_1)$ extends to an $L^-$ automorphism $\sigma'$ on $acl^-(B, C, D_1)$. As $(acl(B, C), \sigma)$ is an amalgamation base, we may find $D$ in $\bar{M}$ and an isomorphism $g$ between $(acl^-(B, C, D_1), \sigma')$ and $(acl^-(B, C, D), \sigma)$, which is the identity on $acl^-(B, C)$. If $d$ is the image of $d_1$ under $g$ then clearly $tp(d/B) = tp(d_1/B)$ and $tp(d/C) = tp(d_2/C)$. The independence of $d$ from $B, c$ over $A$ is also clear.

Recall that (working in a saturated model $N$ of some theory), a type $p(x, b)$ is said to divide over a set $A$ if for some $A$-indiscernible sequence $(b_i : i < \omega)$ of realizations of $tp(b/A)$, $\bigcup_i p(x, b_i)$ is inconsistent. $Th(N)$ is said to be (super)simple if any complete type $tp(a/B)$ ($a$ a finite tuple) does not divide over some subset $A$ of $B$ of cardinality at most $|T|$ ($< \omega$). Any stable theory is simple. We leave as an exercise to check that independence agrees with nondividing in strongly minimal sets.

**Lemma 4.4** *$TA$ is simple. Moreover for any $a, A \subseteq B$, $a$ is independent from $B$ over $A$ iff $tp(a/B)$ does not divide over $A$. Moreover the independence relation extends to variable and parameter sets in $(\bar{M}, \sigma)^{eq}$ (namely the properties listed in Lemma 4.2 hold).*

*Proof.* In [6] it is proved that a relation of independence satisfying (ii), (iii), (iv), (v) of Lemma 4.2, plus the Independence Theorem over elementary substructures must be nondividing. (Alternatively a modification of the proof of Proposition 4.3 shows directly that independence as defined in TA coincides with nondividing.)

**Lemma 4.5** *$TA$ is supersimple.*

*Proof.* It is enough to show that any 1-type does not divide over a finite set. So let $a$ be a single element in $(\bar{M}, \sigma)$, and let $B$ be an algebraically closed subset. If $\{\sigma^i(a) : i \in \mathbf{Z}\}$ is $L^-$-algebraically independent over $B$ then clearly $a$ is independent from $B$ over $\emptyset$. Otherwise let $n$ be minimal such that $(a, \sigma(a), .., \sigma^n(a))$ is $L^-$-algebraically dependent over $B$. This is witnessed by some finite tuple $b$ from $B$. Then it is clear that $(\sigma^i(a) : i \in \mathbf{Z})$ is $L^-$ independent from $B$ over $(\sigma^i(b) : i \in \mathbf{Z})$.

**Exercise 4.6** *Show that over any countable set $A$ there are only countably many quantifier-free types. (Again it is enough to do this for 1-types.) Namely TA is quantifier-free $\omega$-stable.*

**Lemma 4.7** *For any tuple $a$ and algebraically closed $B \subset \bar{M}$ there is a unique smallest algebraically closed subset $A$ of $B$ such that $a$ is independent from $B$ over $A$.*

*Proof.* This follows from our definition of independence and the fact (exercise) that the same thing holds in the structure $\bar{M}$.

**Proposition 4.8** *$TA$ has weak elimination of imaginaries.*

*Proof.* Our proof will first yield geometric elimination of imaginaries, using rather weak hypotheses (the existence of a rudimentary independence relation extending to imaginaries) and then get weak elimination using the Independence Theorem. Let $e$ be an imaginary element, and let $a$ be a finite tuple such that $e = a/E$ for some $\emptyset$-definable equivalence relation. We write $e = f(a)$ for $f$ a $\emptyset$-definable function. Let $p(x) = tp(a/e)$. Let $b$ realize $p(x)$ such that $b$ is independent from $a$ over $e$, and let $c$ realise $p(x)$ such that $c$ is independent from $B = acl(a, b)$ over $e$. By Lemma 4.7, let $A \subseteq B$ be the smallest algebraically closed subset of $\bar{M}$ such that $c$ is independent from $B$

over $A$. However clearly $c$ is independent from $B$ over $acl(a) \cap \bar{M}$ and also over $acl(b) \cap \bar{M}$. Thus $A \subseteq acl(a) \cap acl(b)$. By 4.2 (iv)(for imaginaries), $A \subseteq acl(e)$. On the other hand, by symmetry $e$ is independent from $c$ over $A$. As $e \in acl^{eq}(c)$, by 4.2 (vi) (for imaginaries), $e \in acl^{eq}(A)$. Thus we find some finite tuple $a'$ from $A$ such that $e$ is interalgebraic with $a'$. We have shown geometric elimination of imaginaries. To show weak elimination, it is enough to show that $e \in dcl^{eq}(A)$. So suppose $e_1, e_2$ have the same type as $e$ over $A$. Let $q(x,e) = tp(c/Ae)$, and $q_0(x) = tp(c/A)$. Let $c_i$ realise $q(x,e_i)$ be such that $c_2$ is independent from $c_1$ over $Ae_2$ and thus over $A$ (as $e_2 \in acl^{eq}(A)$). By the Independence Theorem over algebraically closed sets, we can easily find $c_3$ realising $q(x,e_1) \cup q(x,e_2)$. So $e_1 = f(c_3) = e_2$.

**Corollary 4.9** *If $T$ has elimination of imaginaries (for example $T = ACF$), then so does $TA$.*

*Proof.* Let $e$ be an imaginary element. By the previous proposition, there is a real tuple $a$ such that $a \in dcl(e)$ and $e \in acl(a)$. Let $a_1, .., a_n$ be the $e$-conjugates of $a$. As $T$ has elimination of imaginaries, the finite set $\{a_1, .., a_n\}$ is interdefinable with some real tuple $b$. Clearly $b$ is interdefinable with $e$.

The next proposition contains quite a bit of interesting theory.

**Proposition 4.10** *Let $A$ be a fixed set of the form $Fix(f.\sigma)$ (so $f$ is an $L^-$-definable over $\emptyset$ automorphism of $\bar{M}$). Let $A^*$ be $A$ considered as an $L^-$-structure.*
*(i) If $T$ has Galois elimination of imaginaries then any subset of $A^n$ definable (in $(\bar{M},\sigma)$) with parameters in $acl(A)$ is definable in the structure $A^*$.*
*(ii) If $T$ has $EI$ then any subset of $A^n$ definable (with any parameters) in $(\bar{M},\sigma)$ is definable in $A^*$.*

*Proof.* (i) Let $X \subseteq A^n$ be definable in $(\bar{M},\sigma)$ over parameter $d \in acl(A)$, by a formula $\phi(x,d)$. Replacing $\sigma(x)$ by $f^{-1}(x)$, and $d$ by some tuple of iterates of $\sigma$ applied to $d$ and using 3.17, $\phi(x,d)$ can be assumed to be a finite disjunction of formulas $\exists y(\theta(x,y,\sigma(y),..,\sigma^k(y),d))$ where $\theta(x,y_0,.,y_k,d)$ is an $L^-$-formula (with parameter $d$) which implies that each $y_i \in acl^-(x,d)$. We will assume for simplicity that $\phi(x,d)$ is a single such formula.
Note by 3.26 that there is a countable set $\{b_i : i \in I\}$ of finite tuples in $acl^-(A)$, each such tuple is permuted by $\sigma$ and such that whenever $c \in$

$acl^-(A)$ then $c \in dcl^-(A, b_i)$ for some $i$. It follows by compactness (saturation) that we can find some finite tuple $b \in acl^-(A)$ (permuted by $\sigma$) and some $L^-$-definable function $h$ such that whenever $a \in A$ and $\theta(a, c_0, .., c_k, d)$ holds then each $c_i$ is of the form $h(b, e)$ for some $e \in A$. Moreover we can also assume that $d = b$. Thus for $a \in A$, $\models \exists y_0, .., y_k(\theta(a, y_0, .., y_k, d)$ iff there exist $e_0, .., e_k \in A$ such that
$\models \theta(a, h(d, e_0), .., h(d, e_k), d)$.
Thus (for $a \in A$),$\models \exists y(\theta(a, y, \sigma(y), .., \sigma^k(y), d)$ iff there exists $e_0 \in A$ such that
$\models \theta(a, h(d, e_0), h(\sigma(d), f^{-1}(e_0)), .., h(\sigma^k(d), f^{-k}(e_0)), d)$.
By the (trivial) definability of $tp^-(d/A)$ the latter is equivalent to a (qf) $L^-$ formula $\chi(a, e_0)$ over $A$. Thus for $a \in A$, $\models \phi(a, d)$ iff $A^* \models \exists z(\chi(a, z))$. We have completed the proof.
(ii) Let $X \subset A^n$ be definable in $(\bar{M}, \sigma)$. Let $e$ be a canonical parameter for $X$. By Corollary 4.9, $e$ may be assumed to be a tuple from $\bar{M}$. Note that $f.\sigma$ is an automorphism of $(\bar{M}, \sigma)$. As $X$ is fixed setwise by $f.\sigma$, $e$ must be fixed (as a tuple) by $f.\sigma$, so $e \in A$. Now use part (i).

Recall that in a simple theory, the $SU$-rank (on complete types) is the foundation rank for forking: $SU(p) \geq \alpha + 1$ if $p$ has a forking extension $q$ of $SU$-rank $\geq \alpha$. Moreover a theory is supersimple if and only if $SU(-)$ is ordinal-valued on complete types in finitely many variables. Also in a simple theory we have (for $a$ a finite tuple), $SU(tp(a/A)) = SU(tp(a/B))$ iff $tp(a/B)$ does not fork over $A$. The $SU$-rank of a type is 0 iff the type is algebraic, i.e. has only finitely many realizations. We will say a few words about the $SU$-rank in $TA$. By $U^-(-)$ we mean $U$-rank in $T$ (defined likewise in terms of forking in $T$), and the reader should be aware that the $U^-$-rank of a complete type in a model of $T$ is the same as the Morley rank, or "dimension". We will be considering $U^-$-rank of possibly infinite tuples. You should realise that for $a$ a possibly infinite tuple in a model of $T$, $U^-(tp^-(a/A))$ is ordinal valued iff it is finite iff $a$ is contained in the algebraic closure (in $T$) of $A$ together with a finite subtuple of $a$.

Let us first repeat our definition of dependence in $TA$. $A$, $B$ .. will usually denote algebraically closed subsets of $(\bar{M}, \sigma)$.

**Remark 4.11** *Let $A \subseteq B$ be algebraically closed. Let $a$ be a finite tuple.*

*Then a is dependent with B over A (tp(a/B) forks over A, $SU(tp(a/B)) < SU(tp(a/A))$) iff for some n,*
$U^-(tp^-(a, \sigma(a), .., \sigma^n(a))/B)) < U^-(tp^-((a, \sigma(a), .., \sigma^n(a))/A))$.

**Corollary 4.12** *Suppose $U^-(tp^-(cl_\sigma(a)/B)) = n$. Then $SU(tp(a/B) \leq n$.*

**Lemma 4.13** *Suppose that a is an element. Let n be such that $\{a, \sigma(a), .., \sigma^n(a)\}$ is $L^-$-algebraically dependent over B, and suppose that n is minimal such. Then $U^-(tp^-(cl_\sigma(a)/B)) = n$.*

*Proof.* Exercise.

**Corollary 4.14** *Let a be a single element. Then $SU(tp(a/A)) \leq \omega$.*

*Proof.* If $tp(a/B)$ forks over $B$ ($A \subseteq B$) then $cl_\sigma(a)$ must be $L^-$-algebraically dependent over $C$, whereby the hypothesis of the previous lemma holds for some $n$. But then $SU(tp(a/B)) \leq n$ by 4.12. So every forking extension of $tp(a/A)$ has finite $SU$-rank, whereby $SU(tp(a/A)) \leq \omega$.

We will see that various geometric properties of dependence (nontriviality, nonmodularity,..) in $T$ will have model-theoretic consequences (concerning $SU$-rank, instability,...) for $TA$. Here we point out the interpretation of triviality/nontriviality. Recall that $T$ is said to be trivial, if whenever $A$ is a subset of $\bar{M}$ and $\{a_i : i \in I\} \subset \bar{M}$ is pairwise $L^-$-algebraically indepependent over $A$ then it is $L^-$-algebraically independent over $A$.

**Lemma 4.15** *Suppose T is trivial. Then every complete 1-type has SU-rank at most 1.*

*Proof.* Consider $tp(a/A)$. Suppose $a$ forks with $B$ over $A$. So for some $n$, $\{a, \sigma(a), .., \sigma^n(a)\}$ is $L^-$-algebraically independent over $A$, but $L^-$-algebraically dependent over $B$. By triviality this can only happen if some $\sigma^i(a)$ (and thus all $\sigma^i(a)$) are in $acl^-(B) = B$. Namely $SU(tp(a/B)) = 0$.

**Lemma 4.16** *Suppose that T is nontrivial. Then there is a complete 1-type in TA of SU-rank $\omega$ (and so there are 1-types of all finite SU-ranks).*

*Proof.*. Working over a suitable subset $A$ of $\bar{M}$ we may, by nontriviality, find elements $b, c, d$ which are (in $\bar{M}$), pairwise independent but dependent over $A$. Namely each of $b, c, d \notin acl^-(A)$, $U^-(tp(b, c, d/A)) = 2$ and each of $b, c, d$ is in $acl^-$ of $A$ and the other two. We may assume $A$ is algebraically closed in $(\bar{M}, \sigma)$. Let $q(x, y, z) = tp^-((b, c, d)/A)$. Now let $a$ be "transformally generic" over $A$, namely $cl_\sigma(a)$ is $L^-$-algebraically independent over $A$.
*Claim 1.* Let $e$ be such that $tp^-((a, \sigma(a), e)/A) = q$. Then $\{a\} \cup \{\sigma^i(e) : i \in \mathbf{Z}\}$ is $L^-$-algebraically independent over $A$. In particular $a \notin acl(A, e)$.
*Proof.* Fix $n < \omega$ and let $X = \{a\} \cup \{\sigma^i(e) : -n \leq i \leq n\}$, a set of $2n + 2$ elements. Note that $tp((\sigma^j(a), \sigma^{j+1}(a), \sigma^j(e))/A) = \sigma(q)$. It follows from the choice of $q$ that $acl^-(X, A) = acl^-(Y, A)$ where $Y = \{\sigma^{-n}(a), ..., \sigma^{-1}(a), a, \sigma(a), ..\sigma^{n+1}(a)\}$. But $U^-(tp^-(Y/A)) = |Y| = 2n + 2$, so the same is true for $U^-(tp^-(X/A))$, implying that $X$ is also $L^-$-algebraically independent over $A$. This proves Claim 1.

Now, let us define inductively $b_i$ for $i < \omega$, by $b_0 = a$, and $b_{i+1}$ realises $q(b_i, \sigma(b_i), z)$ over $A$. By induction, and Claim 1, we obtain.
*Claim 2.* For each $i$, $b_i$ is transformally generic over $A$, and $b_i \notin acl(A, b_{i+1})$.
Note that for $j > i$, $b_j \in acl(A, b_i)$. Now let us fix $n \geq 0$.
*Claim 3.* $tp(a/acl(A, b_n))$ forks over $A \cup \{b_{n+1}\}$.
*Proof.* As remarked above, $b_n \in acl(A, a)$ but by Claim 2, $b_n \notin acl(A, b_{n+1})$. Thus $tp(b_n/A, a, b_{n+1})$ forks over $A, b_{n+1}$. Now use symmetry.
By Claim 3, we see that for any $n$, and any $i < n$, $tp(a/A, b_i, .., b_n)$ forks over $(A, b_{i+1}, .., b_n)$, whereby $SU(tp(a/A, b_n) \geq n$. Thus $SU(tp(a/A)) = \omega$ (using also Corollary 4.14).

**Corollary 4.17** *(Any completion of) ACFA has SU-rank $\omega$.*

# 5 Stationarity, stability and modularity

Here we obtain some rather deeper (but not difficult) results relating the model theory of $TA$ to the geometry of $T$. We are still working in a saturated model $(\bar{M}, \sigma)$ of $TA$.

**Definition 5.1** *$tp(a/A)$ is stationary, if it has a unique nonforking extension over any $B \supseteq A$. Namely whenever $B \supseteq A$ and $a_1, a_2$ are realisations of $tp(a/A)$, each independent from $B$ over $A$, then $tp(a_1/B) = tp(a_2/B)$.*

**Remark 5.2** *The following are equivalent:*
*(i) $TA$ is stable,*
*(ii) $TA$ is superstable*
*(iii) $TA$ does not have the independence property,*
*(iv) every complete type over an algebraically closed set is stationary.*

*Proof.* (This is well-known by the general theory of simplicity, but we give a proof nevertheless.) (i) implies (ii) is because we know by 4.5, that every 1-type does not fork over some finite set. (ii) implies (iii) is well-known.
(iii) implies (iv): Suppose that $A$ is algebraically closed, $p(x)$ is a complete type over $A$ and $p$ has two distinct nonforking extensions $q_1(x), q_2(x)$ over some $B \supseteq A$. Let $\phi(x, b) \in q_1$, $\neg\phi(x, b) \in q_2$. Let $\{B^i : i \in \omega\}$ be an $A$-independent set of realizations of $tp(B/A)$. Let $b^i$ be the copy of $b$ in $B^i$ and $q_j^i(x)$ the copy of $q_j(x)$ over $B^i$ for $j = 1, 2$. By the Independence Theorem, for each $I \subseteq \omega$ we can find some element $a_I$ which realises $q_1^i(x)$ if $i \in I$ and $q_2^i(x)$ for $i \notin I$. In particular $\models \phi(a_I, b_i)$ iff $i \in I$, yielding the independence property.
(iv) implies (i). Using (iv) and the fact that every complete type does not fork over a countable (in fact finite) set, we conclude that over any model $N$ of $TA$ there are at most $|N|^\omega$ complete types (in finitely many variables), which means that $T$ is stable.

There is a very clear criterion for a type to be stationary.

**Lemma 5.3** *Let $A$ be algebraically closed. Let $b$ be a tuple and let $B = acl(A, b)$. Then $tp(b/A)$ is stationary if and only if, whenever $C \supseteq A$ is algebraicaly closed and independent of $B$ over $A$, them $\sigma|(dcl^-(C, B))$ has a unique extension to an automorphism of $acl^-(C, B)$ up to conjugacy in $Aut^-(acl^-(C, B)/dcl^-(C, B))$.*

*Proof.* First we note that $tp(b/A)$ is stationary iff $tp(b'/A)$ is stationary, where $b'$ enumerates $B = acl(A, b)$ (exercise). So we may assume that $b$ enumerates $B$. The lemma now follows once we remember that $tp(b_1/C) = tp(b_2/C)$ iff there is an isomorphism between $(acl(C, b_1), \sigma)$ and $(acl(C, b_2), \sigma)$ which fixes $C$ pointwise and takes $b_1$ to $b_2$.

**Corollary 5.4** *$TA$ is stable (every completion of $TA$ is stable) iff whenever $A \subseteq B, C$ are algebraically closed sets in a model $M$ of $T$ such that $B$ is independent from $C$ over $A$, then $dcl^-(B, C) = acl^-(B, C)$.*

*Proof.* Suppose that the right hand side holds. Then by Lemma 5.3, every type over an algebraically closed set in $(\bar{M},\sigma)$ is stationary, so $TA$ is stable by Remark 5.2. Conversely, suppose the right-hand side fails, witnessed by $A,B,C$ in some model $M$ of $T$. Assume that $(\bar{M},\sigma)$ is our saturated model of $TA$ and that $Th^{-}(\bar{M}) = Th(M)$ and that $\sigma$ fixes $acl^{-}(\emptyset)$ pointwise. So we may assume that $A,B,C$ are contained in $Fix(\sigma)$ (why?). So $dcl^{-}(B,C)$ is contained in $Fix(\sigma)$. As $acl^{-}(B,C)$ properly contains $dcl^{-}(B,C)$, the identity $(\sigma)$ on $dcl^{-}(B,C)$ can be extended to a nontrivial $L^{-}$-elementary permutation of $acl^{-}(B,C)$, clearly not conjugate to the identity in $Aut^{-}(acl^{-}(B,C)/dcl^{-}(B,C))$. So by Lemma 5.3, $tp(B/A)$ is not stationary, hence by Remark 5.2, $(\bar{M},\sigma)$ is not stable.
(This proof can be modified to show that any completion on $TA$ extending $Th^{-}(M)$ is unstable.)

**Remark 5.5** *The proof above shows that if (some completion of) $TA$ is unstable, then this is witnessed inside $Fix(\sigma)$.*

**Example 5.6** *If $T$ is trivial (such as the theory of an infinite set) then the RHS of Corollary holds so $TA$ is stable. By 4.3, $TA$ has $U$-rank 1 (but not Morley rank 1).*

**Example 5.7** *If $T$ is the theory of a vector space over a field $F$ (in the $F$-module language), then definable closure and algebraic closure coincide in models of $T$ so again the RHS holds and $TA$ is stable.*

**Example 5.8** *Let $T$ be the theory of a "trivial 2-cover" of a vector space over $F$. Namely $T = Th(D)$ where $D$ is equipped with an equivalence relation $E$ each class of which has size 2, and such that $D/E$ is equipped with the structure of an $F$-vector space. Write $a/E$ as $\pi(a)$. Let $V_0 \subseteq V_1, V_2$ be subspaces of $V$ with $V_1, V_2$ linearly independent over $V_0$. Let $A,B,C$ be the preimages under $\pi$ of $V_0, V_1, V_2$ respectively. Let $v_i \in V_i \setminus V_0$, and let $v = v_1 + v_2$. Then $\pi^{-1}(v) \in acl(B,C) \setminus dcl(B,C)$. Hence $TA$ is unstable.*

**Definition 5.9** *$T$ (or rather a fixed completion $Th(M)$ of $T$) is modular (or of modular type, or 1-based) if whenever $A,B$ are algebraically closed subsets of $M$ (where $M$ is assumed saturated) then $A$ is independent from $B$ over $A \cap B$ (in sense of $T$ of course).*

**Remark 5.10** *For an arbitrary stable theory $T$, the definition of $T$ being 1-based is the same except we work in $T^{eq}$. As we know that our (strongly minimal) $T$ has weak elimination of imaginaries the definitions agree. (The notion "locally modular" concerns arbitrary strongly minimal theories (or sets) and means that we have modularity (in the home sort) after adding a generic constant.) There is a similar notion of 1-basedness for a simple theory $T$. For our simple theory $TA$, as we have weak elimination of imaginaries (and "elimination of hyperimaginaries") it just amounts to $A$ and $B$ being independent over $A \cap B$ for any algebraically closed sets $A$ and $B$.*

**Fact 5.11** *Suppose $T$ is non modular. Then there are (in a saturated model of $T$), sets $D_1, D_2$ and tuples $a_1, a_2$ such that*
*(i) $D_i$ are ($L^-$)-algebraically closed and $D_1$ is independent from $D_2$ over $\emptyset$.*
*(ii) $a_1$ is independent from $a_2$ over $\emptyset$.*
*(iii) $a_1, a_2 \in acl^-(D_1, D_2)$, and*
*(iv) $tp^-(a_1, a_2/D_1, D_2) = tp^-(a_2, a_1/D_1, D_2)$.*

*Explanation.* This is a rather nontrivial fact, coming out of a result of Buechler (part of the proof of which was supplied by Hrushovski) that a "pseudo-linear" strongly minimal set is modular. This is Proposition 3.2, Chapter 5, in [9], and the relevant application is Corollary 3.5, Chapter 5, in the same book. I have given (and will repeat) an informal (and rather inaccurate) description of "canonical bases" in strongly minimal sets: let $p(x)$ be a type over an algebraically closed set, then $Cb(p)$ is the smallest algebraically closed subset $A_0$ of $A$ such that $p$ does not fork over $A_0$. Assuming $x$ is a finite tuple, $A_0$ will be the algebraic closure of a finite tuple $c$ say and by the dimension or $U$-rank of $A_0$ we mean $U(tp(c/\emptyset))$. The result of Buechler's alluded to above is that if $T$ is nonmodular, then for any $k < \omega$ there is some type $p(x)$ such that $U(p) = 1$ and $U(Cb(p)) \geq k$. So assuming $T$ nonmodular we can find (after naming parameters) such $p$ with $U(Cb(p)) = 3$. We now assume that $p = p(x, d)$, and $D = acl(d) = Cb(p)$, with $U(p) = 1$, $U(tp(d)) = 3$. Let $a_1, a_2$ be independent (over $D$) realisations of $p(x, d)$. $U$-rank computations show that $a_1$ is independent from $a_2$ over $\emptyset$. Also $tp(a_1, a_2/D) = tp(a_2, a_1/D)$ (by stationarity of $p(x, d)$). Also $U(tp(D/a_1, a_2)) = 1$. Now let $D'$ realise $tp(D/acl(a_1, a_2))$ independently from $D$ over $a_1, a_2$. Then some computations show that $D$ is independent from $D'$ over $\emptyset$, and moreover $tp(a_1, a_2/D, D') = tp(a_2, a_1/D, D')$. As $D$ was the canonical base of $p(x, d)$ it follows that $a_i \in acl(D, D')$ for $i = 1, 2$.

**Corollary 5.12** *Suppose that $T$ is nonmodular. Then $TA$ is unstable.*

*Proof.* The above fact gives us $acl^{-}$-sets $D_1, D_2$ independent over $A = acl^{-}(\emptyset)$, and $a_1 \in acl^{-}(D_1, D_2) \setminus dcl^{-}(D_1, D_2)$. Apply Corollary 5.4 to conclude that $TA$ is unstable.

**Corollary 5.13** *If $T$ has EI, then $TA$ is unstable.*

*Proof.* In [8] it is proved that a strongly minimal theory with EI is nonmodular. (There should be a more direct proof of this Corollary, using the fixed set directly, but I could not find it.)

# 6 Modular types in $TA$ and the dichotomy theorem for $ACFA_0$.

In this section we will give a suitable local version of the main result ($T$ nonmodular implies $TA$ unstable) of the last section. We are concerned with types $p$ of $SU$ rank 1 in $TA$. We would like to prove that if $p$ is stationary then $p$ is modular, but we only obtain a rather weaker version: $p(x)$ "strongly stationary" implies $p$ modular. We then sketch a proof of the Zilber dichotomy for characteristic 0 models of $ACFA$: any $SU$ rank 1 type is nonorthogonal to the fixed field or is (stable and) modular. Maybe it is worthwhile defining (non)orthogonality at this point:

**Definition 6.1** *Let $p(x) \in S(A)$, $q(y) \in S(B)$ be types (with $A, b$ algebraically closed say). We say that $p$ is orthogonal to $q$ if whenever $C \supseteq A \cup B$ and $a, b$ realise $p$, $q$ respectively sch that $a$ isindependent from $C$ over $A$ and $b$ is independent from $C$ over $B$, then $a$ is independent from $b$ over $C$.*

We will be analysing in more detail the condition in Lemma 5.3 for a type to be stationary. This will entail saying a few more words about Galois theory in $\bar{M}$. We will assume in this section that $T$ has $EGI$ (although this could be replaced by working in $T^{eq}$ without much trouble). So for now we are just working in the saturated model $\bar{M}$ of $T$.

**Remark 6.2** *Let $A = dcl^-(A) \subset \bar{M}$, and let $\sigma_0$ be an elementary permutation of $A$. Let $\sigma$ be a fixed extension of $\sigma_0$ to $acl^-(A)$. The following are equivalent:*
*(1) There is a proper finite extension $A \subsetneq B \subseteq acl^-(A)$ which is $\sigma$-invariant (i.e. fixed setwise by $\sigma$).*
*(2) There is a proper finite extension $A \subsetneq B \subseteq acl^-(A)$, and some extension $\tau$ of $\sigma_0$ to $acl^-(A)$ such that $B$ is $\tau$-invariant.*
*(3) As in (1) but with "finite Galois" in place of "finite".*
*(4) As in (2) but with "finite Galois" in place of "finite".*

*Proof.* (2) implies (4): Let $B = dcl(A, b)$. Let $\phi(x, a)$ isolate $tp^-(b/A)$, and let $b = b_1, b_2, .., b_n$ be the solutions of $\phi(x, a)$ (all in $acl^-(A)$). Note that $C = dcl(A, b_1, .., b_n)$ is a finite Galois extension of $A$, so it suffices to show that $C$ is $\tau$-invariant. Note that $\tau(b)$ is a solution of $\phi(x, \sigma_0(a))$, so as $\tau(B) = B$, $b$ and $\tau(b)$ are interdefinable over $A$. Thus every solution of $\phi(x, a)$ is interdefinable over $A$ with a solution of $\phi(x, \sigma_0(a))$, and vice versa. It folllows that for each $i$, $\tau(b_i)$ (a solution of $\phi(x, \sigma_0(a))$) is contained in $C$. So $\tau(C) \subseteq C$. The same argument shows that $C$ is contained in the the definable closure of $A$ together with the solutions of $\phi(x, \sigma_0(a))$, so $\tau(C) = C$.
(4) implies (3): Let $\tau$ and $B$ be as given by (3). Note that $f.\tau = \sigma$ for some $f \in Gal(A)$. As $B$ is assumed to be a Galois extension of $A$, $f(B) = B$, and so $\sigma(B) = B$.
(3) implies (1) and (1) implies (2) are immediate.

We aim to show:

**Proposition 6.3** *Suppose $A = dcl^-(A)$ and $\sigma_0$ is an elementary permutation of $A$. Let $\sigma$ be an extension of $\sigma_0$ to an elementary permutation of $acl^-(A)$. Suppose also that there is no finite (Galois) extension $A \subsetneq B \subseteq acl^-(A)$ which is $\sigma$-invariant. Then all extensions of $\sigma_0$ to $acl^-(A)$ are conjugate under $Gal(A)$ (to $\sigma$).*

*Proof.* The proof will go through a few claims. First, we give a necessary and sufficient condition for all extensions of $\sigma_0$ to $acl^-(A)$ to be conjugate (in $Gal(A)$).
Note that the extensions of $\sigma_0$ to $acl^-(A)$ are precisely things of the form $\sigma.\tau$) as $\tau$ ranges over $Gal(A)$. *Claim 1.* All extensions of $\sigma_0$ to $acl^-(A)$ are conjugate under $Gal(A)$ iff the map $\mu : Gal(A) \to Gal(A)$ defined by $\mu(\tau)$

$= \sigma^{-}.\tau^{-1}.\sigma.\tau =_{def} [\sigma, \tau]$ is onto.
*Proof.* Assume LHS. Let $\rho \in Gal(A)$. So $\sigma.\rho = \tau^{-}.\sigma.\tau$ for some $\tau \in Gal(A)$, whereby $[\sigma.\tau] = \rho$. The converse is likewise.

Note that $\mu$ is surjective iff for every finite Galois extension $A \subseteq B \subseteq acl^{-}(A)$, the induced map $\mu_B : Gal(A) \to Aut^{-}(B/A)$ is surjective. For such $B$, the value of $[\sigma, \tau]|B$ depends only on $\tau|B$ and $\tau^{-}|\sigma(B)$. Thus $\mu$ is surjective iff for each finite Galois extension $B$ of $A$, settimg $C = dcl^{-}(B, \sigma(B)$, the map from $Aut^{-}(C/A)$ to $Aut^{-}(B/A)$: $\tau \to [\sigma|C, \tau]|B$ is surjective. So, bearing in mind Claim 1 we have:
*Claim 2.* All extensions of $\sigma_0$ to $acl^{-}(A)$ are conjugate in $Gal(A)$ iff for each finite Galois extension $B$ of $A$, the map $\tau \to [\sigma|C, \tau]|B$ from $Aut^{-}(dcl^{-}(B, \sigma(B))$ to $Aut^{-}(B/A)$ is surjective.

We now "prove" the proposition. We show the RHS of Claim 2 is true by induction on the cardinality of $Aut^{-}(B/A)$. We use EGI and 3.23. If the cardinality is 1 there is nothing to do. Now given $B$. Let $B' = B \cap \sigma(B)$. Our assumptions imply that $B'$ is a finite Galois extension of $A$ properly contained in $B$. Thus EGI (and 3.23) imply that the cardinality of $Aut^{-}(B_1/A)$ is strictly less than the cardinality of $Aut^{-}(B/A)$. The induction assumption tells us that the relevant map from $Aut^{-}(dcl^{-}(B_1, \sigma(B_1))/A)$ to $Aut^{-}(B_1/A))$ is surjective. We leave it as an exercise, using the Galois coorespondence, to deduce that the map from $Aut^{-}(dcl^{-}(B, \sigma(B))/A)$ to $Aut^{-}(B/A))$ is also surjective.

We now return to our model $(\bar{M}, \sigma)$ of $TA$.

**Definition 6.4** *Let $p(x) = tp(b/A)$ where $A = acl(A)$. Let $B = acl(A, b)$. We say that $p(x)$ is strongly stationary iff whenever $C = acl(A)$ is independent from $B$ over $A$, then $dcl^{-}(B, C)$ has no finite $\sigma$-invariant (Galois) extensions.*

Note:

**Lemma 6.5** *If $p(x)$ is strongly stationary then $p(x)$ is stationary.*

*Proof.* By Lemma 5.3 and Proposition 6.1.

In fact, strong stationarity has a more intuitive characterization which we state now. As a matter of notation, if $p(x) = tp(b/A)$, then for any $k \geq 1$,

$p[k]$ denotes the type of $b$ over $A$ in the structure $(\bar{M}, \sigma^k)$ (which remember is also a model of $TA$).

**Lemma 6.6** *$p(x)$ is strongly stationary iff $p[k]$ is stationary for all $k$.*

The left to right direction is left to the reader. (There are a few subtleties.) Right to left: Assume that $p(x) = tp(b/A)$ ($A = acl(A)$ is not strongly stationary. Let $B = acl^-(Ab)$, let $C \supseteq A$ be algebraically closed and independent from $B$ over $A$, and let $E$ be a proper finite Galois extension of $dcl^-(B, C) = D$ which is $\sigma$-invariant. Note that, for any $k$, in $(\bar{M}, \sigma^k)$ we still have that $A, B, C$ are algebraically closed and $C$ independent from $B$ over $A$. Let $G = Aut^-(E/D)$. Let $k = |G|!$ (so $k > 1$). Note that $\sigma|E$ acts on $G$ by conjugation. Thus $\sigma^k|E$ acts trivially on $G$, namely commutes with every element of $G$. It follows that $\sigma^k|D$ has at least 2 extensions to $E$ up to conjugacy by an element of $G$. Thus $\sigma^k|D$ has at least 2 extensions to $acl^-(D)$ up to conjugacy by an element of $Gal(D)$. By Lemma 5.3, $p[k]$ is not stationary.

**Corollary 6.7** *(i) Let $A \subseteq B$ be algebraically closed. Suppose $b$ is independent from $B$ over $A$. Then $tp(b/A)$ is strongly stationary iff $tp(b/B)$ is strongly stationary.*
*(ii) Let $tp(b/A)$ be strongly stationary and let $c \in acl(A, b)$ then $tp(c/A)$ is strongly stationary.*
*(iii) Let $p(x) = tp(b/A)$ be strongly stationary. Let $b_1, .., b_n$ be an $A$-independent tuple of realizattions of $p$. Then $tp(b_1, .., b_n/A)$ is strongly stationary.*

*Proof.* Note first that by the independence theorem, if $b$ is indepedent from $B$ over $A$, then $tp(b/A)$ is stationary iff $tp(b/B)$ is stationary. Also by the independence theorem if $c \in acl(Ab)$ and $tp(b/A)$ is stationary then so is $tp(c/A)$. Now use these observations together with the lemma above.

**Definition 6.8** *Let $A = acl(A)$.*
*(i) $p(x) = tp(a/A)$ is stable if there is no formula $\phi(x, y)$ over $A$ and tuples $a_i, b_i$ for $i < \omega$ such that each $a_i$ is a tuple of realizations of $p$ and $\models \phi(a_i, b_j)$ iff $i < j$. Equivalently, for any set $B \supseteq A$, $p$ has at most $|B|^\omega$ extensions to complete types over $B$.*
*(ii) $p(x) = tp(a/A)$ is modular if whenever $b$ is a tuple of realizations of $p(x)$, and $C$ is any algebraically closed set containing $A$, then $b$ is independent from $C$ over $acl(A, b) \cap C$.*

**Remark 6.9** *(i) The equivalence in (i) above is routine and left to the reader.*
*(ii) If $p(x)$ is stationary of SU-rank 1, then $p(x)$ is stable with $U(p) = 1$. Namely $p$ is a "minimal type". If in addition $p(x)$ is modular, then Fact 5.11 holds for $p$, namely $a_1, a_2$ are tuples of realizations of $p$, and also $D_i = acl(Ad_i)$ where $d_i$ is a tuple of realizations of $p$, all working in $TA$ as opposed to in $T$.*
*(iii) If $p(x) = tp(a/A)$ is strongly stationary of SU-rank 1, then so is $tp(b/A)$ whenever $b$ is contained in the algebraic closure of $A$ together with a tuple of realizatyions of $p$.*

**Proposition 6.10** *Suppose $p(x) = tp(a/A)$ has SU-rank 1 and is strongly stationary. Then $p(x)$ is modular.*

*Proof.* Note that by the remark above, $p(x)$ is stationary of $U$-rank 1. By replacing $a$ by a suitable tuple $(a, \sigma(a), .., \sigma^k(a))$ we may assume that $acl^-(A, a) = acl^-(A, \sigma(a))$. Assume for sake of contradiction that $p(x)$ is not modular. Let $a_1, a_2, D_1, D_2$ be as given by the remark above (as in Fact 5.11). Let $E = dcl^-(D_1, D_2)$. Now let $c_1, .., c_n$ be set of realizations of $tp^-(a_1/E)$ which are $L^-$-interalgebraic with $a_1$ over $A$. Let $e$ be the imaginary element $\{c_1, .., c_n\}$. Note that $e$ is $L^-$-interalgebraic with $\sigma(e)$ over $A$. Moreover note that if $tp(e/E) = tp(e'/E)$ and $e$ is $L^-$-interalgebraic with $e'$ over $A$, then $e = e'$. It follows that $dcl^-(E, e) = dcl^-(E, \sigma(e))$. Now by EGI of $T$, $e$ is $L^-$interdefinable over $E$ with a real tuple $e_1$. It follows that $dcl(E, e_1)$ is a finite $\sigma$-invariant extension of $E$. By 6.1, Remark 6.8, and the assumption that $p$ is strongly stationary, $e_1 \in dcl^-(E)$. Thus $e \in dcl^-(E)$. As $a_1$ and $a_2$ have the same type over $E$, it follows that $a_1$ is interalgebraic with $a_2$ over $A$, a contradiction.

We now consider what is in a sense the "opposite" situation to strong stationarity.

**Definition 6.11** *$(A = acl(A))$. $tp(b/A)$ is bounded if*
*(i) $\sigma(b) \in acl^-(A, b)$ (or equivalently $acl^-(A, b) = acl(A, b)$), and*
*(ii) for some $k < \omega$, $mult^-(\sigma^i(b)/A, b)$ (= number of solutions of $tp^-(\sigma^i(b)/A, b)$) is at most $k$, for all $i \in \mathbf{Z}$.*

So for example the type of any element in any fixed set is bounded.

**Definition 6.12** *We will say that* $tp(b/A)$ *has finite order iff there is a finite bound to the* $U^-$*-ranks of types* $tp^-(c/A)$ *for* $c \in acl(A,b)$*, (equivalently to types* $tp^-(b, sigma(b), .., \sigma^k(b)/A)$*,* $k \in \omega$*). In this case, the order of* $tp(b/A)$ *is the maximum of those* $U^-$*-ranks.*

**Lemma 6.13** *Assume* $T$ *has* $EI$*. Let* $A = acl(A)$*. Suppose* $tp(b/A)$ *is bounded and of order* 1*. Then there is a finite tuple* $c$ *such that* $dcl^-(A,c) = dcl^-(A,\sigma(c))$ *and* $acl^-(A,e) = acl^-(A,b)$ *(*$= acl(A,b)$*).*

*Proof.* Choose $r > 0$ such that $mult^-(b/\sigma(b), .., \sigma^r(b), A)$ is least possible. Replacing $b$ by $(b, \sigma(b), .., \sigma^r(b))$ we have that $tp^-(b/\sigma(b), A) \models tp^-(b/A \cup \{\sigma^i(b) : i > 0\})$. Note that $tp(b/A)$ is still bounded (why?). In particular for each $n > 0$, $tp^-(b/\sigma(b), A) \models tp^-(b/\sigma(b), \sigma^n(b), A)$, and thus $tp^-(\sigma^n(b)/\sigma(b), A) \models tp^-(\sigma^n(b)/b, \sigma(b), A)$. The same is true after applying any (possibly negative) power of $\sigma$. Now (by boundedness of $tp(b/A)$), let $k < 0$ be such that $mult^-(b/\sigma^k(b), A)$ is maximum possible. Thus, for all $i \leq k$, $tp^-(b/\sigma^k(b), A)$ and $tp^-(b/\sigma^i(b), A)$ have the same (finite) set of solutions, $b_1, .., b_n$ say. Let $e = \{b_1, .., b_n\}$, which we assume to be a real tuple by $EI$ for $T$. So $e \in B = \bigcap\{dcl^-(A, \sigma^i(b)) : i \leq k\}$. Note that $acl^-(A,e) = acl^-(A,b)$ (why?). So in particular $\sigma^k(b) \in acl^-(B)$. Let $c$ be a finite tuple from $B$ such that $tp^-(\sigma^k(b)/A,c) \models tp^-(\sigma^k(b)/B)$. Then, as $B \subseteq dcl^-(A, \sigma^k(b))$, one sees that $B = dcl^-(A,c)$. Note that $\sigma(B) \subseteq B$. Thus, $\sigma(c) \in dcl^-(A,c))$. Finally, we want to see that also $c \in dcl^-(A,\sigma(c))$: Note that $tp(c/A)$ is bounded (as is anything in $acl(A,b)$). If $c \notin dcl^-(A,\sigma(c))$ then $\{\sigma^i(B) : i < \omega\}$ is a strictly descending sequence of definably closed (in $L^-$) subsets of $B$ containing $A$. The Galois theory implies that $mult^-(c/\sigma^i(c), A)$ is unbounded as $i$ gets larger, a contradiction.

For the next result we need it seems necessary to pass to $ACFA$ where we can make use of some of the theory of curves/Riemann surfaces.

**Proposition 6.14** *Work in* $ACFA_0$*. Suppose* $E$ *is an elementary substructure of* $(\bar{M}, \sigma)$*, and* $p(x) = tp(b/E)$ *is bounded and of order* 1*. Then there is* $c$ *such that* $acl^-(A,b) = acl^-(A,c)$ *and* $\sigma(c) = c$*.*

This says that any bounded type of order 1 is nonorthogonal to some type of an element in the fixed field.

I will sketch a proof of Proposition 6.14. All we have to know is that if $C$ is a smooth, complete, algebraic curve over an algebraically closed field $K$, and $\alpha$ is a point on $C(K)$, then for some natural number $m > 0$ the "divisor" $m.\alpha$ on $C$ is "very ample". Very ample means that if $f_0, .., f_n$ is a basis for the (finite-dimensional) $K$-vector space $L(m.\alpha)$ of rational functions $f$ on $C$ such that (i) the only pole (if any) of $f$ is $\alpha$, and (ii) if $\alpha$ is a pole of $f$ then it is a pole of order at most $m$, *then* the map from $C$ to $\mathbf{P}^n$ which takes $a \in C$ to $[f_0(a) : .... : f_n(a)]$ is an isomorphism of $C$ with a closed subvariety of $\mathbf{P}^n$.

We now proceed with the proof of 6.14. By Lemma 6.13 we may assume that $dcl^-(E,b) = dcl^-(A,\sigma(b))$. $b$ is a generic point over (the algebraically closed field) $E$ of a smooth, complete algebraic curve $X$ over $E$. Then $\sigma(b) = f(b)$ where $f$ is an isomorphism (defined over $E$) between $X$ and $Y = \sigma(X)$. As $E$ is an elementary substructure of $(\bar{M},\sigma)$, there is a point $d \in X(E)$ such that $f(d) = \sigma(d)$. By the remarks above $md$ is a very ample divisor on $X$ for some $m > 0$. Let $f_0, .., f_n$ be an $E$-basis for $L(md)$. Let $\phi$ be the corresponding embedding of $X$ into $\mathbf{P}^n$, defined over $E$. Then $\sigma(\phi).f$ is another embedding of $X$ into $\mathbf{P}^n$, corresponding to the basis $\sigma(f_0).f, .., \sigma(f_n).f$ of $L(m.d)$. This second basis is thus the imasge of the first basis under an element $A \in PGL(n)$, defined over $E$. Thus $A.\phi = \sigma(\phi).f$ on $X$. As $(E,\sigma)$ is a model of $ACFA$ there are $B, C \in PGL(n,E)$ such that $B = C.A$ and $C = \sigma(B)$ (why?) (i.e. $B = \sigma(B).A$). Now $\psi = B.\phi$ is another embedding of $X$ into $\mathbf{P}^n$ defined over $E$.
We claim that $\psi(b)$ is fixed by $\sigma$: $\sigma(\psi(b)) = \sigma(B.\phi(b)) = \sigma(B).\sigma(\phi)(\sigma(b)) = \sigma(B).\sigma(\phi).f(b) = \sigma(B).A.\phi(b) = B.\phi(b) = \psi(b)$. So putting $c = \psi(b)$ we clearly have that $dcl^-(E,c) = dcl^-(E,b)$ and $\sigma(c) = c$.

**Remark 6.15** *In the case $ACFA_p$ the same proof shows that $c$ can be found in some fixed field.*

The next result completes the dichotomy theorem for types of order 1 for $ACFA_0$.

**Proposition 6.16** *(Work in $ACFA_0$.) Let $E = acl(E)$. Suppose that $tp(a/E)$ has order 1 and is unbounded. Then $tp(a/E)$ is strongly stationary.*

The proof will depend also on some facts about algebraic curves. I will give the background as briefly as possible. We work with algebraic geometry in characteristic 0. Fix an algebraically closed field $E$. By a function field of transcendence degree 1 over $E$ we mean a field $L > E$ of transcenfdence degree 1 over $E$ and finitely generated over $E$. $L$ is then the function field of a smooth projective curve $C$ defined over $E$. (If $L = E(a)$ then $a$ is a generic point of $C$ over $E$.) Now suppose that $L'$ is a finite extension of $L$. $L' = E(b)$ is the function field of a smooth projective curve $C'$ defined over $E$ and the "map" $b \to a$ extends to a surjective finite-to-one morphism $\pi$ from $C'$ to $C$ also defined over $E$. Assuming $L'$ to be an extension of $L$ of degree $n$. $\pi$ is almost everywhere $n$ to 1, namely for all but finitely many $x \in E$, $\pi^{-1}(x)$ has cardinality $n$ (and for all points $x$, $\pi^{-1}(x)$ has cardinality at most $n$). The ramification divisor of $\pi$ is the set of points $x \in C$ such that $\pi^{-1}(x)$ has cardinality strictly less than $n$. This is a finite subset of $C(E)$. A point $y \in C'$ will be said to ramify over $C$ (with respect to $\pi$) if $\pi(y)$ is in the ramification divisor of $\pi$.

Now suppose that $E < F$ are algebraically closed fields, $C$ is a (smooth projective) curve defined over $E$, $C'$ a curve defined over $F$, and $\pi : C' \to C$ a surjective morphism defined over $F$. We will say that $\pi : C' \to C$ descends to $E$ if there is a curve $C''$ defined over $E$, and an isomorphism $f : C'' \to C'$ defined over $F$ such that the surjective morphism $\pi.f : C'' \to C$ is defined over $E$.

**Fact 6.17** *With above notation, let $F(C) < F(C')$ be ther inclusion of function fields induced by $\pi : C' \to C$. Then $\pi$ descends to $E$ iff $F(C') < F.acl(E(C))$.*

The main tool we will use is the following result, originating, I think, with Riemann. (Namely the Riemann surface case is due to Riemann).

**Fact 6.18** *Suppose that $E < F$ are algebraically closed fields, $C$, $C'$ are smooth projective curves defined over $E, F$ respectively, and $\pi : C' \to C$ is a surjective morphism defined over $F$. Suppose that the ramification divisor of $\pi$ is contained in $E(C)$. Then $\pi$ descends to $E$.*

We also use:

**Fact 6.19** *(i) Let $E$ be algebraically closed. Supppose $C_0, C_1, C_2, C_3$ are smooth projective curves over $E$ with generic points $a, b, c, (b, c)$ respectively where $E(a) < E(b)$ and $E(a) < E(c)$. Let $\pi_1 : C_1 \to C_0$, $\pi_2 : C_2 \to C_0$, $\pi_3 : C_3 \to C_2$ and $\pi_4 : C_3 \to C_0$ be the corresponding morphisms. Suppose $a' \in C_0$ is not in the ramification divisor of $\pi_1$. Let $(b', c') \in \pi_4^{-1}(a')$. Then $c'$ is not in the ramification divisor of $\pi_3$*
*(ii) Let $C_2 \to C_1 \to C_0$ be surjective morphisms of curves over $E$. Let $a \in C_0$ and $b$ some preimage of $a$ in $C_1$. Then $a$ is not in the ramification divisor of $C_2 \to C_0$ iff $a$ is not in the ramification divisor of $C_1 \to C_)$ and $b$ is not in the ramification divisor of $C_2 \to C_1$.*

We now proceed with the proof of 6.16. Assume the hypotheses of 6.16. We may assume that $a$ is a single element and so is the generic point of $\mathbf{P}^1$. Suppose by way of contradiction that $tp(a/E)$ is not strongly tationary. So there is some algebraicaly closed $F$, independent from $a$ over $E$ such that $acl(E(a)).F$ has a finite (Galois) $\sigma$-invariant extension $L$. Write $K = acl(E(a)).F$. Let $L_1 = F(b)$ be a finite Galois extension of $F(a)$ such that $K.L_1 = L$. So $L_1$ is not contained in $K$. Let $C$ be the (smooth projective) curve over $E$ corresponding to $E(a))$ (which we can take to be $\mathbf{P^1}$ as remarked above), let $V$ be the curve over $F$ corresponding to $L_1$ and let $\pi : V \to C$ be the corresponding morphism. By Fact 6.18 the ramification divisor of $\pi$ is not contained in $C(E)$. Let $S$ be those points of the ramification divisor of $\pi$ which are not $E$-rational. So $S$ is a finite nonempty subset of $C(F)$. Note that $tp(a/F)$ is unbounded, so also $tp(b/F)$ is unbounded. (We may assume $b$ is a generic point of $V$ over $F$.) So we may choose $k$ such that $N = mult^-(\sigma^k(b)/F(b)) > |\pi^{-1}(S)|$. Let $W$ be the curve over $F$ whose generic point is $(b, \sigma^k(b))$, so the function field of $W$ over $F$ is $L_2 = (L_1, \sigma^k(L_1)$. Now let us fix an element $d \in \pi^{-1}(S)$. Let $\rho : W \to V$ be induced by $(b, \sigma^k(b)) \to b$. Let $a' = \pi(d)$
*Claim 1.* $d$ is not in the ramification divisor of $\rho$.
*Proof.* As $L_2 < K.L_1$, there is some $c \in acl(E(a))$ such that $L_2 < F(c).L_1$. $c$ is the generic point of a curve $C'$ over $E$ and the corresponding morphism from $C'$ to $C$ is defined over $E$. $(b, c)$ is the generic point of a curve $C''$ defined over $F$. Consider now the extensions of $F$: $F(a)$, $F(c)$, $F(b) = L_1$ and $F(c, b)(> F(b, \sigma^k(b)))$. Let $a' \in S$ with $\pi(d) = a'$. So $a'$ is a generic point of $C$ over $E$, thus is not in the ramification divisor of $C' \to C$. By Fact 6.19, $d$ is not in the ramification divisor of $C'' \to V$. Thus $d$ is not in the

ramification divisor of $W \to V$.

Now let $C_1, C_2$ be the curves (defined over $E$) with generic points (over $F$) $\sigma^k(a)$, $(a, \sigma^k(a))$ respectively. As the corresponding morphisms are defined over $E$, and $a'$ is a generic point of $C$ over $E$, we have
*Claim 2.* $a'$ is not in the ramification divisor of $C_2 \to C$.

By claim 1, $\rho^{-1}(d) = \{e_1, .., e_N\} \subset W$ has cardinality $N$. We have a canonical morphism $W \to C_1$, factoring through $C_2$. Let $f_1, .., f_n$ be the images of the $e_i$ under this morphism
*Claim 3.* Each $f_i$ is generic in $C_1$ over $E$ and hence is not in the ramification divisor of $C_2 \to C_1$.
*Proof.* $f_i$ is interalgebraic with $a'$ over $E$.

*Claim 4.* Each $f_i$ is in the ramification divisor of $W \to C_1$.
*Proof.* Suppose not. Then as $W \to C_1$ factors through $h : W \to C_2$, $h(e_i)$ is not in the ramification divisor of $h$. The image of $h(e_i$ under $C_2 \to C$ is $a'$, hence (why?) $a'$ is not in the ramification divisor of $W \to C$. So $a'$ is not in the ramification divisor of $\pi : V \to C$, a contradiction.

*Claim 5.* Each $f_i$ is in the ramification divisor of $\sigma^k(\pi) : \sigma^k(V) \to C_1$.
*Proof.* As in the proof of Claim 1 (using 6.19), each element of $(\sigma^k(\pi))^{-1}(f_i)$ is not in the ramification divisor of $W \to \sigma^k(V)$. Now use claim 4.

By Claims 5 and 3, each $f_i$ is in $\sigma^k(S)$. Let $g_i$ be the image of $e_i$ under $W \to \sigma^k(V)$. (so $f_i = \sigma^k(\pi)(g_i)$. As the $e_i$'s al have the same image ($d$) under $\rho$, the $g_i$ must be all different. So the preimage of $\sigma^k(S)$ under $\sigma^k(\pi)$ has cardinality $N > |\pi^{-1}(S)|$ a contradiction.

So we have:

**Corollary 6.20** *($ACFA_0$) Let $p(x) \in S(E)$ be of order* 1 *(so of $SU$-rank* 1*). Then $p(x)$ is either modular and of $U$-rank !, or $p$ is nonorthogonal to some type of an element in $Fix(\sigma)$.*

Finally in this section we want to extend the above Corollary to all types of $SU$-rank 1 (for $ACFA_0$). I will just describe the ingredients. There should be some transparent model-theoretic-geometric intuition behind what is going on, but I did not find it.

**Lemma 6.21** *Suppose $E$ is algebraically closed and $p(x) = tp(a/E)$ has finite order $> 1$. Suppose also $p(x)$ is not strongly stationary. Then for some $k > 0$, $SU_k(p[k]) > 1$.*

*Sketch proof.* Let $F$ be algebraically closed and independent from $a$ over $E$, such that $F.acl(E(a))$ has a proper finite $\sigma$-invariant (Galois) extension $L$. Show that we can assume $F$ to have finite transcendence degree over $E$, so of the form $acl(E(b))$ where $acl^{-}(E, b) = acl^{-}(E, \sigma(b))$ and $mult^{-}(\sigma(b)/E, b) = mult^{-}(\sigma^k(b)/E, b, .., \sigma^{k-1}(b))$ for all $k$, and similarly for $mult^{-}(b/E, \sigma(b))$. Also we may assume that $E[b]$ is integrally closed in $E(b)$ and that the affine variety $V$ over $E$ of which $b$ is the generic point is nonsingular. Let $E' = acl(E(a))$ and let $L_1$ be a finite $\sigma$-invariant Galois extension of $E'(b)$. Let $\alpha \in L_1$ be such that $L_1 = E'(b, \alpha)$ and $E'[b, \alpha]$ is the integral closure of $E'[b]$ in $L_1$. Then $(b, \alpha)$ is the generic point over $E'$ of a normal variety $W$ and we have a canonical finite surjective morphism $\pi : W \to V$. Then, as in 6.17, $\pi$ does not descend to $E$ and thus (using a higher dimensional version of 6.18), the ramification divisor of $\pi$ (a codimension 1 subset of $V$) is not defined over $E$, so has a component $U$ not defined over $E$. $U$ corresponds to a discrete rank 1 valuation $v$ on $E'(b)$. Arguments as in the proof of 6.16 show that we can find such $U$ and a valuation $w$ on $E'(\sigma^i(b) : i \in \mathbf{Z})$ extending $v$ such that $\sigma^k(w) = w$ for some $k > 0$.
We have:
(i) $w$ is 0 on $E'$.
(ii) $w$ is 0 on $E(\sigma^i(b) : i \in \mathbf{Z})$ (as $v$, being not defined over $E$ is 0 on $E(b)$, so on $acl(E(b))$).
Let $E''$ be the residue class field of $w$, and $\phi$ the corresponding map into $E''$ (union "infinity"). As $w$ is invariant under $\sigma^k$, $\phi(\sigma^k)$ is an automorphism $\tau$ of $E''$. By (i) and (ii) above, the restriction of $\phi$ to $(E(\sigma^i(b) : i \in \mathbf{Z}), \sigma^k)$ and to $(E', \sigma^k)$ are isomorphisms (with difference subfields of $(E'', \tau)$). $(E'', \tau)$ embeds in a saturated model $N$ of $ACFA$. We may assume $\phi(E) = E$. We have
(iii) $tp(\phi(a)/E)$ in $N$ equals $tp_k(a/E)$, and
(iv) $qftp_k(b/E) = qftp(\phi(b)/E)$.
Now, $\phi(E'(b))$ is the function field of $U$ over $E'$, so $tr.deg(\phi(E'(b)/\phi(E')) = tr.deg(E(b)/E) - 1$ $(= tr.deg(\phi(E(b))/\phi(E)))$.
So (*) $tr.deg(\phi(E(b)).\phi(E(a))/E) = tr.deg((\phi(E(a))/\phi(E)) + tr.deg(\phi(E(b))/\phi(E)) - 1$.

Thus (*) $tr.deg(\phi(E(a))/\phi(E(b)) = tr.deg(\phi(E(a))/\phi(E)) - 1$.
As $\phi$ was an isomorphism between $(E', \sigma^k)$ and its image in $(E'', \tau)$, we see from (*) that in $(\bar{M}, \sigma^k)$ there is algebraically closed $K > E$ such that $tr.deg(E(a)/K) = tr.deg(E(a)/E) - 1$. So, as $tr.deg(E(a)/E) > 1$ by hypothesis, this shows that $SU_k(a/E) > 1$, completing the proof sketch.

**Theorem 6.22** *($ACFA_0$) Let $SU(tp(a/E)) = 1$. Then either $tp(a/E)$ is strongly stationary, (of $U$-rank 1) and modular, or $tp(a/E)$ is nonorthogonal to the type of some element in the fixed field (in which case $tp(a/E)$ has order 1).*

*Proof.* We may assume $E$ to be a saturated model. We prove the theorem by induction on (the finite) order of $tp(a/E)$). If the order is 1 we are finished by 6.16. Suppose order is $r > 1$. If $tp(a/E)$ is strongly stationary, it is modular of $U$-rank 1 by 6.10. So suppose not and we will get a contradiction. By the previous lemma, $SU_k(a/E) > 1$ for some $k$. Standard "coordinatization" methods show that $p[k]$ (= type of $a$ over $E$ in $(\bar{M}, \sigma^k)$ is nonorthogonal to a type of $SU$-rank 1. So there is $E' > E$ indepemdent from $a$ over $E$, and $c$ such that $SU_k(tp(c/E') = 1$ and $c \in acl(E'(a))$. Note that the order of $tp_k(c/E')$ is $< r$. So we can apply the induction hypothesis. As $a \in acl^-(E'(c, \sigma(c), ...)$ it easily follows that $tp_k(c/E')$ is not strongly stationary. Thus it is nonorthogonal to $fic(\sigma^k)$. We may assume $e'$ is a model and it is then not difficult to find $d$ such that $\sigma^k(d) = d$ and in $(\bar{M}, \sigma^k)$, $c$ is interalgebraic with $d$ over $E'$. So $d \not{in} E'$ and $d \in acl(E', a)$. So $\{d, \sigma(d), .., \sigma^{k-1}(d)\}$ is in $acl(E'(a))$ and by elimination of imaginaries there is $e \in Fix(\sigma)$, $e \notin E'$ with $e \in acl(E, a)$. As $SU(tp(a/E')) = 1$ it follows that $tp(a/E)$ has order 1, a contadiction.

# 7 Limit structures - preliminaries

We start looking at the material in paper 2. A method is described for proving the dichotomy theorem for $ACFA$ in all characteristics. There are a few ingredients: the construction, from a type $p$ of $SU$-rank 1 of a "limit structures". This will not be a first order structure but a $qf$-universal structures which lives in some strange way in our model $(\bar{M}, \sigma)$, and has a canonical structure of Zariski geometry. The Zariski geometry theorem is claimed to apply in this situation. Nonmodulairity of $p$ implies an infinite field is definable in the limit structure. One has then to show rthat this field is connected

in some way to a fixed field back in $(\bar{M}, \sigma)$ which is nonorthogonal to $p$.
The construction of the limit structure is somewhat ring-theoretic. I will try to do this construction in the $TA$ context, and so isolate its model-theoretic content.
In this section we cover the "preliminaries" section of [2] in the general $TA$ situation. So $T$ is strongly minimal with QE, let's say countable and complete, and $TA$ is as before. $(\bar{M}, \sigma)$ is a big saturated model of $TA$. We first give a proof of Exercise 4.6.

**Lemma 7.1** *$TA$ is $qf$-$\omega$-stable. That is over any countable subset of $(\bar{M}, \sigma)$ there are only countably many complete quantifier-free $n$-types, for all $n < \omega$.*

*Proof.* It is enough to prove this for $n = 1$. Fix countable $A$, say algebraically closed. Consider $qftp(b/A)$. Note that $qftp(b/A)$ is given by the (quantifier-free) $L^-$-type of the infinite tuple $(b, \sigma(b), ...\sigma^n(b), ....)$ over $A$.If $b$ is "transformally generic" over $A$, namely $\{\sigma^i(b) : i < \omega\}$ is $L^-$-algebraically independent over $A$, then this data determines $qftp(b/A)$. Similarly if $b \in A$ there is nothing to do. So we assume $b$ has finite order $> 0$ over $A$. So for some $n \geq 0$, $\sigma^{n+1}(b) \in acl^-(A, b, .., \sigma^n(b))$. We may choose $n$ large enough so that $mult^-(\sigma^{n+1}(b)/A, b, .., \sigma^n(b))$ is smallest possible. Let $q(x_0, .., x_{n+1}) = tp^-(b, \sigma(b), .., \sigma^{n+1}(b)/A)$. We claim that $q$ determines $qftp(b/A)$. So suppose that $tp^-(c, \sigma(c), .., \sigma^{n+1}(c)/A) = q$. We want to see that for all $m > n$, $tp^-(c, \sigma(c), ..., \sigma^m(c)/A) = tp^-(b, ..., \sigma^m(c)/A)$. Assume it is true for $m$ and prove it for $m + 1$. Note that $tp^-(\sigma^{m-n}(b), .., \sigma^m(b), \sigma^{m+1}(b)/A) = \sigma^{m-n}(q) = tp^-(\sigma^{m-n}(c), .., \sigma^{m+1}(c)/A)$. Our assumptions mean that

$tp^-(\sigma^{m+1}(b)/A, \sigma^{m-n}(b), .., \sigma^m(b))$ has a unique extension over $A, b, .., \sigma^m(b)$.
The same is thus true with $c$ in place of $b$ and we are finished.

**Remark 7.2** *(i) The same proof shows that if $b$ is an arbitrary finite tuple, then $qftp(b/A)$ is determined by the following data: $tp^-(b, ...., \sigma^{n+1}(b)/A)$ where $n$ is chosen so that first the Morley rank, and secondly the Morley degree of $tp^-(\sigma^{n+1}(b)/A, b, .., \sigma^n(b))$ is minimized.*
*(ii) For $A$ any $\sigma$-closed subset, and $b$ a tuple, the right limit degree of $b$ over $A$ ($rld(b/A)$), is defined to be $min\{mult^-(tp^-(\sigma^n(b)/A, b, ..., \sigma^{n-1}(b)) : n < \omega\}$ if this exists. Note that that $rld(b/A)$ is an invariant of $cl_\sigma(A, b)$. One can also define left limit degree. If $b \in acl^-(A)$, then $rld(b/A) = lld(b/A) = ld(b/A)$.*

*(iii) With notation as in (ii). Assume* $c \in cl_\sigma(A, b)$*, and let* $C = cl_\sigma(A, c)$*. Show that* $ld(b/A) = ld(b/C).ld(c/A)$*.*
*(iv) Let* $A$ *be* $\sigma$*-closed. Let* $B = cl_\sigma(A, b)$ *where* $b \in acl^-(A)$*. Then* $B$ *is of the form* $dcl^-(A, b')$ *for some finite tuple* $b'$ *if and only if* $ld(b/A) = 1$*.*

In these preliminaries we study extensions of $\sigma$ from a $\sigma$-closed set $A$ to its algebraic closure, and then define the eventual $S$-rank.

**Definition 7.3** *Suppose* $A$ *is* $\sigma$*-closed, and* $a \in acl^-(A)$*. Then* $a$ *is said to be benign over* $A$*, if for any* $n$*,* $tp^-(a/A) \cup tp^-(\sigma(a)/A) \cup ... \cup tp^-(\sigma^n(a)/A) \models tp^-(a, \sigma(a), .., \sigma^n(a)/A)$*.*

**Exercise 7.4** *If* $a$ *is benign over* $A$ *then all extensions of* $\sigma|A$ *to* $cl_\sigma(A, a)$ *are isomorphic over* $A$*.*

*Proof.* Let $\tau$ be an extension of $\sigma$ to $cl_\sigma(A, a)$. Define $f$ to be the identity on $A$, and $f(\sigma^i(a)) = \tau^i(a)$ for $i \in \mathbf{Z}$.

**Lemma 7.5** *(Assume* $T$ *has EGI.) Suppose* $A \subseteq B \subseteq C$ *are* $\sigma$*-closed sets with both* $B$ *and* $C$ *Galois (not necessarily finite) extensions of* $A$*. Let* $C_0$ *be the union of all* $\sigma$*-closed subsets of* $C$ *which contain* $B$ *and are of the form* $dcl^-(B, c)$ *for some finite tuple* $c$ *from* $C$*. Then*
*(i)* $C_0$ *is a Galois extension of* $A$*.*
*(ii) Suppose also that* $C = cl_\sigma(B, c)$ *for some finite tuple* $c$*. Then* $C_0 = dcl^-(B, c')$ *for some finite tuple* $c'$*, and there is a sequence* $C_0 \subseteq C_1 \subseteq ...C_m = C$ *of* $\sigma$*-closed sets, each Galois over* $A$ *such that* $C_i$ *is benign over* $C_{i-1}$ *for each* $i \geq 1$*.*

*Proof.* (i) Suppose $dcl^-(B, c)$ is $\sigma$-closed. Let $c = c_1, c_2, .., c_n$ be the solutions to $tp^-(c/B)$ (all in $C$, by hypothesis). By the proof of 6.2 ((2) implies (4)), $dcl^-(B, c_1, .., c_n)$ is $\sigma$-closed, hence is contained in $C_0$.
(ii) First we leave it as an exercise to show that $C_0 = cl_\sigma(B, c')$ for some finite tuple $c'$. But then by definition of $C_0$ by expanding $c'$ we have $C_0 = dcl^-(B, c')$.
For the rest , we may assume that $C \neq C_0$. Thus (using Remark 7.2, (iii) and (iv)),
(*) for each finite $b \in C \setminus C_0$, $ld(b/C_0) > 1$.
We construct $C_1$. Let $D$ be such that $C_0 \subset D \subseteq C$, $D$ is $\sigma$-closed and Galois

over $A$, and $ld(D/C_0) = N$ is least possible (it has to be $> 1$ by (*)). Now let $C_1$ be largest such that $D \subseteq C_1 \subseteq C$, $D$ is $\sigma$-closed, and $ld(C_1/D) = 1$. As in (i), $C_1$ is Galois over $A$. Moreover by Remark 7.2 (iii), $ld(C_1/C_0) = N$, and the same is true for any $\sigma$-closed $D'$ contained in $C_1$ and properly containing $C_0$ which is Galois over $A$.
*Claim.* $C_1$ is benign over $C_0$.
Choose $\alpha$ such that $C_1 = cl_\sigma(C_0, \alpha)$, $mult^-(\sigma(\alpha)/C_0, \alpha) = N$, $dcl^-(C_0, \alpha)$ is Galois over $A$, and $mult^-(\alpha/C_0)$ is least possible. Consider $E = dcl^-(C_0, \alpha) \cap dcl^-(C_0, \sigma(\alpha))$. If $E = C_0$ then it follows that $C_1$ is benign over $C_0$. (In fact it just follows that $tp^-(\alpha/C_0) \cup tp^-(\sigma(\alpha)/C_0) \models tp^-(\alpha, \sigma(\alpha)/C_0)$ which is a bit weaker than what is required, but a similar argument will deal with this case too, I think.) So assume $E$ properly contains $C_0$. Note that $E$ is Galois over $A$, as is its $\sigma$-closure. Let $E = dcl^-(C_0, \beta)$. Then $mult^-(\sigma(\alpha)/E) = N$ (why?, model theory plus $EGI$). On the other hand, by the choice of $D$ and $C_1$, $mult^-(\sigma(\beta)/E) = N$. As $\sigma(\beta) \in dcl^-(C_0, \sigma(\alpha))$ it follows (Galois theory?) that $\sigma(\alpha) \in dcl^-(C_0, \beta, \sigma(\beta))$, whereby $C_1$ is also equal to $cl_\sigma(C_0, \beta)$. But clearly, $mult^-(\beta/C_0) < mult^-(\alpha/C_0)$, so we contradict minimal choice of the latter. This proves the claim (modulo the parenthetical remark above).

Now we can continue, as in the construction above to find $C_0 \subset C_1 \subset C_2 \ldots$ inside $C$, each $C_{i+1}$ being $\sigma$-closed, and Galois over $A$ and benign over $C_i$. We have to reach $C$ after a finite number of steps. (Why?)

Here is a preliminary application.

**Lemma 7.6** *(Assume $T$ has EGI.) Let $A$ be $\sigma$-closed. Let $B$ be a finite Galois extension of $A$ and let $C$ be the $\sigma$-closed set generated by $B$. Then for any $\tau \in Gal(B/A)$ there is $k < \omega$ and an extension $\tau'$ of $\tau$ to $C$ such that $\tau'$ commutes with $\sigma^k$.*

*Proof.* Let $A \subseteq C_0 \subset C_1 \subset \ldots \subset C_n = C$ be as given by the previous lemma. Namely these are all $\sigma$-closed Galois extensions of $A$, $C_0$ is a finite Galois extension of $A$, and $C_{i+1}$ is benign over $C_i$. By enlarging $B$ we may assume that $B$ contains $C_0$.
Let $Gal(C_0/A)$ have cardinality $m$. $\sigma$ acts by conjugation on $Gal(C_0/A)$, so $\sigma^m$ commutes with $\tau|C_0$. Now we want to extend $\tau|C_0$ to $\tau_1 \in Gal(C_1/A)$ in such a way that $\tau_1$ and $\tau$ are compatible (in the obvious sense, namely there union is elementary) and such that some power of $\sigma$ commutes with $\tau_1$.

(Note that $\tau_1$ will be compatible with $\tau$ just if the restriction of $\tau_1$ to $C_1 \cap B$ agrees with $\tau$.) We can find $B_1$, finite Galois over $A$ such that $B \cap C_1 \subseteq B_1$, $C_1 = cl_\sigma(B_1)$, and such that $\bigcup tp^-(\sigma^j(B_1)/C_0) : j \in \mathbf{Z} \models tp^-((\sigma^j(B_1)_j/C_0)$ (namely witnessing benignness of $C_1$ over $C_0$).
Let $D_1 = dcl^-(B_1, \sigma(B_1), ..\sigma^m(B_1))$, a finite Galois extension of $C_0$, containing $B \cap C_1$. Let $\tau'$ be an extension of $\tau|(B \cap C_1)$ to (an automorphism of) $D_1$. We will extend $\tau'$ to $\tau_1 \in Gal(C_1/A)$: on $\sigma^{jm}(D_1)$, $\tau_1$ will be $\sigma^{jm}.\tau'.\sigma^{-jm}$. Note that, as $\sigma^{jm}|C_0$ commutes with $\tau|C_0$, $\tau_1$ agrees with $\tau$ on $C_0$. Moreover, the assumption on $B_1$ (witnessing benignness of $C_1$ over $C_0$) shows that $\tau' \in Gal(C_1/A)$.
A similar argument allows us to construct inducively $\tau_i \in Gal(C_i/A)$ compatible with $\tau$ and commuting with a suitable power of $\sigma$, for $i = 2, .., m$.

In section 6, given $p(x) = tp(b/A)$ we defined $p[k]$ to be the type of $b$ over $A$ in the reduct $(\bar{M}, \sigma^k)$. Similarly if $q = qftp(b/A)$, we let $q[k]$ denote the quantifier-free type of $b$ over $A$ in $(\bar{M}, \sigma^k)$, and note this depends just on $q$ (rather than $b$). With above notation, $SU(p)[k]$ (funny notation) denotes the $SU$-rank of $p[m]$ in the structure $(\bar{M}, \sigma^k)$ (rather than the $SU$-rank of the partial type $p[k]$ in $(\bar{M}, \sigma)$. We will also define $order_k(b/A)$ to be the $U$-rank in $\bar{M}$ of $tp^-((\sigma^{jk}(b))_{j \in \mathbf{Z}}/A)$ (so the "transcendence degree" of $A(\sigma^{jk}(a) : j \in \mathbf{Z})$ over $A$).

**Exercise 7.7** *($ACFA_0$) (i) Let $a$ be a generic solution over $E$ of $\sigma^2(a) = a$, and let $p(x) = tp(a/E)$. Then $SU(p) = 2$ and $SU(p)[2] = 1$.*
*(ii) Let $a$ be a generic solution over $E$ of $\sigma^2(x) = x^2$, and $p(x) = tp(a/E)$. Then $SU(p) = 1$ and $SU(p)[2] = 1$.*
*(iii) Let $a$ be as in (i) and let $b = \sigma(a)$. Let $q(x, y) = tp(a, b/E)$. Then $SU(q) = 1$ and $SU(q)[2] = 2$.*

**Proposition 7.8** *(T has EGI.) Let $A$ be $\sigma$-closed, and let $a$ be an element in $acl^-(A)$. Then there is $k$ such that whenever $tp^-(b/A) = tp^-(a/A)$ then $qftp(a/A)[k] = qftp(b/A)[k]$.*

*Proof.* Let $B$ the finite Galois extension of $A$ generated by $a$, and let $C$ be $cl_\sigma(B)$. Let $\tau \in Gal(B/A)$ such that $\tau(a) = b$. By Lemma 7.6, $\tau$ has an extension to $\tau' \in Gal(C/A)$ which commutes with $\sigma^k$ for some $k$. It follows that $a$ and $b$ have the same quantifier-free type over $A$ in $(C, \sigma^k)$ (and so in $(\bar{M}, \sigma^k)$.

**Proposition 7.9** *(T has EGI.) Let A be σ-closed. Let B be a finite Galois extension of A and let* $C = cl_\sigma(B)$*. Then there is k such that whenever* $\sigma_1, \sigma_2$ *are extensions of σ in* $Gal(C/A)$ *then* $(C, (\sigma_1)^k)$ *and* $(C, (\sigma_2)^k$ *are isomorphic over A.*

*Proof.* Let $C_0$ be as in the proof of Lemma 7.6. So
(i) $C_0$ is a finite Galois extension of $A$ which is $\sigma$-closed, and is maximal such among subsets of $C$, and
(ii) $C_0$ has no finite proper (Galois) $\sigma$-closed extensions inside $C$.
By (i), $C_0$ is $\sigma_i$ closed for $i = 1, 2$. Let $\tau_i$ be the restriction of $\sigma_i$ to $C_0$. So $\tau_i$ is an elementary permutation of $C_0$ agreeing with $\sigma$ on $A$. We leave it to the reader (using finiteness of $Gal(C_0/A)$) to prove that for some $k$, $(\tau_1)^k = (\tau_2)^k$. Then, using (ii) and copying the proof of 6.3, one sees that $(C, (\sigma_1)^k)$ is isomorphic to $(C, (\sigma_2)^k)$ over $A$. Or one can do this directly using the $C_i$.

Here is a more useful version of the above proposition. In fact it seems that the next two propositions are all that matters. Maybe even the benignness business is not needed.

**Proposition 7.10** *(T has EGI.) Let A be σ closed. Let f be an isomorphism of A with some* $A' \subseteq (\bar{M}, \sigma)$*. Let C be a Galois extension of A which is finitely generated as a difference set. Then for some k, f extends to an isomorphism of* $(C, \sigma^k)$ *with* $(C', \sigma^k)$ *for some* $C' \subset \bar{M}$*.*

*Proof.* Let $C_0 \subset C_1 \subset ... \subset C_m = C$ be as before. $C_0$ is a difference set which is a finite Galois extension of $A$. $f$ extends to an $L^-$ isomomorphism $f'$ betweeen $C_0$ and some $C_0'$. As $Gal(C_0/A)$ is finite, $f'$ commutes with $\sigma^k$ for some $k$. Thus $f'$ is an isomorphism between $(C_0, \sigma^k)$ and $(C_0', \sigma^k)$. We want to extend to $C_1$. As $C_1$ is benign over $C_0$, it is $\sigma^k$ generated by some $b \in acl^-(C_0)$ such that the types $tp^-(\sigma^{jk}(b)/C_0)$ are weakly orthogonal. The same is thus true of the images of these types under $f'$. $f'$ then clearly extends to an isomorphism $f''$ between $(C_1, \sigma^k)$ and some $(C_1', \sigma^k)$. Continue. (Is there an easier proof using the proof of 6.3?)

**Proposition 7.11** *(T has EGI.) Suppose f is an isomorphism between* $(A, \sigma)$ *and* $(A', \sigma)$ *where A and A′ are σ-closed. Let* $B = cl_\sigma(A, b)$ *for some finite tuple b. Then for some k, f extends to an isomorphism between* $(B, \sigma^k)$ *and* $(B', \sigma^k)$ *for some* $A' \subset B' (\subset \bar{M})$*.*

*Proof.* Let $b'$ be the infinite tuple $(\sigma^j(b))_j$. Then $tp^-((b'/acl^-(A))$ is definable over some $C = cl_\sigma(A, \alpha)$ for some finite tuple $\alpha \in acl^-(A)$. (Exercise.) Moreover we may assume that $C$ is Galois over $A$. By the previous proposition, $f$ extends to an isomorphism $f'$ between $(C, \sigma^k)$ and $(C', \sigma^k)$ for some $A' \subseteq C' \subseteq \bar{M}$. Let $B_0 = dcl^-(C, b')$. Note $B_0$ is closed under $\sigma$ hence under $\sigma^k$. Let $B'_0$ be a substructure of a model of $T$ which contains $C'$ and $\tau$ an elementary permutation of $B'_0$ extending $\sigma^k|C'$, and $f''$ an extension of $f'$ to an isomorphism between $(B_0, \sigma^k)$ and $(B'_0, \tau)$. $tp^-(B'_0/C')$ clearly has a unique extension to an $L^-$ type over $acl^-(C')$ $(= acl^-(A'))$. It follows that $\tau$ and $\sigma^k|acl^-(A')$ have a common extension $\tau'$ to a substructure $B''$ of a model of $T$ where $B' = dcl^-(acl^-(A'), B'_0)$. The properties of $TA$ imply that we can assume that $B' \subset \bar{M}$ and that $\tau'$ is precisely $\sigma^k|B''$. In particular the restriction of $f''$ to $B$ works.

We assume now that $T$ has $EGI$. We now want to define the "eventual $SU$-rank". We need a few remarks first.

**Lemma 7.12** *Let $A$ be $\sigma$-closed. Suppose that $k \geq 1$ and $order(b/A) = order_k(b/A) < \omega$. Then $SU(b/A) \leq SU(b/A)[k]$.*

*Proof.* Note first that for any $\sigma$-closed $B \supset A$, $order(b/B) = order_k(b/B)$ (why?). Now suppose $SU(b/A) = n$. So there are $\sigma$-closed sets $A = A_0 \subset A_1 \subset A_2 \subset ...A_n$ such that $order(b/A_{i+1}) < order(b/A_i)$ for $i = 0, .., n-1$. By the previous remark the same is then true with $order_k$ replacing $order$. Hence, $SU(b/A)[k] \geq n$.

**Lemma 7.13** *Let $A$ be $\sigma$-closed. Suppose $ord(b/A)$ is finite. Then $lim_{n\geq 1}SU(b/A)[n!]$ exists.*

*Proof.* Note first that if $1 \leq k \leq l$ and $k$ divides $l$ then $order_k(b/A) \geq order_l(b/A)$. Fix $k$ such that $order_k(b/A) = r$ is least possible. So by the previous lemma (applied to reducts $(\bar{M}, \sigma^{m!})$ for $m \geq k$, we see that for each $k \leq m \leq n$, $SU(b/A)[m!] \leq SU(b/A)[n!] \leq r$. So the limit exists.

**Corollary 7.14** *Suppose $k \geq 1$, $A$ is $\sigma^k$ closed and $order_k(b/A)$ is finite Then $lim_{n\geq k}SU(b/A)[n!]$ exists.*

*Proof.* Apply the previous lemma to $(\bar{M}, \sigma^k)$ (essentially).

This the following definition makes sense.

**Definition 7.15** *Suppose that $A$ is $\sigma^k$ closed and $order_k(b/A)$ is finite for some $k \geq 1$. Then we define the eventual SU-rank of $b$ over $A$, $evSU(b/A)$ to be $lim_{n \geq k} SU(b/A)[n!]$.*

**Remark 7.16** *(i) Let $b$ be a generic solution of $\sigma^2(x) = x$ over $\sigma$-closed $A$. Then $SU(b/A)[m] = 2$ if $m$ is odd and $= 1$ if $m$ is even. So $lim_{n \geq 1} SU(b/A)[n]$ does not exist, but $evSU(b/A) = 1$.*
*(ii) Suppose $tp(b/A)[m] = tp(c/A)[m]$ for some $m$. Then $evSU(b/A) = evSU(c/A)$.*
*(iii) $evSU(b/A) = evSU(b/acl^{-}(A)$.*
*(iv) Let $A$ be sigma-closed, $SU(a/A) < \omega$ and $SU(b/A) < \omega$. Let $B = \bigcap\{acl^{-}(cl_{\sigma^n}(A, b)) : n \geq 1\}$. Then $evSU(a, b/A) = evSU(a/B) + evSU(b/A)$. (Exercise using additivity of SU-rank.)*

Here is the rather surprising conclusion of the results of this section.

**Proposition 7.17** *Suppose $A$ is $\sigma$-closed and $order(b/A)$ is finite. Then $evSU(b/A)$ depends only on $qftp(A, b)$. Namely, suppose $f$ is an isomorphism of $cl_\sigma(A, b)$ with $cl_\sigma(A', b')$ which takes $A$ onto $A'$ and $b$ to $b'$. Then $evSU(b/A) = evSU(b'/A')$.*

*Proof.* Suppose that $evSU(b/A) \geq n + 1$. Let $m$ be such that $evSU(b/A) = SU(b/A)[m]$ and for any $r > m$ such that $m$ divides $r$, $order_m(b/A) = order_r(b/A)$. Working now in $(\bar{M}, \sigma^m)$ let $c$ be such that $SU(b/A, c)[m] \geq n$, and $b$ forks with $c$ over $A$ in $(\bar{M}, \sigma^m)$. A Morley sequence argument shows that we may assume that $SU(c/A)[m]$ is finite. So we may assume that $\sigma^m(c) \in acl^{-}(A, c)$. So, as $b$ forks with $c$ over $A$ in $(\bar{M}, \sigma^m)$, our assumptions above on $m$ imply
(*) $cl_{\sigma^r}(A, b)$ forks with $A, c$ over $A$ in the model $\bar{M}$ of $T$, for all $r$ divisible by $m$.
By 7.12, $evSU(b/cl_{\sigma^m}(A, c)) \geq n$. By 7.11 $f$ extends to an isomorphism $f'$ of $(cl_{\sigma^m}(A, b, c), \sigma^{lm})$ with a $\sigma^{lm}$-closed substructure of $\bar{M}$, for some $l$. Let $f(c) = c'$. By (*), $b'$ forks with $c'$ over $A$ in $(\bar{M}, \sigma^{mn})$ whenever $l$ divides $n$. On the other hand, by the induction hypothesis, $evSU(b'/cl_{\sigma^{ml}}(A', c')) \geq n$. It follows that $evSU(b'/A') \geq n + 1$.
We have shown that $evSU(b/A) \geq n + 1$ implies $evSU(b'/A') \geq n + 1$. By symmetry we get equality of $evSU(b/A)$ and $evSU(b'/A')$.

**Problem 7.18** *Is it the case that for $b$ of finite order over $A$, $SU(b/A)$ depends only on $qftp(b/A)$?*

# 8 Limit structures

In this section $A$ will be algebraically closed (in $(\bar{M}, \sigma)$), and $(M, \sigma)$ an $|A|$-saturated (but still small) elementary substructure of $(\bar{M}, \sigma)$ containing $A$.

**Definition 8.1** *(i) The complete quantifier-free type $p(x) = qftp(a/A)$ is said to be basic if (a) ($ALG_m$) for some $m \geq 1$, $\sigma^m(a) \in acl^-(A, a)$, and (b) $evSU(a/A) = 1$.*
*(ii) $p(x) = qftp(a/A)$ is said to be semi-basic if it satisfies (a) above, and also there exist $a_1, .., a_n$, such that $qftp(a_i/A)$ is basic, the $a_i$ are independent over $A$ in the sense of $T$, and $acl^-(A, a) = acl^-(A, a_1, .., a_n)$.*

Note that by 7.17, (i) makes sense. (Eventual rank of $tp(a/A)$ depends only on $qftp(a/A)$).

**Definition 8.2** *Let $p$ be semi-basic. Then $\chi_p(M)$ is the set of those $a \in M$ such that $a$ realises $p[m]$ for some $m \geq 1$. $\chi_p$ is the same thing but allowing any $a$ (not just in $M$)*

**Remark 8.3** *(i) $\chi_p$ depends only on the $\sim$-equivalence class of $p$ where we put $p \sim q$ if $p[m] = q[m]$ for some $m$. (Zoe calls the $\sim$-class of a basic type $p$ a virtual basic type.) Similarly for a virtual semibasic type.*
*(ii) Suppose $p(x)$ is semi-basic. Let $a, a_1, .., a_n$ be as in Definition 8.1. Let $b \in \chi_p(M)$. Then there are $b_1, .., b_n$, $L^-$-independent over $A$, each realising a basic type over $A$, such that $acl^-(A, b) = acl^-(A, b_1, .., b_n)$.*
*(iii) Let $p$ be as in (ii). Then $evSU(p) = n$.*

*Proof.* (ii) Suppose $a$ satisfies $AGL_m$. Let $l$ be divisible by $m$ be such that $b$ realises $p[l]$. By 7.11, there are $k$ and $b_1, .., b_n$ such that $qftp(a, a_1, .., a_n/A)[kl] = qftp(b, b_1, .., b_n/A)[kl]$. This is enough.
(iii) Left to you.

We now want to define a structure on $\chi_p(M)$. For some reason, they consider a many-sorted structure with sorts corresponding to the semi-basic types. They introduce various "coordinate rings" and virtual ideals. Working in $TA$ we want to do this with formulas. Why do they work with complete quantifier-free types rather than quantifier-free formulas? Probably to get "smoothness" of the Zariski geometries in the $ACFA$ case. Anyway, here is my tentative definition.

**Definition 8.4** *(i) $\chi(M)$ will be a many-sorted structure whose sorts are the $\chi_p(M)$ for $p$ semi-basic, in the following language $L^*$. Let $p_1(x_1), .., p_n(x_1)$ be semi-basic types. Let $\phi(x_1, .., x_n)$ be a quantifier-free formula over $A$. Then $R_\phi$ will be a relation symbol in $L^*$, with interpretation in $\chi(M)$ as follows: let $Q_\phi$ be the set of complete quantifier-free types $q(x_1, .., x_n)$ over $A$ containing $\phi(x_1, .., x_n)$. Then for $(a_1, .., a_n) \in \chi_{p_1}(M) \times ... \times \chi_{p_n}(M)$, $(a_1, .., a_n) \in R_\phi$ iff for some $q \in Q$ and some $m$, $(a_1, .., a_n)$ realises $q$.*
*(ii) Similarly, we define a structure on $\chi_p(M)$ for a fixed $p$.*

I will work with $\chi_p(M)$, where $p$ is basic. We would like to prove:

**Theorem 8.5** *Suppose $p(x)$ is a basic type over $A$. Then $\chi_p(M)$, as an $L^*$-structure, is qf-saturated, qf-homogeneous, and qf-strongly minimal.*

**Remark 8.6** *(i) qf-homogeneity means that any partial isomorphism between small subsets of the $L^*$-structure $\chi_p(M)$ extends to an automorphism. $qf$-saturation means that any collection of quantifier-free formulas over a small subset of $\chi_p(M)$, evety finite subset of which is realised in $\chi_p(M)$, is itself realised in $chi_p(M)$. Qf-strong minimality means that any qf-definable subset of $\chi_p(M)$ is finite or cofinite. Moreover there should be a finite bound on the finite sets so defined, as the parasmeters vary, for a given qf-formula of $L^*$.*
*(ii) As we pointed out before $(\bar{M}, \sigma)$ is qf-$\omega$-stable: there are only countably many quantifier-free types over any countable set. One can then define Morley rank etc. for qf-formulas, and try to work inside a qf-strongly minimal set. One will obtain qf-saturation (from saturation of the ambient structure) but not qf-homogenity.*
*(iii) Essentially all model theory goes through in the qf-saturated, qf-homogeneous context.*

From now on, $p$ is basic. Underlying what will be done now is quantifier-free stability, which I will use freely. The main point, as far as I can see is:

**Lemma 8.7** *Suppose $b, c$ are tuples of the same length from the $L^*$-structure $\chi_p(M)$. Then $b$ and $c$ have the same quantifier-free $L^*$-type in $\chi_p(M)$ iff for some $m$, $qftp(b/A)[m] = qftp(c/A)[m]$ (namely $b$ and $c$ have the same quantifier-free type over $A$ in $(\bar{M}, \sigma^m)$).*

*Proof.* Right to left is easy, but I'll do it nevertheless. Suppose $qftp(b/A)[m] = qftp(c/A)[m]$. Let $\phi(y)$ be a quantifier-free formula over $A$ (in $L$). Suppose that $R_\phi(b)$. This means that there is a complete qf-free type $q(y)$ over $A$ containing $\phi(y)$, such that $b$ satisfies $q[l]$ for some $l$. Note that $qftp(b/A)[ml] = qftp(c/A)[ml]$, so $qftp(c/A)[ml] = q[ml]$. Thus $c$ satisfies $R_\phi$ too.
Left to right is a bit more problematic.
Suppose $b$ and $c$ have the same qf-$L^*$-type in $\chi_p(M)$. To make life easy I will assume that $b = (b_1, b_2)$ ($b_i \in \chi_p(M)$) and similarly for $c$.
*Claim 1.* $evSU(b/A) = evSU(c/A)$.
*Proof.* Exercise.
If $evSU(b/A) = 2$, then for some $m$, $SU(p)[m] = 1$ and $SU(b/A)[m] = SU(c/A)[m] = 2$. The qf-theory implies that $qftp(b/A)[m] = qftp(c/A)[m]$. Similarly if $evSU(b/A) = 0$ then $b \in A$ and there is nothing to do. So suppose $evSU(b/A) = 1$. Without loss of generality $SU(p)[k] = 1$ for all $k$, and for $a$ realising $p$, $\sigma(a) \in acl^-(A, a)$. (Pass to $(\bar{M}, \sigma^r)$ for sufficiently large $r$.) So $\sigma(b_2) \in acl^-(A, b_1)$. Then there is a qf-formula over $A$, $\psi(x_1, x_2)$, witnessing this, and such that there is a unique nonalgebraic qf-type $q(x_1, x_2)$ over $A$ containing $\psi$. Then $R_\psi$ holds of $b$, so also of $c$. As $evSU(c/A) = 1$ it follows that $c$ realises $q[m]$ for some $m$.

**Lemma 8.8** *(Qf-$\omega$-homogeneity of $chi_p(M)$.) Suppose that $b, c$ are finite tuples from $\chi_p(M)$ of the same length with the same quantifier-free $L^*$-type. Let $d \in \chi_p(M)$. Then there is $e \in chi_p(M)$ such that $(b, d)$ and $(c, e)$ have the same qf-$L^*$-type.*

*Proof.* By the previous lemma, for some $m$, $qftp(b/A)[m] = qftp(c/A)[m]$. So there is an isomorphism $f$ over $A$ between $(cl_{\sigma^m}(A, b), \sigma^m)$ and $(cl_{\sigma^m}(A, c), \sigma^m)$ fixing $A$ pointwise and taking $b$ to $c$. Let $B = cl_{\sigma^m}(A, b, d)$. By 7.11, there is $r$ such that $f$ extends to an isomorphism $f'$ between $(cl_{\sigma^m}(B), \sigma^{mr})$ and $(B', \sigma^{mr})$ for some $B' \subset \bar{M}$. Let $e = f'(d)$. By saturation of $(M, \sigma)$ we may assme that $e \in M$, and so clearly $e \in \chi_p(M)$. Note that $qftp(b, d/A)[mr] = qftp(c, e/A)[mr]$ and so by the previous lemma, $(b, d)$ and $(c, e)$ have the same qf-$L^*$-type in $\chi_p(M)$.

**Lemma 8.9** *(Qf-$\omega$-saturation of $\chi_p(M)$.) Suppose that $b$ is a finite tuple from $\chi_p(M)$. Let $\Sigma(z)$ be a set of qf-$L^*$-formulas over $b$ which is finitely satisfiable in $\chi_p(M)$. Then $\Sigma(z)$ is satisfiable in $\chi_p(M)$.*

I will start things over a bit. To make a life a bit easier I will assume that the complete qf type $p(x)$ over $A$ has order 1. $p(x) = qftp(a/A)$, $U(tp^{-}(a/A)) = 1$, and $\sigma(a) \in acl^{-}(A, a)$.

**Lemma 8.10** *Let $\phi(x_1, .., x_n)$ be a quantifier-free formula over A. Let Q be the set of complete qf types $q(x_1, .., x_n)$ over A such that $q(x_1, .., x_n)$ contains $p(x_i)$ for each i. Fix m. Then there is a quantifier-free formula $\psi(x_1, .., x_n)$ in $L|\sigma^m$ such that for any realizations $b_1, .., b_n$ of $p[m]$, $qftp(b_1, .., b_n/A)[m] = q[m]$ for some $q \in Q$ if and only $(b_1, .., b_n)$ satisfies $\psi$.*

*Proof.* Again to make life easy, let's assume that $\phi(x_1, x_2)$ implies that $x_1$ and $x_2$ are interalgebraic (in either equivalent sense) over $A$. Fix $m$, and let $\Psi_m(x_1, x_2)$ be all formulas over $A$ in $L|\sigma^m$ which are consequences of $\phi(x_1, x_2)$. So there is $\psi(x_1, x_2) \in \psi_m$ which implies $x_1$ and $x_2$ are interalgebraic over $A$. So we can choose such $\psi$ such that there are $b_1, b_2$ realising $p[m]$ with $\psi(b_1, b_2$ and such that the (finite) set of solutions of $\psi(b_1, x_2)$ is minimized. This formula should work (??)

I will come back to this stuff sometime. I believe these limit structures can be constructed in $TA$.

# 9 Examples in $ACFA$

I will look at the interesting examples from section 6 of [1]. These concern the model theory of finite rank types in $ACFA$. So we will work in some saturated model $(\bar{M}, \sigma)$ of $ACFA$.
We start by looking at the difference equation $\phi_b(x)$: $\sigma(x) = x^2 + b$, in the characteristic 0 case. We will prove

**Proposition 9.1** *($ACFA_0$.) Suppose that $\sigma(b) \neq -b^2/2$. Then $\phi_b(x)$ is strongly minimal and is trivial.*

We give some explanation. To say that $\phi_b(x)$ is strongly minimal means that it cannot be partitioned into two infinite definable subsets. This is equivalent to, for every set $E$ containing $b$, there is a unique nonalgebraic type over $E$ containing $\phi_b(x)$. To be trivial, means that whenever $a_1, .., a_n, c$ are solutions of $\phi_b(x)$, and $c \in acl(b, a_1, .., a_n)$ then $c \in acl(b, a_i)$ for some $i$. It would be equivalent if we replaced $b$ by any set containing $b$. The proof of the proposition will go through several remarks and observations.

newline Note first that $\phi_b(x)$ is an order 1 equation, thus every nonalgebraic complete type extending the formula has $SU$-rank 1. (In other words $\phi_b(x)$ has $SU$-rank 1.)

**Remark 9.2** *$\phi_b(x)$ is qf-strongly minimal. Namely, $\phi_b(x)$ cannot be partioned into two qf-definable infinite sets.*

*Proof.* What this amounts to is that over any $E$ containing $b$, there is a unique nonalgebraic complete quantifier-free type containing $\phi_b(x)$. So let $E$ be algebraically closed, and $a$ realise $\phi_b(x)$ with $a \notin E$. Now $tp^-(a/E)$ is uniquely determined. So as $\sigma(a) = a^2 + b$, $tp^-(a, \sigma(a), \sigma^2(a), .., /E)$ is uniquely determined. So therefore so is $tp^-((\sigma^i(a)_i/E)$, but the latter is precisely $qftp(a/E)$.

$E$ will denote an algebraically closed set containing $b$. Note that for $a \notin E$ satisfying $\phi_b(x)$, $tp(a/E)$ is unbounded (as for negative $k$, $mult^-(\sigma^k(a)/E, a)$ is arbitrarly large). So by Proposition 6.16 we have $tp(a/E)$ is (strongly) stationary, thus of $U$-rank 1. This does not necessarily imply strong minimality. We will first prove:

**Lemma 9.3** *Let $a$ be a nonalgebraic (over E) solution of $\phi_b(x)$. Suppose $\sigma(b) \neq -b^2/2$. Let $K = cl_\sigma(E, a)$. Then $K$ has no finite $\sigma$-invariant (Galois) extension.*

*Proof.* Suppose for sake of contradiction that $L$ is a (proper) finite Galois $\sigma$-invariant extension of $K$. $L$ has the form $K(c)$ for some finite tuple $c$. Now $tp^-(c/K)$ is determined by some $L^-$-formula $\psi(y, e)$. As $e \in K = cl_\sigma(E(a))$, it follows that for some $n$, $\sigma^n(e) \in E(a)$, and $\psi(\sigma^n(b), \sigma^n(e))$ holds. Let $L_1 = E(a, \sigma^n(b))$. It follows that $L_1$ is a Galois extension of $E(a))$, that $[L : K] = [L_1, E(a)]$ and that $L = K.L_1$. Moreover $\sigma(L_1) \subseteq L_1$, $\sigma^{-1}(L_1) = E(\sigma^-(a)).L_1$ and $K$ is linearly disjoint from $L_1$ over $E(a)$.
We will now aim for a contradiction by looking at ramification issues, as in section 6. $E(a)$ is the function field of $\mathbf{P}^1(E)$ with generic point $a$. $L_1$ is the function field of a smooth projective curve $X$, and the inclusion $E(a) < L_1$ induces a surjective finite-to-one morphism $\pi : X \to \mathbf{P}^1$ defined over $E$. Throw away from $X$ the preimage of $\infty$ under $\pi$, and we have an induced surjective morphism $pi$ from $X'$ onto $\mathbf{A}^1$, where $X'$ is now affine. As $L_1$ is a proper extension of $E(a)$, $\pi$ is generically $n$ to 1 for some $n > 1$. As $\mathbf{A}^1(\mathbf{C})$ is

simply connected, $\pi$ could not be a covering map, and thus the set of points $\alpha \in E$ such $|\pi^{-1}(\alpha)| < n$ is nonempty (and finite). Let $T$ denote this set. The inclusion $E(a) < E(\sigma^{-}(a))$ corresponds to the surjective morphism $f : \mathbf{A}^1 \to \mathbf{A}^1$ defined by $f(x) = x^2 + \sigma^{-1}(b)$. Note that the only point in $\mathbf{A}^1(E)$ ramifying for $f$ is $\sigma^{-}(b)$. $\sigma^{-1}(L_1)$ which is remarked above to be equal to $E(\sigma^{-}(a)).L_1$ is the function field of the curve $Y = \sigma^{-1}(X)$ which projects onto $\mathbf{P}^1$ by $\sigma^{-1}(\pi)$ and also onto $X$ by $g$ (and onto $\mathbf{P}^1$ by $\pi.g$). Any element $c$ of $Y$ is determined by its images under $\sigma^{-1}(\pi)$ and $g$.All these surjective morphisms of curves are Galois.

*Claim 1.* Suppose $\alpha \in T$ and $\alpha \neq \sigma^{-1}(b)$. Then $\pm\sqrt{\sigma(\alpha) - b} \in T$.

*Proof.* $\alpha$ has two distinct preimages under $f$: $\pm\sqrt{\alpha - \sigma^{-1}(b)}$. They must both both ramify under $\sigma^{-1}(\pi)$ (why?). Apply $\sigma$ to see that $\pm\sqrt{\sigma(\alpha) - b}$ ramifies for $\pi$.

*Claim 2.* Suppose $\sigma^{-1}(b) \in T$ with ramification index $e \neq 2$. Then $0 \in T$.

*Proof.* The unique preimage of $\sigma^{-}(b)$ under $f$ is 0. Then the number of preimages of 0 under $\sigma^{-1}(\pi)$ must be $n/e$ or $n/(e/2)$ (by counting points in the preimage of $\sigma^{-}(b)$ under $\pi.g$). In particular 0 ramifies under $\sigma^{-}(\pi)$. So by applying $\sigma$, 0 ramifies under $\pi$.

For each $m$ let $p_m(x)$ be the polynomial over $E$ such that $\sigma^m(a) = p_m(a)$. Note that all monomials in $p_m$ have even degree, and so $p_m$ is symmetric ($p_m(x) = p_m(-x)$). In particular:
(*) if both $\alpha$ and $-\alpha$ are solutions of $p_m(x) = \sigma^m(x)$, then $\alpha = 0$.
We aim towards showing that $T = \{\sigma^{-1}(b)\}$ (from which we shall derive a contradiction). Let $S = T \setminus \{\sigma^{-1}(b)\}$.

*Claim 3.* There is $m$ such that for all $\alpha \in S$, $p_m(\alpha) = \sigma^m(\alpha)$.

*Proof.* Let $R$ be the following (directed) binary relation on $S$: $\alpha R \beta$ iff $\beta^2 = \sigma(\alpha) - b$. By Claim 1 (and the assumption that $b \neq 0$, if $\alpha \in S$ then there is $\beta \in S$ such that $\alpha R \beta$. Fix $\alpha \in S$. It is not hard to show that if $\alpha = \alpha_1, \alpha_2, ..., \alpha_k$ is a maximal sequence of distinct elements of $S$ such that $\alpha_i R \alpha_{i+1}$ for $i = 1, .., k-1$ then $\alpha_k R \alpha$. It follows easily that $p_{k+1}(\alpha) = \sigma^{k+1}(\alpha)$, and the same for any multiple of $k+1$ in place of $k+1$. As $S$ is finite, we can find suitable $m$.

We obtain, from Claim 3 and (*):

*Claim 4.* If both $\alpha$ and $-\alpha$ are in $S$, them $\alpha = 0$.

*Claim 5.* $T$ contains $\sigma^{-1}(b)$.
*Proof.* If not, then as $T$ is nonempty, so is $S$. Let $m$ be as in Claim 3. Let $\alpha \in S$. As in the proof of Claim 3, we have an $R$-path in $S = T$ from $\alpha$ to itself. In particular there is $\beta \in S$ such that $\beta R \alpha$. By Claim 1, $-\alpha$ is also in $S$. By (*), $\alpha = 0$. By Claim 1, $\pm\sqrt{-b}$ are in $S$. So, as we have just shown $b = 0$, a contradiction.

*Claim 6.* The ramification index of $\sigma^{-1}(b)$ under $\pi$ is 2.
*Proof.* Suppose not. Then by (Claim 4 and) Claim 2, $0 \in T$. So $0 \in S$. As in the proof of Claim 3, there is $\alpha \in S$ such that $\alpha R 0$. But then $\sigma(\alpha) = b$ so $\alpha = \sigma^{-1}(b)$, a contradiction.

*Claim 7.* $T = \{sigma^{-1}(b)\}$
*Proof.* Note by the proof of Claim 3 and Claim 4 that $S$ is a subset of $\{0, -\sigma^{-1}(b)\}$. We will first suppose that $0 \in S$ (and get a contradiction). Then $\pm\sqrt{-b} \in T$ by Claim 1. So, by Claim 4, one of them, say $+\sqrt{(}-b)$ equals $\sigma^{-1}(b)$. Then easily (*) $\sigma(b) = -b^2$. Then $-\sigma^{-1}(b) \in S$. So, by claim 1, $\pm\beta \in T$, where $\beta^2 = \sigma(-\sigma^{-1}(b)) - b = -2b \neq 0$. So $\beta = \sigma^{-1}(b)$, and we see that $\sigma(b) = -b^2/2$, contradicting (*). Now suppose that $-\sigma^{-1}(b) \in S$. As before, we see that $(\sigma^{-}(b))^2 = \sigma(-\sigma^{-1}(b)) - b$ and so $\sigma(b) = -b^2/2$, contradicting our assumption.

By Claim 7 and Claim 6, $\pi$ it is apparently standard (Hurwitz theorem??) to deduce that $L_1 = E(a, \sqrt{a - \sigma^{-1}(b)})$. Given this, as $\sigma(a) = a^2 + b$, $a = \sigma^{-1}(a^2) + \sigma^{-1}(b)$, whereby $\sigma^{-1}(a)$ is a square root of $a - \sigma^{-1}(b)$. It follows that $L_1 = E(\sigma^{-1}(a))$, contradicting their linear disjointness over $E(a)$. This proves the lemma.

Now suppose that $a_1, a_2$ are nonalgebraic (over $E$) solutions of $\phi_b(x)$ $(b \neq 0)$. As remarked before there is an isomorphism $f$ between $(cl_\sigma(E(a_1))), \sigma)$ and $(cl_\sigma(E(a_2))), \sigma)$ which is the identity on $E$ and takes $a_1$ to $a_2$. By the lemma, and Proposition 6.3, $f$ extends to an isomorphism between $(acl(E(a_1)), \sigma)$ and $acl(E(a_2)), \sigma)$, whereby $a_1$ and $a_2$ have the same type over $E$. This proves the strong minimality of $\phi_b(x)$.

Before completing the proof of Proposition 9.1 (by showing the triviality of $\phi_b(x)$) we point out.

**Lemma 9.4** *Suppose that for some odd prime e, and primitive eth roots of* *unity* $\zeta$, $\sigma(\eta) = (\eta)^2$. *Then the formula* $\phi_0(x)$ ($\sigma^2(x) = x^2$) *is not strongly minimal.*

*Proof.* We follow notation as in the proof of the lemma. ($a$ is a generic solution of $\phi_0(x)$ over $E$, where $E$ is algebraically closed, and $K = cl_\sigma(E(a))$). Let $L = K(b)$ where $b$ is an *eth* root of $a$ ($e$ as in the hypothesis). Note that $E(a, b)$ is linearly disjoint from $K$ over $E(a)$. As $\sigma(a) = a^2$, we obtain an automorphism $\sigma'$ of $L$ extending $\sigma$ by defining $\sigma'(b) = b^2$. Note that $Gal(K(b)/K) = \{\tau_j : j = 1, .., e\}$ where $\tau_j(b) = \zeta^j.b$. It is clear that $\sigma'$ commutes with each $\tau_j$ in $Aut(K(b)$. Hence, $\sigma$ has $e$ extensions to $Aut(K(b))$, up to isomorphism over $K$. This shows that there are at least $e$ completions of $\phi_0(x)$ to a complete nonalgebraic type over $E$. So $\phi_0(x)$ is not strongly minimal.

We now return to the case where $\sigma(b) \neq -b^2/2$.

**Lemma 9.5** *Suppose that* $c \in acl(E(a)) \setminus E$*, and* $c$ *also satisfies* $\phi_d(x)$ *for some* $d \in E$*. Then* $c = \sigma^k(a)$ *for some* $k \in \mathbf{Z}$*.*

*Proof.* We may replace $c$ by $\sigma^r(c)$ whenever we want. Let $K = cl_\sigma(E(a))$. Suppose, by way of contradiction that $mult^-(c/K) = s > 1$. We can find $m < 0$ such that $mult^-(c/E(\sigma^m(a)) = s$. So replacing $c$ by $\sigma^m(c)$ we have that $mult^-(c/E(a)) = s$, and moreover $mult^-(\sigma(c)/E(a)) = s$ too. Thus $E(a, c) = E(a, \sigma(c))$. It follows, by automorphism, that $cl_\sigma(E(a, c)) = cl_\sigma(E(a))(c)$, so we have a proper finite algebraic extension of $K$ which is $\sigma$-invariant. This contradicts Lemma 9.3 We have shown that $c \in K$.
*Claim.* $E(c) = E(\sigma^k(a))$ for some $k$.
*Proof.* There is minimal $j \in \mathbf{Z}$ such that $\sigma^j(a) \in E(c)$ (why???). We may assume $j = 0$, so $E(a) \subseteq E(c)$. We want to show that $E(c) \subseteq E(a)$. Note that there is a smallest nonnegative $k$ such that $E(c) \subset E(\sigma^{-k}(a))$. Suppose for the sake of contradiction that $k > 0$. In fact, suppose for simplicity that $k = 1$ (similar argument in general). So $c \in E(\sigma^{-1}(a))$, and thus $\sigma(c) \in E(a)$.Now $mult^-(c/E(\sigma(c)) = 2$. Also $mult^-(a/E(\sigma(c)) = 2$. (Note that $\sigma(a) \in E(\sigma(c))$, so $mult^-(a/E(\sigma(c)) \leq 2$. If the latter is 1, then also $mult^-(\sigma^{-1}/E(a)) = 1$, contradicting minimal choice of $j$ above.) Now $mult^-(a/E(c, \sigma(c)))) = 1$, so we conclude that $mult^-(c/E(\sigma(c), a)) = 1$. Thus $mult^-(c/E(a)) = 1$, which is either a contradiction or what we want. OK.

By the Claim we may suppose that $E(a) = E(c)$. Thus $c$ is obtained from $a$ by a linear fractional transformation over $E$: $c = (\alpha.a + \beta)/(\gamma.a + \delta)$ for some $\alpha, \beta, \gamma, \delta \in E$ such that $\alpha.\delta - \beta.\gamma \neq 0$. Applying $\sigma$ and looking at the poles of the correspondinf=g rational function, we see that $\gamma = 0$, so we may assume that $c = \alpha.a + \beta$ ($\alpha \neq 0$). Applying $\sigma$ again we obtain
$\alpha^2.a^2 + 2\alpha\beta a + \beta^2 + d = \alpha.\hat{2} + \alpha.b + \beta$.
Fromm this we conclude that $\beta = 0$ and $\alpha = 1$, so $c = a$. This completes the proof of the lemma.

We now complete the proof of the proposition (triviality of $\phi_b(x)$). Let $a_1, .., a_n, c$ be solutions of $\phi_b(x)$, such that $c \in acl(b, a_1, .., a_n)$. We may assume (by induction) that $c \notin acl(b, a_1, .., a_{n-1})$. Let $E = acl(b, a_1, .., a_{n-1})$. By the previous lemma, $c \in acl(b, a_n)$. Good.

# References


[1] Z. Chatzidakis and E. Hrushovski, The Model theory of difference fields, to appear in Transactions AMS.

[2] Z. Chatzidakis, E. Hrushovski, and Y. Peterzil, Model theory of difference fields II, preliminary version.

[3] Z. Chatzidakis and A. Pillay, Generic structures and simple theories, Annals of Pure and Applied Logic, 1998.

[4] Cohn, Difference algebra.

[5] E. Hrushovski and B. Zilber, Zariski geometries, Journal of AMS 1996.

[6] B. Kim and A. Pillay, Simple theories, Annals of Pure and Applied Logic, 1997.

[7] A. Pillay, Model theory of algebraically closed fields, in *Model Theory and Algebraic Geometry*, ed. E. Bouscaren, Lecture Notes in Math. 1696, 1998.

[8] A. Pillay, Some remarks on modular regular types, Journal of Symbolic Logic, 56 (1003-1011), 1991.

[9] A. Pillay, Geometric Stability Theory, Oxford University Press 1996.