

On compactifications and the topological dynamics of definable groups

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Abstract

For G a group definable in some structure M , we define notions of “definable” compactification of G and “definable” action of G on a compact space X (definable G -flow), where the latter is under a definability of types assumption on M . We describe the universal definable compactification of G as $G^*/(G^*)_M^{00}$ and the universal definable G -ambit as the type space $S_G(M)$. We also point out the existence and uniqueness of “universal minimal definable G -flows”, and discuss issues of amenability and extreme amenability in this definable category, with a characterization of the latter. For the sake of completeness we also describe the universal (Bohr) compactification and universal G -ambit in model-theoretic terms, when G is a topological group (although it is essentially well-known).

1 Introduction and preliminaries

Given a topological (Hausdorff) group G , a compactification of G is a pair (H, f) where H is a compact topological group and $f: G \rightarrow H$ a continuous homomorphism with dense image. There is a *universal* such compactification, called the *Bohr compactification*. Let us note immediately that a compactification of the topological group G is a special case of continuous action of G on a compact space X , where X has a distinguished point x_0 with dense orbit under G (a so-called G -ambit.) Again there is a universal G -ambit. There is reasonably comprehensive account of abstract

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topological dynamics in [1].

On the other hand, given a group G definable in a structure M , a saturated elementary extension M^* of M , G^* the interpretation of the formulas defining G in M^* and a type-definable over M , normal subgroup N of G^* of index at most $2^{|M|}$, G^*/N , equipped with the so-called *logic topology* is a compact Hausdorff group, and the identity embedding of G in G^* induces a homomorphism from G into G^*/N with dense image. There is a smallest such N which is called $(G^*)_M^{00}$, a kind of *connected component* of G^* . More generally we have the action of G on the space $S_G(M)$ of complete types over M concentrating on G , where $S_G(M)$ is the Stone space of the Boolean algebra of definable subsets of G . And there is a canonical dense orbit: $G \cdot \text{tp}(1_G/M)$.

Here we will relate these two theories, in the context of two categories: first the classical case of topological groups, and secondly the new case of definable groups. The case of compactifications of topological groups was explicated by Robinson and Hirschfeld ([15], [5]) in the language of nonstandard analysis. For the more general case of G -flows, the explication of the universal G -ambit via the *Samuel compactification* of G ([16], [19]), with respect to the right uniformity on G , is basically equivalent to the model-theoretic account that we give below. As far as definable groups are concerned, we give appropriate definitions (definable compactification, definable G -flow), obtaining in a sense a theory of *tame* topological dynamics, although we make a *definability of types* assumption on the model concerned, in the case of group actions. A free group \mathbb{F}_n ($n \geq 2$), considered as a first order structure (\mathbb{F}_n, \cdot) , will be extremely amenable as a definable group, although considered as a discrete topological group, it will not be such.

A big influence on this paper is work of Newelski (for example [8], [9]) on trying to use the machinery of topological dynamics to extend stable group theory to tame unstable contexts. The present paper is quite soft, aiming partly at putting Newelski's ideas into a formal framework, *definable, or tame, topological dynamics*. On the other hand the treatment of *fs* groups in [13] and the case analysis of $\text{SL}(2, \mathbb{R})$ in [4] from this point of view, are rather harder.

In the remainder of this introduction, we recall some very basic model-theoretic notions and constructions. A good reference for the kind of model theory in the current paper is [14] as well as [6]. Whenever we take about a topological space we assume Hausdorffness.

We fix a complete theory T in language L , a model M of T and a very saturated elementary extension M^* of M , for example κ -saturated of cardinality κ where $\kappa > 2^{|M|+|L|}$. For X a definable set in M^* , definable over M (or even a set definable in M), $S_X(M)$ denotes the Stone space of complete types over M which concentrate on (the formula defining) X . By a type-definable over M set in M^* we mean the common solution set in M^* of a collection of formulas over M , equivalently an in-

tersection of sets, definable in M^* over M . Sometimes L_M denotes the language L expanded by constants for elements of M .

We first recall the logic topology on bounded hyperdefinable sets (in a saturated model).

Definition 1.1. Let X be a definable set in the structure M^* , definable with parameters from M .

- (i) Suppose E is a type-definable over M equivalence relation with a *bounded* number of classes, that is $< \kappa$ (equivalently $\leq 2^{|M|+|L|}$) many, and $\pi: X \rightarrow X/E$ the canonical surjection. The *logic topology* on X/E is defined as follows: $Z \subseteq X/E$ is closed if $\pi^{-1}(Z) \subseteq X$ is type-definable over M .
- (ii) Suppose C is a compact space, and $f: X \rightarrow C$ a map. We say that f is *definable over M* , if for any closed subset D of C , $f^{-1}(D)$ is type-definable over M .

The following is well-known (see e.g. [6]).

Lemma 1.2. Let X, E be as in Definition 1.1 and C be a compact space.

- (i) The set X/E equipped with the logic topology is a compact space, and $\pi: X \rightarrow X/E$ is definable over M .
- (ii) Conversely, suppose $f: X \rightarrow C$ is an M -definable map. Then:
 - (a) $D = f(X)$ is closed in C ,
 - (b) the equivalence relation $E(x, y) \Leftrightarrow f(x) = f(y)$, for $x, y \in X$, is type-definable over M and bounded,
 - (c) f induces a homeomorphism between X/E equipped with the logic topology and D .
- (iii) Let again $f: X \rightarrow C$ be M -definable. Then f factors through the tautological map $a \mapsto \text{tp}(a/M)$ from X to $S_X(M)$, and C is equipped with the quotient topology.

Fact 1.3. Suppose G is a group definable in M and G^* is the interpretation in M^* of the formulas defining G .

- (i) Let N be a normal, type-definable over M , bounded index subgroup of G^* . Then G^*/N equipped with the logic topology is a compact group.
- (ii) There is a smallest, type-definable over M , subgroup of G . It is called $(G^*)_M^{00}$, and is normal in G^* .

We also recall:

Definition 1.4. (i) A type $p(x) \in S(M)$ is *definable* if for any $\varphi(x, y) \in L$, $\{b \in M: \varphi(x, b) \in p\}$ is a definable set in M .

- (ii) If $A \supseteq M$, and $q(x) \in S(A)$, then q is said to be *finitely satisfiable in M* (or equivalently a *coheir of $q|M$*) if every formula in q is satisfied by some element or tuple from M .
- (iii) If $A \supseteq M$ and $q(x) \in S(A)$, then q is said to be an *heir of $q|M$* if for any L_M -formula $\varphi(x, y)$ such that $\varphi(x, a) \in q$ for some $a \in A$, there is $m \in M$ such that $\varphi(x, m) \in q$.

Fact 1.5. (i) $\text{tp}(a/M, b)$ is a coheir of $\text{tp}(a/M)$ if and only if $\text{tp}(b/M, a)$ is an heir of $\text{tp}(b/M)$.

(ii) $p(x) \in S(M)$ is definable if and only if p has a unique heir over any $A \supseteq M$.

The property, for a given theory T , that all complete types over all models are definable (equivalently all complete types over a sufficiently saturated model are definable), is very strong, and equivalent to *stability* of T . On the other hand there are certain important structures M over which all complete types are definable, even though $\text{Th}(M)$ is unstable. Examples are $(\mathbb{R}, +, \cdot)$ and $(\mathbb{Q}_p, +, \cdot)$. There is also a class of first order theories, properly containing the class of stable theories, for which certain “mild” expansions of arbitrary models have the property that all types over them are definable: T is (or has) *NIP*, if (working in a saturated model M^*) for any L -formula $\varphi(x, y)$, indiscernible sequence $(a_i : i < \omega)$ and $b \in M^*$, the truth value of $\varphi(a_i, b)$ stabilizes as $i \rightarrow \infty$. Given a model M of any theory T , by M^{ext} we mean the structure whose universe is M and which has predicates for all *externally definable* subsets of M^n : where $X \subseteq M^n$ is said to be externally definable, if there is a formula $\psi(x)$ possibly with parameters from M^* such that $X = \{a \in M^n : M^* \models \psi(a)\}$. Note that M^{ext} has (essentially) constants for all elements. The following is due to Shelah [17], but see also [12] for a simplified account.

Fact 1.6. Assume that T has *NIP*. Then for any model M of T , $\text{Th}(M^{\text{ext}})$ has *quantifier elimination* and *NIP*. Moreover all complete types (in any number of variables) over M^{ext} are definable.

In fact Shelah suggests, in Thesis 0.5 of [18], that in studying an *NIP* theory T , one should from the start work with the class of externally definable sets. This in some sense provides some justification for our “definability of types” assumption when we introduce “definable topological dynamics” in Section 3.

2 The topological case

We give a brief model-theoretic description of the topological case. We refer the reader to [19] for a nice treatment of topological dynamics.

Let G be a topological group. As mentioned in the introduction G has a unique “universal compactification”, called *Bohr compactification*. Here is a summary of a

model-theoretic account. Let M be a structure whose universe is G , and has a symbol for the group operation, and, for each open subset of G a predicate symbol P_U whose interpretation in M is U (and maybe more). We suppose the language L of M has cardinality at most $2^{|M|}$. Let M^* be a very saturated elementary extension of M and G^* the corresponding group. Let U^* denote the interpretation of P_U in M^* . The following proposition immediately follows from [3, Theorem 5.6].

Proposition 2.1. *Let N be the smallest subgroup of G^* which has bounded index, and is an intersection of some of the U^* 's. Then:*

- (i) *N is normal (hence G^*/N with the logic topology is a compact group),*
- (ii) *The mapping f from G to G^*/N induced by the identity embedding of G in G^* is continuous and is the universal compactification of the topological group G .*

Now we consider topological dynamics. G remains a topological group. By a G -flow (X, G) we mean a continuous action of G on a compact space X (i.e. an action of G on X such that the corresponding function $G \times X \rightarrow X$ is continuous). Note that for X a compact space, the space $\text{Homeo}(X)$ of homeomorphisms of X equipped with the compact-open topology, is a compact group. Moreover a G -flow (X, G) is precisely given by a continuous homomorphism from G to $\text{Homeo}(X)$. Note that if $x \in X$, then the map taking $g \in G$ to $g \cdot x \in X$ is continuous. By a G -ambit (X, x_0, G) we mean a G -flow (X, G) together with a point $x_0 \in X$ such that the orbit $G \cdot x_0$ is dense in X . There is a *universal G -ambit*, sometimes called $S(G)$, with distinguished point e say: for every G -flow (X, G) and $p \in X$ there is a unique map of G -flows from $S(G)$ to X which takes e to p . The universal G -ambit is unique, by definition. The following model-theoretic (nonstandard analytic) account of the universal G -ambit is easily seen to follow from the account of $S(G)$ as the ‘‘Samuel compactification’’ of the uniform space (G, \mathcal{R}) where \mathcal{R} is the canonical right uniformity on G (see [19, Section 2]). The basic entourages of \mathcal{R} are $\{(x, y) \in G^2 : xy^{-1} \in V\}$, where $V \subseteq G$ range over open sets containing the identity 1 of G .

Proposition 2.2. *Let G be a topological group, and M a structure as described at the beginning of this section. Let E be the finest bounded, type-definable over M equivalence relation on G^* which satisfies the following condition:*

whenever $g, h \in G^$ and $gh^{-1} \in U^*$ for every U which is a neighbourhood of the identity, then $E(g, h)$.*

Then E is G -invariant, and G^/E with the logic topology and distinguished point $1/E$ is the universal G -ambit.*

Proof (sketch). $S(G)$ is the completion of G with respect to the finest precompact uniformity \mathcal{U} which is coarser than \mathcal{R} . Such a uniformity corresponds to a type-definable equivalence relation E on G^* . Since \mathcal{U} is precompact, E has boundedly many classes. Also \mathcal{U} is G -invariant, as \mathcal{R} is. The completion of G with respect to \mathcal{U} is exactly G^*/E with the logic topology. \square

We give a few additional remarks. The universal G -ambit $S(G)$ has a canonical semigroup structure (which in the proof of Theorem 2.1 in [19] is seen to follow directly from the universality property of $S(G)$). This semigroup structure is in turn used to prove the uniqueness of the universal minimal G -flow: A G -flow (X, G) is *minimal* if X has no proper G -subflow (i.e. closed G -invariant subset). A universal minimal G -flow is a minimal G -flow (X, G) such that for any (minimal) G -flow (Y, G) there is some map of G -flows $f: X \rightarrow Y$. So uniqueness of the universal minimal G -flow is not built into the definition, but is nevertheless true. Computation of the universal minimal flow $M(G)$ for various topological groups is an important enterprise. We have been told that for no locally compact noncompact topological group G , has $M(G)$ been explicitly described. The topological group G is said to be (extremely) amenable if for every G -flow (X, G) , there is a G -invariant Borel probability measure on X (X has a G -invariant point). Clearly it suffices for the universal G -ambit to have this property.

3 The definable case

The aim in this main section of the paper is to try to give appropriate analogues of compactification and action on a compact space, for definable groups rather than topological groups. Many of our formulations can possibly be improved or modified. We also ask a few questions. We fix a first order structure M , not necessarily saturated. By a definable set in M we mean a subset Y of M^n definable *with parameters from M* . M^* denotes a very saturated elementary extension of M . For Y a definable set in M , we often use Y^* to denote the interpretation in M^* of the formula which defines Y in M .

We first give an analogue of Definition 1.1(ii) in the nonsaturated case:

Definition 3.1. Let Y be a definable set in M and C a compact space. By a *definable map* f from Y to C we mean a map f such that for any disjoint closed subsets C_1, C_2 of C there is a definable subset Y' of Y such that $f^{-1}(C_1) \subseteq Y'$ and $Y' \cap f^{-1}(C_2) = \emptyset$.

Lemma 3.2. *Suppose Y is a definable set in M .*

- (i) *Suppose $f: Y \rightarrow C$ is a definable map from Y to a compact space C . Then f extends uniquely to an M -definable map from Y^* to C (in the sense of Definition 1.1(ii)).*
- (ii) *Conversely, suppose f^* is an M -definable map from Y^* to the compact space C (in the sense of Definition 1.1(ii)). Then the restriction f of f^* to Y is a definable map from Y to C in the sense of Definition 3.1.*

Proof. (i) We first define f^* . Let $c \in Y^*$ and let $p(y) = \text{tp}(c/M)$. For $\varphi(y)$ a formula in p , let $\overline{f(\varphi(M))}$ denote the closure of $f(\varphi(M))$ in C .

Claim. $\bigcap_{\varphi \in p} \overline{f(\varphi(M))}$ is a singleton in C .

Proof of Claim. Suppose for a contradiction that $a \neq b$ are both in $\bigcap_{\varphi \in p} \overline{f(\varphi(M))}$. Let C_1, C_2 be disjoint closed neighbourhoods in C of a, b respectively. So there is $\varphi(y)$ over M such that $f^{-1}(C_1) \subseteq \varphi(M)$, and $\varphi(M) \cap f^{-1}(C_2) = \emptyset$. Without loss of generality $\varphi(y) \in p(y)$. Now $f(\varphi(M))$ is disjoint from C_2 hence (as C_2 contains an open neighbourhood of b), $\overline{f(\varphi(M))}$ does not contain b , a contradiction. \square

So define $f^*(c)$ to be the unique element in $\bigcap_{\varphi \in p} \overline{f(\varphi(M))}$. A similar argument to the claim shows that f^* is definable over M (if $D \subseteq C$ is closed, let $\Sigma(y)$ be the set of formulas $\varphi(y)$ over M such that $f^{-1}(D) \subseteq \varphi(M)$, and show that if c satisfies $\Sigma(y)$ in Y^* then $f^*(c) \in D$). Uniqueness of f^* is also clear.

(ii) Let C_1, C_2 be disjoint closed subsets of C . Let $X_i = (f^*)^{-1}(C_i)$ for $i = 1, 2$. As X_1 and X_2 are disjoint, type-definable over M subsets of Y^* , by compactness (or saturation of M^*) they are separated by an M -definable set Z . So $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are separated by $Z(M)$, a definable set in M . \square

We start with the rather easy case of group compactifications.

Definition 3.3. Let G be a group definable in M . By a *definable compactification* of G we mean a definable homomorphism from G to a compact group C with dense image.

Proposition 3.4. *Let G be a group definable in M . Then there is a (unique) universal definable compactification of G , and it is precisely $G^*/(G^*)_M^{00}$ (where the homomorphism from G to $G^*/(G^*)_M^{00}$ is that induced by the identity embedding of G in G^*).*

Proof. Let $f: G \rightarrow C$ be a definable compactification of G . By Lemma 3.2(i), f lifts uniquely to an M -definable map f^* from G^* onto C . We claim that f^* is a group homomorphism. The proof of Lemma 3.2(i) shows that for any $a \in G^*$, $f^*(a) = \bigcap_{\varphi \in p} \overline{f(\varphi(M))}$, where $p = \text{tp}(a/M)$. Let $b \in G^*$ and $q = \text{tp}(b/M)$. Then

$$\bigcap_{\varphi \in p, \psi \in q} \overline{f(\varphi(M) \cdot \psi(M))} = \bigcap_{\varphi \in p, \psi \in q} \overline{f(\varphi(M)) \cdot f(\psi(M))} = \bigcap_{\varphi \in p} \overline{f(\varphi(M))} \cdot \bigcap_{\psi \in q} \overline{f(\psi(M))}$$

$= f^*(a) \cdot f^*(b)$. This is enough to show that f^* is a homomorphism. So the kernel of f^* is a normal type-definable (over M) subgroup of G^* of bounded index. Also by 1.2, the topology on C coincides with the logic topology (on G^*/N). So if $(G^*)_M^{00}$ is the smallest type-definable over M subgroup of G^* of bounded index, then we have a canonical surjection map $\pi: G^*/(G^*)_M^{00} \rightarrow C = G^*/N$, which is clearly a continuous homomorphism and commutes with the canonical homomorphisms from G . \square

Now for definable group actions. Let again G be a group definable in a structure M . Our aim is to give an account of “definable actions of G on compact spaces X ” which is analogous to the topological case and is a generalization of the case where G is a discrete topological group. Now we have the action of G on the type space $S_G(M)$, and the orbit of 1 is dense. If all subsets of G are definable, then $S_G(M)$ is precisely

βG , the space of ultrafilters on the Boolean algebra of subsets of G . Moreover it is well-known that $(\beta G, 1)$ coincides with the universal G -ambit when G is considered as a discrete topological group. So back in the situation of G being a definable group in an arbitrary structure M , we would like a notion of “definable action of G on a compact space X ” such that (i) if Y is a closed G -invariant subset of X , then the induced action of G on Y is definable, and (ii) $(S_G(M), 1)$ is the universal definable G -ambit. We note now that these demands already place some constraints on G and M : Let $p \in S_G(M)$ and let $Y \subseteq X$ be the closure of the orbit $G \cdot p$. By (i) and (ii), there should be a continuous (G -invariant) map $f : S_G(M) \rightarrow Y$, taking $g \in G$ to gp . By 3.2 (essentially) the restriction of f to G should be definable, in the sense of Definition 3.1. But then

(*) for any formula $\phi(x)$ over M (defining a subset of G), $\{g \in G : g \cdot \phi(x) \in p\}$ should be a definable subset of G .

This would be guaranteed if p were a definable type. So below we make the assumption that all types in $S_G(M)$ are definable, and with suitable definitions develop the desired theory. But the reader can check that in fact (*), for any $p \in S_G(M)$, is enough to make the theory work.

As remarked in the introduction important unstable structures such as the field \mathbb{R} of real numbers, or the field \mathbb{Q}_p of p -adic numbers, do have the feature that all types over them are definable. So our theory will apply to real and p -adic semialgebraic Lie groups. Given an *NIP* theory T , an arbitrary model M of T and group G definable in M , it would be natural to apply the theory we develop to G as a group definable in the expansion M^{ext} . In fact the third author already did this successfully for *fs* groups in [13]

ASSUMPTION. M is a first order structure, G is a group definable in M , and all types in $S_G(M)$ are definable.

So note we do not assume that ALL complete types over M are definable, just complete types extending the formula $x \in G$.

We now give a somewhat strong definition of a *definable action* of G on a compact space X , suitable for our purposes, under the ASSUMPTION above.

Definition 3.5. Let X be a compact space.

- (i) By a definable action of G on X (or definable G -flow) we mean that G acts on X by homeomorphisms, and for each $x \in X$, the map $f_x : G \rightarrow X$ taking g to $g \cdot x$ is definable.
- (ii) A definable G -ambit is a definable G -flow (X, G) with a distinguished point $x_0 \in X$ such that the orbit $G \cdot x_0$ is dense in X .

Remark 3.6. (i) The second clause in Definition 3.5(i) is equivalent to saying that the induced map from G to the compact space X^X is definable.

- (ii) If G acts definably on the compact space X and Y is a closed G -invariant subset of X , then clearly the action of G on Y is also definable.
- (ii) Itai Ben Yaacov suggested to the third author that the topologically correct definition in (i) is that in addition to G acting by homeomorphisms, the map from G to the space $\text{Homeo}(X)$ of homeomorphisms of X , where $\text{Homeo}(X)$ is equipped with the compact-open topology, is definable when considered as a map from G into the Roelcke compactification of $\text{Homeo}(X)$. Under our definability of types assumption, definable G -ambits will have this property.

Lemma 3.7. (i) *The action of G on $S_G(M)$ is definable.*

- (ii) *Moreover for any bounded type-definable over M , G -invariant equivalence relation E on G^* , the action of G on G^*/E (equipped with the logic topology) is definable.*

Proof. (i) Clearly G acts on $S_G(M)$ by homeomorphisms. As $S_G(M)$ is totally disconnected, the second clause in Definition 3.5(i), says that for any L_M -formula $\varphi(x)$, and $p(x) \in S_G(M)$, $\{g \in G : \varphi(x) \in gp\}$ is a definable subset of G . And this follows by definability of $p(x)$.

(ii) This follows from (i): Let $X = G^*/E$ and π the (continuous) G -invariant surjection from $S_G(M)$ to X . Suppose $a \in X$ and C_1, C_2 are disjoint closed subsets of X . Let $D_i = \pi^{-1}(C_i)$ for $i = 1, 2$. So D_1, D_2 are disjoint closed subsets of $S_G(M)$, and by compactness (E being type-definable over M), there are M -definable sets D'_1, D'_2 , containing D_1, D_2 respectively, such that $\pi(D'_1)$ and $\pi(D'_2)$ are disjoint. Let $p(x) \in \pi^{-1}(a)$. So as in (i) $\{g \in G : gp \in D'_1\}$ is a definable subset Y of G . We see that Y separates $\{g \in G : g \cdot a \in C_1\}$ and $\{g \in G : g \cdot a \in C_2\}$, as required. \square

Proposition 3.8. *There is a (unique) universal definable G -ambit, which is precisely the type space $S_G(M)$, under the natural action of G , where the distinguished element of $S_G(M)$ is the identity element 1 of G . Namely for any other definable G ambit (X, G, x_0) there is a unique continuous (necessarily surjective) map $h: S_G(M) \rightarrow X$ of definable G flows with $f(1) = x_0$.*

Proof. Let (X, G, x_0) be a definable G -ambit. Let $f: G \rightarrow X$ be $f(g) = g \cdot x_0$. By Lemma 3.2(i), f extends uniquely to an M -definable map f^* from G^* to X , and thus a continuous map $h: S_G(M) \rightarrow X$. Note that h is surjective as its image is compact and contains $G \cdot x_0$. Also note that $h(1) = x_0$. It remains to see that h is a map of G -spaces. This follows from the construction of f^* in the proof of 3.2: for $p \in S_G(M)$, and $g \in G$, $gh(p) = g \overline{\bigcap_{\varphi \in p} f(\varphi(M))} = \overline{\bigcap_{\varphi \in p} gf(\varphi(M))} = \overline{\bigcap_{\varphi \in p} f(g\varphi(M))} = \overline{\bigcap_{\psi \in gp} f(\psi(M))} = h(gp)$ (using the fact that $f(\varphi(M)) = \varphi(M) \cdot x_0$). \square

We now discuss the semigroup structure on $S_G(M)$

Definition 3.9. Let $p, q \in S_G(M)$. Then by $p*q$ we mean $\text{tp}(ab/M)$ where b realizes q and a realizes the unique coheir of p over M, b . (Equivalently a realizes p and b realizes the unique heir of q over M, a).

So note that if $p = g$ is in G (namely a realized type), then $p * q$ is just gq , so the operation $*$: $S_G(M) \times S_G(M) \rightarrow S_G(M)$, extends the group action $G \times X \rightarrow X$.

Lemma 3.10. (i) $*$ is a semigroup operation on $S_G(M)$, continuous on the left, namely for given $q \in S_G(M)$ the map taking p to $p * q$ is a continuous map from $S_G(M)$ to itself.

(ii) Given $q \in S_G(M)$, let r_q be the unique continuous function from $S_G(M)$ to itself which takes 1 to q (given by Proposition 3.8). Then for any $p \in S_G(M)$, $r_q(p)$ is precisely $p * q$.

Proof. Left to the reader. □

Definition 3.11. (i) By a *minimal definable G -flow* we mean a definable G -flow (X, G) with no proper closed G -invariant subset (i.e. with no proper G -subflow).

(ii) By a *universal minimal definable G -flow*, we mean a minimal definable G -flow (I, G) such that for any minimal definable G -flow (X, G) there is a G -map from I to X . (So in fact for every definable G -flow X , minimal or not, there is a G -map from I to X).

Note first that any minimal G -subflow I of $S_G(M)$ will be a universal minimal definable G -flow. (Because for any definable G -flow (X, G) there is by Proposition 3.8 a G -map from $S_G(M)$ to X whose restriction to I is a G map from I to X .) On the other hand, there is no a priori reason for there to be a unique universal minimal definable G -flow (in the obvious sense). But in fact uniqueness of the the universal minimal definable G -flow *is* true. This makes use of the semigroup structure on $S_G(M)$, and is well-known. But for completeness we go briefly through the steps, following Section 3 of [19]. Let I be a minimal subflow of $S_G(M)$:

Claim. (i) Left ideals (with respect to $*$) of $S_G(M)$ coincide with closed G -subflows.

(ii) I contains an idempotent p_0 and $q * p_0 = q$ for all $q \in I$.

(iii) Every G -map $f : I \rightarrow I$ has the form $p \rightarrow p * q$ for some $q \in I$.

Proof. Consider $h = f \circ r_{p_0} : S_G(M) \rightarrow I$. Then $h = r_t$ where $h(1) = t \in I$. As $r_{p_0}|I$ is the identity (by (ii)), f and $h = r_t$ have the same restriction to I . □

Claim. (iv) Every G -map $f : I \rightarrow I$ is a bijection.

Proof. By (iii) $f = r_t|I$ for some $t \in I$. Now $I * t$ being an ideal contained in I coincides with I (by minimality of I) hence $s * t = p_0$ for some $s \in I$. Let $g : I \rightarrow I$ be the map $r_s|I$. Then $f \circ g = id$. As f was arbitrary all G -maps from I to I are surjective. In particular g is hence f is a bijection. □

We conclude from (iv):

Proposition 3.12. *There is a unique (up to isomorphism of G -spaces) universal minimal definable G -flow, and it coincides with some (any) minimal G -subflow of $S_G(M)$.*

Proof. Let I be a given minimal subflow of $S_G(M)$, and J any other universal minimal G -flow. There are G -maps $f : I \rightarrow J$ and $g : J \rightarrow I$, both necessarily surjective. The composition $g \circ f : I \rightarrow I$ is bijective by (iv) above, whereby f is injective, hence a homeomorphism. \square

So part of the content of [4] was to describe the universal minimal definable G -flow when $G = SL(2, \mathbb{R})$ and $M = (\mathbb{R}, +, \cdot)$.

Finally we discuss amenability. Our definability of types assumption remains in place.

Definition 3.13. (i) G is *definably amenable* if for every definable G -flow (X, G) there is a G -invariant Borel probability measure on X .

(ii) G is *definably extremely amenable* if for every definable G -flow (X, G) , X has a fixed point.

Remark 3.14. By Proposition 3.8, definable (extreme) amenability of G is equivalent to $S_G(M)$ having a G -invariant Borel probability measure (fixed point)

Question 3.15. Suppose G is definably amenable. Is there then a *definable* G -invariant Borel probability measure on $S_G(M)$? Where definability of μ means: for any L -formula $\varphi(x, y)$, and closed disjoint subsets C_1, C_2 of $[0, 1]$ $\{b \in M : \mu(\varphi(x, b)) \in C_1\}$, and $\{b \in M : \mu(\varphi(x, b)) \in C_2\}$ are separated by a definable set in M .

In the papers [6], [7], the expression “definable amenability” of a definable group was used, without any definability of types assumption. It simply meant the existence of a G -invariant Keisler measure on G , equivalently a G -invariant Borel probability measure on $S_G(M)$. So we ask:

Question 3.16. Assume T has *NIP*, $M \models T$ and G is a group definable in M (no definability of types assumption).

(i) Suppose there is a G -invariant Borel probability measure on $S_G(M)$. Is there also one on $S_G(M^{ext})$?

(ii) Likewise, if there is a fixed point in $S_G(M)$ is there also one in $S_G(M^{ext})$?

Question 3.16(i) has a positive answer when T is *o*-minimal, essentially by the characterization in [2] as well as [13]. We are not sure about (ii), even in the *o*-minimal case.

Finally we return to our definability of types assumption on $S_G(M)$, and point out that Pestov’s characterization ([10], Theorem 8.1) of extreme amenability passes suitably to the definable category. We say that a definable subset Y of G is *left generic* if finitely many left translates of Y cover G .

Proposition 3.17. *G is (definably) extremely amenable if and only if for each definable left generic subset Y of G , $YY^{-1} = G$*

Proof. Suppose first that there is a G -invariant type $p \in S_G(M)$. We will prove the right hand side without assuming any definability of types assumption. Let X be a left generic definable subset of G . So for some $g \in G$, $gX \in p$. By G -invariance of p , $X \in p$. As p is G -invariant, $XX^{-1} = G$.

Now for the converse. Given a definable subset Y of G we feel free to identify Y with the clopen subset of $S_G(M)$ (and of any closed subset of $S_G(M)$) it defines. So for $p \in S_G(M)$ we may write $p \in Y$, although this corresponds to the formula defining Y being in the type p . Assume the right hand side. Let \mathcal{M} be a minimal closed G -invariant subset of $S_G(M)$. We will prove that \mathcal{M} is a singleton, by showing that for any $p \in \mathcal{M}$, and $g \in G$, $gp = p$. Suppose for a contradiction that for some $g \in G$ and $p \in S_G(M)$, $gp \neq p$. Then there is a formula (i.e. definable subset of G) $Y \in p$ such that $Y \cap g(Y) = \emptyset$. Note that the union U of all gY must cover \mathcal{M} , Because if not then $\mathcal{M} \setminus U$ is closed G -invariant and nonempty, contradicting minimality of \mathcal{M} . By compactness of \mathcal{M} , it is covered by $g_1Y \cup \dots \cup g_kY$ for some $g_1, \dots, g_k \in G$. Let $Y_1 = \{h \in G : Y \in h(p)\}$. By definability of p , Y_1 is a definable subset of G .

Claim (I). $g_1Y_1 \cup \dots \cup g_kY_1 = G$ (so Y_1 is left generic in G).

Proof of Claim (I). Let $h \in G$. So $hp \in \mathcal{M}$ so $hp \in g_iY$ for some $i = 1, \dots, k$. Hence $g_i^{-1}h \in Y_1$, and so $h \in g_iY_1$. \square

Claim (II). $Y_1 \cap gY_1 = \emptyset$ (so $Y_1Y_1^{-1} \neq G$).

Proof of Claim (II). If not, let $h \in Y_1$ such that $gh \in Y_1$. So $hp \in Y$ and $ghp \in Y$ hence $hp \in Y \cap g^{-1}Y$ implying that $gY \cap Y \neq \emptyset$, contradicting choice of Y . \square

Claims (I) and (II) contradict the right hand side, completing the proof. \square

Remark 3.18. (i) The proof above (which was adapted from Pestov's proof of Theorem 7.1 in [10]) generalizes to show that the kernel of the action of G on its minimal subflow is precisely the intersection of the sets YY^{-1} for $Y \subseteq G$ definable and left generic.

(ii) If $Th(M)$ is stable, then G is definably extremely amenable iff G is connected, in which case there is a unique invariant type.

(iii) One might try to generalize Proposition 3.17 to the general situation with no definability of types assumption. As an example the following holds: Every minimal subflow of $S_G(M)$ is a singleton if and only if whenever $Y \subseteq G$ is definable and (as a clopen subset of $S_G(M)$) meets some minimal subflow of $S_G(M)$ then $YY^{-1} = G$.

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