Gödel lecture: First order theories

Anand Pillay

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Introduction I

A theme of this lecture is to give some justification for the thesis that stable theories are the logically perfect (first order) theories, or maybe rather the fundamental "tame" (first order) theories:

- This is a variant on Zilber's "thesis" that categorical theories are the logically perfect theories (but why one, why not three?).
- It may seem like just a marketing or self-serving enterprise, but ten years ago I would have said something different:
- for example that "tame" model theory is "multicultural" or "multipolar" (stable, simple, o-minimal, c-minimal,...).
- Thirty years ago things looked even more different; stable theories were considered an exception or singularity, with little or no bearing on other examples around "tame" model theory such as Henselian valued fields.

Introduction II

- Let us distinguish at the start "foundational" theories (ZFC, PA, second order arithmetic, ...) which purport to encode all of mathematics or large parts of mathematics, from
- "tame" theories which encode smaller chunks of (interesting) mathematics and tend to be decidable.
- We will be mainly concerned with tame theories, but will at the end consider also *pseudofinite theories* which can be foundational.
- I would like to define or describe model theory as *the study of first order theories*, although this is not only controversial but empirically wrong, as a lot of what goes under the name of model theory is outside the first order context.
- But let's see where it goes.

First order? I

- By a first order theory I mean, to begin with, a consistent collection T of $L_{\omega,\omega}$ sentences (where L is some finitary language).
- Traditionally there is a basic distinction between syntax and semantics, with mathematical structures entering the picture at the semantic level (as models).
- However a first order theory T is already a mathematical object. For example it can be identified with the category $Def(T)$ (definable sets) whose objects are formulas $\phi(x_1,..,x_n)$ up to equivalence mod T and with morphisms "definable" (mod T) functions.
- Replacing a formula ϕ by the space $S_\phi(T)$ of complete types containing ϕ, we can view T as a category $S(T)$ of (totally disconnected) compact spaces with certain open continuous maps as morphisms.

First order? II

- So compactness is not just a property that first order logic happens to possess but is at the centre of the notion of a first order theory.
- We could generalize or weaken this picture by for example allowing arbitrary compact spaces as objects of $S(T)$, and a coherent account of such a generalization goes under the name of *continuous* logic or model theory (where formulas are real-valued) and should also be considered as part of first order logic.
- Returning to the "standard" context, there are various other categories, invariants, etc., associated to a first order theory T, one being its category $Mod(T)$ of models (with elementary embeddings as morphisms). The attempt to recover T (i.e. $Def(T)$) from $Mod(T)$ has been an influential enterprise, both for categorical logic and model theory.

Applications

- Although my main aim here is to discuss logic, I should briefly mention applications. I have often heard it said that model theory is naturally concerned with applications so is closer to current mathematics. But it is a result of the choices that we make on what to think about/work on.
- Model theory could have easily become an appendage of set theory and/or recursion theory, and this what was expected in the 1970's.
- In any case, for certain specific theories T, $Def(T)$ is a familiar category in mathematics (e.g. $T = ACF_0$, with $Def(T)$ being essentially the category of complex algebraic varieties defined over $\mathbb{Q}$), and the tools/ideas developed by model-theorists (including things discussed subsequently) turn out to be meaningful in such concrete contexts.

Stable theories

- We now restrict ourselves to consideration of a complete theory T in some language L, countable if you wish.
- A formula $\phi(x, y) \in L$ (where x, y are finite tuples of variables) is *stable* (for T) if there does not exist a model M of T and tuples a_i, b_i in M for $i < \omega$, such that $M \models \phi(a_i, b_j)$ if and only if $i < j$.
- T is stable if every formula $\phi(x, y) \in L$ is stable for T.
- So stability of T means the non-interpretability in any model of T of certain bipartite graphs.
- Examples are the theory of an infinite set in the empty language, the theory of algebraically closed fields of some fixed characteristic, the theory of differentially closed fields of characteristic 0, the theory of any abelian group (in the language of groups), as well as any theory T with $< 2^{\kappa}$ models of cardinality κ for some uncountable κ.

Canonical ultrafilter I

- I will describe a (or *the*) characteristic feature of stable theories, suppressing a few (important) details.
- Fix a κ-saturated and homogeneous model M of a stable theory T ($\kappa > 2^{\aleph_0}$ say), and let $X \subseteq M^n$ be a definable set, defined over a countable set $A \subset M$ of parameters.
- Then the main point is (with a few provisos), the existence of a *canonical* ultrafilter q_X on the Boolean algebra of definable (in M with parameters from M) subsets of X.
- We first define the *ideal* I_X which should correspond to q_X: $Y \in I_X$ if for some indiscernible sequence $(Y_i : i < \omega)$ of $Aut(M/A)$-images of Y, $\cap_i Y_i = \emptyset$. (Y divides/forks over A or Y is "A-small")
- For I_X to be the ideal of an ultrafilter we need for example, that for any Y at least one of Y, $X \setminus Y$ is in I_X.

Canonical ultrafilter II

- As any $Y \subseteq X$ which is defined over A is not in I_X, we should at least work rather with $X = p^M$ the set of solutions of a *complete type* $p(x)$ over A.
- On the other hand, if $X = p^M$ is *finite* and of cardinality > 1, then for $a \neq b \in X$, the ultrafilters q_1, q_2 concentrating on a, b respectively will both have ideals containing I_X.
- Hence nontriviality of the profinite group $Aut(acl(A)/A)$ (of cardinality $\leq 2^{\aleph_0}$) is a potential obstruction to I_X being the ideal of an ultrafilter.
- In fact this is the *only* obstruction:
- Theorem 1. if A is "algebraically closed" and $p(x)$ is a complete type over A, then there is a *unique complete type* (ultrafilter) $q(x)$ over M which extends p and contains no A-small formula (definable set).

Stable groups I

- When $X = G$ is a *definable group* defined over A (rather than the set of realizations of a complete type over A), there is a somewhat simpler picture taking account of the group structure/action.
- The ideal $I_{G,ng}$ of small or nongeneric definable subsets of G is defined by $Y \in I_{G,ng}$ if some G-translate of Y is in I_G (is A-small).
- If G has a proper definable subgroup of finite index then each of its translates is generic. Hence letting G^0 be the intersection of all definable subgroups of G of finite index, nontriviality of the profinite group G/G^0 is an obstruction to $I_{G,ng}$ being the ideal of an ultrafilter.
- Again this is the ONLY obstruction: If $G = G^0$ (G is "connected") then G has a unique "generic" type, i.e. unique ultrafilter on definable subsets of G avoiding the nongeneric definable sets.

Stable groups II

- For a possibly nonconnected definable group G, the "generic types" of G are in $1-1$ correspondence with elements of G/G^0 (cosets of G^0 in G) and in fact
  Theorem 2. G^0 can be recovered as the *stabilizer* of some (any) generic type, in the obvious sense.
- For $G = G^0$, the generic type is in fact the unique type (ultrafilter on definable subsets of G) which is left (right) G-invariant.
- Note the formal analogy with uniqueness of Haar measure on compact groups (see later).

Stable theories: conclusion, example

- We have described above the key ingredients of both classical and geometric stability theory (behind counting models as well as applications). We have also seen the appearance of other pervasive objects/notions/invariants in the study of first order theories: Galois groups, and connected components of definable groups.
- Theorem 1 sometimes goes under the name of *uniqueness of free 2-amalgamation over algebraically closed sets*. Explain!
- Example for those familiar with naive algebraic geometry: for $X \subseteq \mathbb{C}^n$ an (absolutely) irreducible complex algebraic variety, the "canonical ultrafilter" material above (Theorem 1) gives rise to the "generic point" of X viewed as a scheme.

Simple theories I

- I want now to consider unstable theories and to some extent "tame" ones which include main the main examples of applications, and try to describe the role of stability.
- I will start with "simple" theories (which include stable theories). I will not give the combinatorial definition, but roughly speaking these are theories of the form "stable theory enriched by some random relations", and include the theory of the random graph, completions of the (common) theory of finite fields, and the theory of algebraically closed fields equipped with a "random" automorphism, all of which are unstable.
- Theorem 1 in the stable case (uniqueness of free 2-amalgamation) is replaced by a free 3-amalgamation theorem which I will not spell out. (But explain!)

Simple theories II

- A key technical observation is that given a base set A and formulas $\phi(x, y)$, $\psi(x, z)$, the relation $R(y, z)$ which by definition holds of (b, c) if $\phi(x, b) \wedge \psi(x, c)$ is A-small (divides over A) is an $Aut(M/A)$-invariant *stable relation* in a suitable sense, and a "local" version of the uniqueness theorem Theorem 1 plays a crucial role in 3-amalgamation.
- The only obstruction to 3-amalgamation is a *compact, not necessarily profinite* Galois group $Aut(bdd(A)/A)$. (Explain!)
- If G is a definable group (over A), and p is a "generic" type of G i.e. avoids the ideal $I_{G,ng}$ described earlier, then let $S(p) = \{g \in G : p \cup g \cdot p$ avoids $I_{G,ng}\}$. Then using 3-amalgamation, one has the following generalization of Theorem 2.
- Theorem 3. $S(p) \cdot S(p) =$ "$Stab(p)$" is the smallest type-definable subgroup of G of "small" index, defined over A, which we call G^{00}. G/G^{00} is a compact group.

NIP theories I

- T is (or has) NIP if it is NOT the case that there is a model $M \models T$, formula $\phi(x,y) \in L$ and $\{a_i : i < \omega\}$ from M, and $\{b_s : s \subseteq \omega\}$ from M such that $M \models \phi(a_i, b_s)$ iff $i \in s$ (for all i, s).
- Stable theories have NIP and key unstable examples include RCF (and more generally o-minimal theories), Presburger, theory of the p-adic field, theory of algebraically closed nontrivially valued fields (of a given characteristic).
- The main point I want to make is that stability is present (in an NIP theory) at the level of *measures* rather than types, namely in a probabilistic fashion.
- Fix a tuple x of variables and A a set of parameters. A (Keisler) measure μ over A is a finitely additive probability measure on formulas $\phi(x)$ with parameters from A (A-definable sets).

NIP theories II

- I will not give the definition of when μ is a stable measure, but a key property (analogous to Theorem 1) is that μ has a unique extension to a measure μ' on formulas with parameters from M (ambient saturated model) such that the A-small formulas (those which divide over A) get μ' measure 0.
- There is also an appropriate notion of a "measure-stable" definable group (G, μ') and a key feature is that μ' is the unique left (right) G-invariant measure on definable subsets of G. This is the common generalization of uniqueness of Haar measure on compact groups and uniqueness of invariant types in connected stable groups, which I still find fascinating.
- In a sense which has not yet been worked out properly, stable measures control the structure in NIP theories (and likewise measure-stable groups control the structure of definable groups).

NIP theories III

- But the ubiquity of stable measures in NIP theories is exemplified in the following.
- If $I = (a_i : i \in [0,1])$ is an indiscernible sequence (indexed by the real unit interval), and we fix a parameter set A containing the a_i, then let the measure μ_I on formulas $\phi(x)$ over A be defined by: $\mu_I(\phi(x)) =$ Lebesgue measure of $\{i \in [0,1] : M \models \phi(a_i)\}$. Then μ_I is stable.
- Let $T = RCF$ and let $A = \mathbb{R}$ the standard model. Let $X \subseteq \mathbb{R}^n$ be a semialgebraic set and μ a Borel probability measure on (the topological space) X. Then the induced measure (on $\mathbb{R}$ definable sets) is stable.

Pseudofinite theories and definable sets I

- The L-theory T is *pseudofinite* if $T = Th(M)$ where M is an ultraproduct of finite L-structures. Likewise we can speak of an L-formula $\phi(x)$ being pseudofinite for a theory T if $T = Th(M)$ where M is an ultraproduct of structures M_i and the interpretation of $\phi(x)$ in each M_i is a finite set.
- Pseudofinite theories can be foundational: for example let M be a nonstandard model of true arithmetic, $n \in M$ nonstandard, and let T be the theory of $[0, n]$ with induced structure from M.
- (T arbitrary.) If M is a saturated model of T and $X = \phi^M$ a ($\emptyset$)-definable pseudofinite set, the counting measures on the finite approximations to X give rise to a measure μ on the Boolean algebra of definable subsets of X, which, via some expansion of the language, can be assumed to be $Aut(M)$-invariant.

Pseudofinite theories and definable sets II

- One can conclude (just by virtue of μ being an invariant measure) that if $\psi(x,y)$, $\chi(x,y)$ are L-formulas implying $\phi(x)$, then the relation $R(y,z)$ which is defined to hold of (b,c) if $\mu(\psi(x,b) \wedge \chi(x,c)) = 0$, is an $Aut(M)$-invariant stable relation.
- This allows the results in the case of simple theories (3-amalgamation, Theorem 3) to go through in suitable forms. In particular:
- Let G be a $\emptyset$-definable group in M and X an (infinite) pseudofinite $\emptyset$-definable subset of G. Call X an *approximate subgroup* of G if $|X \cdot X^{-1} \cdot X| \leq k|X|$ for some finite k (makes sense). Let $\tilde{G}$ be the subgroup of G generated by X (which is in general $\bigvee$-definable rather than definable).

Pseudofinite theories and definable sets III

- Under these assumptions, Hrushovski proves, following Theorem 3, and using $Stab(p)$ for suitable p (concentrating on X), that $\tilde{G}$ has a normal type-definable subgroup $\tilde{G}^{00}$ of bounded (at most continuum) index (and $\tilde{G}/\tilde{G}^{00}$ is locally compact under the "logic topology").
- I just want to finish by saying that this result, together with the structure of locally compact groups, introduces new methods and ideas into the asymptotic study of *finite* approximate subgroups of arbitrary groups ("additive combinatorics") and has, I understand, led to solutions of outstanding problems in the subject (Breuillard, Green, Tao, 2011, building on Hrushovski, 2009).

References

- Big influences on the material discussed here are Shelah (who invented stability), Zilber (who initiated the equivariant theory, albeit in a finite Morley rank context), and Hrushovski.
- In addition, Byunghan Kim and Pierre Simon were involved in some of the joint work referred to above around simple theories and NIP theories respectively.
- More detailed references will be given in a later posting or write-up.
- Thanks.