

Topological dynamics and definable groups

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January 11, 2012

Abstract

We give a commentary on Newelski's suggestion or conjecture [8] that topological dynamics, in the sense of Ellis [3], applied to the action of a definable group $G(M)$ on its type space $S_G(M)$, can explain, account for, or give rise to, the quotient G/G^{00} , at least for suitable groups in *NIP* theories. We give a positive answer for measure-stable (or *fs*) groups in *NIP* theories. As part of our analysis we show the existence of "externally definable" generics of $G(M)$ for measure-stable groups. We also point out that for G definably amenable (in a *NIP* theory) G/G^{00} can be recovered, via the Ellis theory, from a natural Ellis semigroup structure on the space of global f -generic types.

1 Introduction and preliminaries

This paper concerns the relationship between two "theories" or "bits of mathematics". On the one hand that of a group G and its actions, by homeomorphisms, on compact spaces, i.e. abstract topological dynamics. On the other hand, that of the existence and properties of a certain canonical quotient G/G^{00} for G a group definable in a saturated model of a (suitable) first order theory T .

This relationship has been explored in a series of papers by Newelski, including [8] and [9] which are most relevant to the considerations of this

*Supported by EPSRC grant EP/I002294/1

paper. For *stable groups*, namely groups definable in stable theories, there is a good match, which we will briefly recall below, and the issue is whether this extends to more general contexts.

A subtext of this paper as well as of Newelski’s work is whether there exists a reasonably robust theory of *definable topological dynamics*, namely of actions of *definable* groups on compact spaces. For example in the same way as amenability of a (discrete) group G is equivalent to the existence of a G -invariant Borel probability measure on the compact space βG (under the natural action of G), *definable* amenability of a group G (definable in some theory T), as defined in [4] for example, is equivalent to the existence of a $G(M)$ -invariant Borel probability measure on the type space $S_G(M)$, for M some (any) model of T . It might then be natural to call a definable group *definably extremely amenable* if for a saturated model M of T the action of $G(M)$ on $S_G(M)$ has a fixed point. And it would be also natural to ask (by analogy) whether this is equivalent to the action of $G(M)$ on $S_G(M)$ having a fixed point, for *some* model M of T . When T is stable this is indeed the case, and is equivalent to G being *connected*. On the other hand, the Ellis theory suggests that it might be better to consider the space $S_{G,ext}(M)$ of external types (with the natural action of $G(M)$) as being the *definable* analogue of βG . Exploration of these issues will be left to subsequent work.

Let us now give a brief description of the problem as posed by Newelski and of our main results, where definitions will be given later. To begin with let T be a complete first order theory with *NIP* say, and G a \emptyset -definable group. Let M be *any* model of T , not necessarily saturated, and $X = S_G(M)$ the Stone space of complete types over M concentrating on G . So $G(M)$ acts on X on the left say, by homeomorphisms. Let $(E(X), \cdot)$ be the enveloping Ellis semigroup of X , I a minimal left ideal of $E(X)$ and $u \in I$ an idempotent. The Ellis theory yields that $u \cdot I$ is a group and the question is whether this group coincides with G/G^{00} . We will give a positive answer (Theorem 3.8) when the group G is *measure-stable*. In all cases $E(X)$ coincides with $S_{G,ext}(M)$, the Stone space of the Boolean algebra of “externally definable” subsets of $G(M)$, and in the measure-stable case, we will also show the existence of “generics” of $S_{G,ext}(M)$ and in fact point out a one-one correspondence between these external generics over M and global generic types of G (Theorem 3.4). In the special case where G is a definably compact group in an *o*-minimal theory, these results were obtained by Newelski [9].

We also discuss briefly in section 2 a natural Ellis semigroup structure on the space of global f -generic types for G a definably amenable group in an NIP theory and point out that G/G^{00} coincides with $u \cdot I$ (I a closed left ideal and $u \in I$ an idempotent). See Proposition 2.5.

In the rest of this introduction we describe key aspects of the model-theoretic and topological dynamics contexts, as well as their interaction. We will be repeating some observations from [8], [9], but hopefully this will help to popularize the nice ideas.

T will denote a complete first order theory in a language L which for simplicity will be assumed to be countable. $x, y, ..$ will usually range over finite tuples of variables. \bar{M} will usually denote a saturated model of T (say κ -saturated of cardinality κ where κ is inaccessible). G will usually denote a \emptyset -definable group, often identified with its points $G(\bar{M})$ in \bar{M} . However sometimes we pass to a larger saturated model \bar{M}' in which types over \bar{M} can be realized. In general “definability” means with parameters unless stated otherwise. For a model M , $S_G(M)$ denotes the set (space) of complete types $p(x)$ over M which contain the formula $\phi(x)$ say which defines G . Identifying $G(M)$ with the collection of “realized types” in $S_G(M)$, we see that $G(M)$ is a dense subset of $S_G(M)$.

As usual we often identify a formula with the set it defines in \bar{M} .

We first recall G^{00} . Let A be a “small” set of parameters from \bar{M} . Then there is a smallest type-definable over A subgroup of G which has index $< \kappa$ (equivalently index at most $2^{|A|+\omega}$). We call this group G_A^{00} . The quotient map $G \rightarrow G/G_A^{00}$ factors through the type space $S_G(M)$ for some (any) small model M containing A and equips G/G_A^{00} with the structure of a compact (Hausdorff) topological group. When T has NIP (see below), G_A^{00} does not depend on A so coincides with G_\emptyset^{00} and we simply call it G^{00} . So the compact group G/G^{00} is a basic invariant of the definable group G .

Fix a model M . By an *externally definable* subset of $G(M)$ we mean a set of form $X \cap G(M)$ where X is a definable subset of G (defined with parameters possibly outside M). The collection of externally definable subsets of $G(M)$ is a Boolean algebra and we denote its Stone space by $S_{G,ext}(M)$, the space of “external types” over M concentrating on G . Let $S_{G,M}(\bar{M})$ denote the (closed) subset of $S_G(\bar{M})$ consisting of types which are finitely satisfiable in

M .

Fact 1.1. (i) Let $p(x) \in S_{G,ext}(M)$. Then the collection of definable subsets X of G such that $X \cap G(M) \in p(x)$ is a complete type in $S_{G,M}(\bar{M})$ which we call $p^{\bar{M}}$.

(ii) The map taking p to $p^{\bar{M}}$ establishes a homeomorphism between $S_{G,ext}(M)$ and $S_{G,M}(\bar{M})$.

By a *Keisler measure* μ on G over M we mean a finitely additive probability measure on the collection of subsets of G defined over M , or equivalently on the collection of definable subsets of $G(M)$. When $M = \bar{M}$ we speak of a global Keisler measure. G is said to be *definably amenable* if it has a global left-invariant Keisler measure. From section 5 of [4] G is definably amenable if and only if for some model M there is a $G(M)$ -invariant Keisler measure on G over M .

A definable subset X of G is said to be left generic if finitely many left translates of by elements of G cover G . Likewise for right generic. A type $p(x) \in S_G(M)$ is said to (left, right) generic if every formula in p is.

In the body of this paper we will consider suitable groups G in an *NIP* theory T . T is said to be (or have) *NIP* if for any formula $\phi(x, y)$, indiscernible sequence $(a_i : i < \omega)$, and b the truth value of $\phi(a_i, b)$ stabilizes as $i \rightarrow \infty$. If T has *NIP* then for any definable group G , G^{00} exists (i.e. does not depend on the choice of a parameter set A). See [4] and [5] for background on *NIP* theories. A very special case of an *NIP* theory is a stable theory, and by a *stable group* one just means a group definable in a stable theory. A characteristic property of a stable theory T is that *externally definable sets are definable*. Much of the work on definable groups in *NIP* theories attempts to generalize aspects of the stable case. See Chapter 1 of [10] for an exposition of stable group theory. In a stable group, left generic coincides with right generic (we just say generic) and generic types exist. Also G^{00} coincides with G^0 , the intersection of all \emptyset -definable subgroups of finite index, whereby G/G^{00} is a profinite group. Moreover what one might call the “fundamental theorem of stable group theory” is:

Fact 1.2. (*T* stable.) Let M be any model. Then the set $S_{G,gen}(M)$ of generic types, a closed subset of $S_G(M)$, is homeomorphic to G/G^0 .

In section 2 we will consider definably amenable groups in *NIP* theories, and in section 3, what we will call *measure-stable* groups in *NIP* theories. The latter also go under the name of *fs* groups or groups generically stable for measure. They are now seen to be the right generalization of *stable group* in the *NIP* setting. Definitions will be given in section 3.

We finish these model-theoretic preliminaries with a discussion of “invariant” types (and forking). Suppose M_0 is a (small) model, $M > M_0$ is saturated with respect to M_0 (e.g. $M = \bar{M}$), and $p(x) \in S(M)$. We say that p is M_0 -invariant if for any L -formula $\phi(x, y)$ and $b \in M$, whether or not $\phi(x, b) \in p(x)$ depends only on $tp(b/M_0)$. If $N > M$ is a bigger model, we can then define a canonical extension $p|N \in S(N)$ of $p(x)$, by defining for $\phi(x, y) \in L$ and $b \in N$, $\phi(x, b)$ to be in $p|N$ if and only if for some (any) $b' \in M$ realizing $tp(b/M_0)$, $\phi(x, b') \in p$. An important example of an M_0 -invariant type is $p(x) \in S(M)$ which is *finitely satisfiable* in M_0 .

When T is *NIP*, $p(x) \in S(M)$ is M_0 -invariant if and only if $p(x)$ *does not fork over* M_0 . The latter means that whenever $\phi(x, b) \in p(x)$ and (b_0, b_1, b_2, \dots) is an indiscernible over M_0 -sequence with $b_0 = b$ then $\{\phi(x, b_i) : i < \omega\}$ is consistent. In a stable theory T these notions give rise to a notion of *independence* with good properties. For example, $tp(a/M_0, b)$ does not fork over M_0 if and only if $tp(b/M_0, a)$ does not fork over M_0 and we say a and b are independent over M_0 .

We now pass to topological dynamics. Our references are [3] as well as [1].

Definition 1.3. (i) By an Ellis semigroup we mean a semigroup (S, \cdot) which is a compact (Hausdorff) topological space such that \cdot is continuous in the first coordinate, namely for each $b \in S$ the map taking x to $x \cdot b$ is continuous. (ii) By a closed left ideal of such an Ellis semigroup we mean a nonempty closed subset I of S such that $a \cdot I \subseteq I$ for all $a \in S$.

Note that by the continuity assumptions any minimal left ideal of an Ellis semigroup S is closed, and moreover such things exist.

Fact 1.4. Let (S, \cdot) be an Ellis semigroup. Let J be the set of idempotents of S (i.e. $a \in S$ such that $a \cdot a = a$). Then (i) for any closed left ideal I of S , $I \cap J$ is nonempty.

- (ii) If I is minimal and $u \in I \cap J$ then $(u \cdot I, \cdot)$ is a group.
- (iii) Moreover, as I, u vary in (ii), the groups $u \cdot I$ are isomorphic.

Following Newelski we may call the group $u \cdot I$ above the “ideal group” of S .

We now consider a “ G -flow” (X, G) , namely a group G and a (left) action of G on a compact space X by homeomorphisms. For $g \in G$, let $\pi_g : X \rightarrow X$ be the corresponding homeomorphism of X . By a subflow of (G, X) we mean some (G, Y) where Y is a nonempty closed subspace of X , closed under the action of G (so (G, Y) is itself a G -flow).

Fact 1.5. *Given a G -flow (X, G) let $E(X)$ be the closure of $\{\pi_g : g \in G\}$ in (the compact space) X^X . Then*

- (i) $(E(X), \cdot)$ is an Ellis semigroup, where \cdot is composition, and is called the enveloping Ellis semigroup of (X, G) .
- (ii) $(E(X), G)$ is also a G -flow, where the action G on $E(X)$ is $\pi_g \circ f$.
- (iii) The minimal closed left ideals of $(E(X), \cdot)$ coincide with the minimal subflows of $(E(X), G)$.

Hence from a G -flow (X, G) , by Facts 1.4 and 1.5 we obtain a unique (up to isomorphism) group (i.e. $(u \cdot I, \cdot)$ where I is a minimal left ideal of $E(X)$ and u is an idempotent in I).

We now begin connecting the two points of view. Let T be a complete first order theory, G a \emptyset -definable group, and M a model. Then $(S_G(M), G(M))$ is a $G(M)$ -flow, $G(M)$ acting on the left. It will be convenient, now and throughout the rest of the paper, to denote by \cdot the group operation on G as well as the action of $G(M)$ on $S_G(M)$. We will also use \cdot to denote the semigroup operation on $E(S_G(M))$ but as we point out there should be no ambiguity. As Newelski [9] observes:

Fact 1.6. (i) *There is a natural homeomorphism between the compact spaces $E(S_G(M))$ and $S_{G,M}(\bar{M})$ (global types concentrating on G which are finitely satisfiable in M), and hence by Fact 1.1 also $S_{G,ext}(M)$.*

(ii) *Under this homeomorphism, the Ellis semigroup operation on $E(S_G(M))$, becomes the following operation \cdot on $S_{G,M}(\bar{M})$: Given $p, q \in S_{G,M}(\bar{M})$, let b realize q and let a realize $p|(\bar{M}, b)$. Then $p \cdot q = tp(a \cdot b/\bar{M})$.*

Commentary. Concerning (i): Let $p \in S_{G,M}(\bar{M})$ and let a realise p . Then we have a well-defined map $\pi_p : S_G(M) \rightarrow S_G(M)$, given by: let $q \in S_G(M)$ be realized by $b \in G(\bar{M})$. Then $\pi_p(q) = tp(a \cdot b/\bar{M})$. The map π_p is well-defined precisely because p is M -invariant. Suppose $q_1, \dots, q_n \in S_G(M)$, realized by $b_1, \dots, b_n \in G(\bar{M})$, and $\phi_1(x), \dots, \phi_n(x)$ are formulas over M such that $\phi_i(a \cdot b_i)$ for $i = 1, \dots, n$. Then as p is finitely satisfiable in M there is $a' \in G(M)$, such that $\phi_i(a' \cdot b_i)$ for $i = 1, \dots, n$. This shows that π_p is in the closure of $\{\pi_g : g \in G(M)\}$. On the other hand, by compactness, any $f : X \rightarrow X$ in the closure of $\{\pi_g : g \in G(M)\}$ has the form π_p for some $p \in S_{G,M}(\bar{M})$. It remains to see that $p \in S_{G,M}(\bar{M})$ is determined uniquely by π_p and this is left to the reader, as well as (ii).

In the following we will identify freely $E(S_G(M))$, $S_{G,M}(\bar{M})$ and $S_{G,ext}(M)$, denoting them by S , and denote by \cdot the Ellis semigroup structure. As remarked earlier there is a natural embedding of $G(M)$ in S and the group operation on $G(M)$ is precisely the restriction of the semigroup structure on S . So there is no ambiguity in denoting this semigroup operation by \cdot . The following is not needed for the rest of the paper but we state it just for the record:

Remark 1.7. *Let $S^* = S \setminus G(M)$. Then $(S, G(M), S^*)$ is a classical Ellis semigroup in the sense of Definition 5.2 (and Chapter 6) of [1]. Namely S is an Ellis monoid (with identity e the identity of $G(M)$), $G(M)$ is an open dense submonoid, in fact subgroup, such that the restriction of the semigroup operation \cdot to $G(M) \times S$ is continuous, S^* is a closed subset of S such that $G(M) \cup S^* = S$, and moreover $S \cdot S^* \cdot S = S^*$.*

We finish this introductory section by summarizing how the Ellis theory applies to stable groups.

Fact 1.8. *Suppose T is stable (and as above G a \emptyset -definable group, and M any model). Then*

(i) $S_G(M) = E(S_G(M))$.

(ii) *The semigroup operation on $S_G(M)$ is: given $p, q \in S_G(M)$, let a, b realize p, q respectively such that a and b are independent over M . Then $p \cdot q = tp(a \cdot b/M)$.*

(iii) $S_G(M)$ has a unique minimal closed left ideal (also the unique minimal closed right ideal) I and I is already a subgroup of $S_G(M)$.

(iv) I is precisely the collection of generic types over M .

(iv) I (with its induced topology) is a compact topological group, isomorphic to G/G^0 .

2 Definably amenable groups

Here we give a rather soft result for definably amenable groups G in *NIP* theories. The result is that the class of global right f -generic types of G is, under the natural operation \cdot , an Ellis semigroup S whose corresponding “ideal group” (from 1.4) is precisely G/G^{00} (even as a topological group). In fact in this case S has no proper closed left ideals.

We first recall the relevant facts from [5] about definably amenable groups in *NIP* theories. We assume T has *NIP*. Let us fix a countable submodel M_0 of \bar{M} . A definable subset X of G (or the formula defining X) is said to be left f -generic if for all $g \in G$, $g \cdot X$ does not fork over M_0 . By [2] we can replace “does not fork” by “does not divide”. A global type $p \in S_G(\bar{M})$ is said to be *left f -generic* if every formula in p is left f -generic (equivalently, by *NIP* for all $g \in G$, gp is $\text{Aut}(\bar{M}/M_0)$ -invariant). Likewise for right f -generic. Note that $p \in S_G(\bar{M})$ is left f -generic if and only if p^{-1} is right f -generic. The existence of a left (right) f -generic type is by 5.10 and 5.11 of [5] equivalent to the definable amenability of G .

Fact 2.1. *Suppose $p(x) \in S_G(\bar{M})$ is right f -generic. Then*

(i) *so is $p|_{\bar{M}'}$ for any saturated \bar{M}' containing \bar{M} , as well as $p \cdot g$ for any $g \in G$.*

(ii) *G^{00} is the right-stabilizer of p , i.e. $\{g \in G : p \cdot g = p\}$.*

We now assume G to be definably amenable (equivalently as mentioned above right f -generic types of G exist).

Lemma 2.2. *Let S be the set of global right f -generic types of G . For $p, q \in S$ define $p \cdot q$ to be $tp(a \cdot b/\bar{M}) \in S_G(\bar{M})$, where b realizes q and a realizes $p|_{\bar{M}'}$ where \bar{M}' is a saturated model containing \bar{M}, b . Then with the induced topology from $S_G(\bar{M})$, (S, \cdot) is an Ellis semigroup.*

Proof. Note first that S is a closed subset of $S_G(\bar{M})$ so is compact.

Secondly we show that $\cdot : S \times S \rightarrow S_G(\bar{M})$ is continuous in the first coordinate. Let $\phi(x)$ be a formula over \bar{M} , say over a countable model M

containing M_0 . Let $q \in S$ (or even in $S_G(\bar{M})$) and let $b \in G = G(\bar{M})$ realize $q|M$. Then for $p \in S$, $\phi(x) \in p \cdot q$ if and only if $\phi(b \cdot x) \in p$.

Thirdly we show that S is closed under \cdot . Let $p, q \in S$, let b realize q and a realize $p|\bar{M}'$ as in the statement of the lemma. By Fact 2.1(i), $tp(a \cdot b/\bar{M}')$ is right f -generic, hence so is $p \cdot q = tp(a \cdot b/\bar{M})$.

Finally we need to know that \cdot is associative. This amounts to showing that if $p, q, r \in S$, and a, b, c realize p, q, r respectively such that b realizes $q|\bar{M}c$ and a realizes $p|\bar{M}$, b, c then $a \cdot b$ realizes $(p \cdot q)|\bar{M}c$, and this is straightforward. \square

Lemma 2.3. *S has no proper left ideals (closed or otherwise).*

Proof. Let I be a left ideal of S .

Claim. $I \cap G^{00} \neq \emptyset$, namely there is $p \in I$ such that $p(x) \models x \in G^{00}$.

Proof of claim. Let $q \in I$. So q determines a coset say C of G^{00} in G . Then the coset C^{-1} (as an element of G/G^{00}) also contains a right f -generic type $r \in S$. Let $p = r \cdot q$. So $p \in I$, and $p(x) \models x \in G^{00}$.

Now let $q \in S$. By Fact 2.1(ii), $q \cdot p = q$, so $q \in I$. \square

Note that an idempotent of S is precisely any element of $S \cap G^{00}$ (by Fact 2.1 for example).

Lemma 2.4. *Let $p \in S$ be an idempotent. Then $p \cdot S$ meets every coset of G^{00} in G in exactly one element.*

Proof. First note that if $q \in S \cap G^{00}$ then $p \cdot q = p$ by Fact 2.1(ii). Hence $p \cdot S$ meets G^{00} in exactly one element. On the other hand we know that $p \cdot S$ is a subgroup of the semigroup S , and as $p \cdot p = p \in p \cdot S$, p is its identity element. Now suppose that $q, r \in p \cdot S$ are in the same coset of G^{00} . So working in the group $p \cdot I$, $q^{-1} \cdot r$ is in G^{00} so by what we have just seen must $= p$. But then $p \cdot q = r$, so $q = r$. \square

It follows from Lemma 2.4 that the “ideal group” $p \cdot S$ is isomorphic to G/G^{00} , under the map taking $q \in p \cdot S$ to the unique coset of G^{00} containing q . But in fact this is tautologically an isomorphism of topological groups where $p \cdot S$ is given the *quotient topology* (with respect to the map from the compact space I to $p \cdot I$ taking q to $p \cdot q$). This is because we know in advance that the topology on G/G^{00} is precisely that by the map $S_G(\bar{M}) \rightarrow G/G^{00}$ and in fact also by its restriction to the compact subspace S . So summarizing, we have:

Proposition 2.5. *Suppose T has NIP, G is definably amenable. Let S be the space of global right f -generic types of G under the operation \cdot (as in Lemma 2.2). Then, (S, \cdot) is an Ellis semigroup, is itself a minimal (closed) left ideal, and for some (any) idempotent $u \in S$, the group $u \cdot S$ (with the quotient topology) is homeomorphic to G/G^{00} .*

3 Measure-stable groups

We again assume that T has NIP and G is a \emptyset -definable group.

Fact 3.1. *The following are equivalent:*

- (i) *There is some $p(x) \in S_G(\bar{M})$ such that for some countable model M_0 and any $g \in G$, $g \cdot p$ is finitely satisfiable in M_0 ,*
- (ii) *There is a global left G -invariant Keisler measure μ concentrating on G such that μ is generically stable, i.e. for some countable model M_0 , μ is both definable over and finitely satisfiable in M_0 .*

Commentary. We discuss the notions of generic stability in (ii). To say that μ is definable over M_0 , means that for any L -formula $\phi(x, y)$ and closed set $I \subset [0, 1]$, the set of $b \in \bar{M}$ such that $\mu(\phi(x, b)) \in I$ is type-definable over M_0 . To say that μ is finitely satisfiable in M_0 means that any formula over \bar{M} with positive μ -measure is realized by a tuple from M_0 . When μ is a type $p(x)$ we get the notion of a generically stable type. A characteristic property of stable theories is that every global type is generically stable: definable over and finitely satisfiable in some countable model M_0 .

Let us also remark that in both parts (i) and (ii) above we can replace “some countable model M_0 ” by “any countable model M_0 ”.

Groups satisfying the equivalent conditions in Fact 3.1 were first called *fsg* (for “finitely satisfiable generics”) groups, and later groups which are *generically stable for measure*. Here we rebaptize them as *measure-stable* groups. Among measure-stable groups are stable groups, as well as definably compact groups in \mathcal{o} -minimal structures and certain valued fields (algebraically closed, real closed, p -adically closed). The Keisler measure μ in Fact 3.1(ii) is in fact the unique global left-invariant Keisler measure on G and also the unique right-invariant Keisler measure on G (Theorem 7.7 of [5]).

Fact 3.2. *Assume G to be measure-stable, let μ be as in 3.1(ii), and let X be a definable subset of G . Then the following are equivalent:*

- (i) $\mu(X) > 0$,
- (ii) X is left generic,
- (iii) X is right generic,
- (iv) every left G -translate of X is satisfiable in M_0 (i.e. meets $G(M_0)$),
- (v) every right G -translate of X is satisfiable in M_0 .

Note in particular that the family of non generic definable subsets of G is an ideal (in the Boolean algebra of definable subsets of G).

Let us fix now a small model M , which may or may not be M_0 .

Definition 3.3. (i) *Let $X \subseteq G(M)$ be externally definable. We call X left-generic in $G(M)$ if finitely many left translates $g \cdot X$ of X by elements $g \in G(M)$ cover $G(M)$. Likewise for right-generic.*

(ii) *An external type $p \in S_{G,ext}(M)$ is said to be left-generic if every set in p is left-generic. Likewise for right-generic.*

One of our main results is:

Theorem 3.4. *Assume G is measure stable. Then*

(i) *Let $X \subseteq G$ be definable (with parameters from \bar{M}). Then X is generic in G if and only if $X \cap G(M)$ is (left, right) generic in $G(M)$.*

(ii) *The natural map taking definable $X \subseteq G$ to $X \cap G(M)$ induces a bijection between left (right) generic types in $S_G(\bar{M})$ and left (right) generic types in $S_{G,ext}(M)$.*

(iii) *In particular left and right generic types in $S_{G,ext}(M)$ coincide and such things exist.*

We work towards a proof of Theorem 3.4. The main point is (i). We assume now that G is measure-stable. The easy “direction” is:

Lemma 3.5. *Suppose $X \subseteq G$ is definable and $X \cap G(M)$ is (left) generic. Then X is (left, so also right) generic.*

Proof. Let $g_1, \dots, g_n \in G(M)$ be such that $G(M) = g_1 \cdot (X \cap G(M)) \cup \dots \cup g_n \cdot (X \cap G(M))$. Let $Z = \cup_i g_i \cdot X$ (a definable subset of $G = G(\bar{M})$). Hence Z contains $G(M)$, whereby the complement Z^c of Z in G must be nongeneric (by Fact 3.2). Hence Z is generic, whereby X is generic too. \square

Note that it follows from Lemma 3.5 that if $p \in S_{G,ext}(M)$ is left generic, then $p^{\bar{M}}$ (with notation from Fact 1.1) is a global generic type.

For the other direction we will make use of “generic compact domination” from [6], as well as the following result proved in [7] (Proposition 3.2 and its proof, as well as Corollary 3.3):

Fact 3.6. (*G* measure-stable) (i) A global type is (left, right) generic iff it is (left, right) *f*-generic.

(ii) Moreover, suppose $W \subseteq G$ is definable and nongeneric. Let p be a global generic type. Let $M' > M$ be a model over which W is defined. Then for some n , if (g_1, \dots, g_n) realizes $p^{(n)}|M'$ then $\cap_i g_i \cdot W = \emptyset$ (and also $\cap_i W \cdot g_i = \emptyset$).

Lemma 3.7. Let $X \subseteq G$ be definable and generic. Then $X \cap G(M)$ is left and right generic

Proof. First generic compact domination (Proposition 5.8 of [6]) gives some coset C of G^{00} in G and some nongeneric definable subset W of G such that $C \subseteq X \cup W$. Now C is type-definable over M , hence by compactness there is definable subset D of G , defined over M such that $C \subset D$ and $D \subseteq X \cup W$. Note that as $C \subset D$, D is generic. Note also that $D \cap W$ is nongeneric, hence $D \cap X$ is generic. Replacing X by $D \cap X$ and W by $D \cap W$ we may assume that $D = X \cup W$. Let us suppose that all the data are defined over a model $M' > M$. Let p be a global generic type of G such that $p(x) \models x \in G^{00}$. Let (g_1, \dots, g_n) be a realization of $p^{(n)}|M'$ as in Fact 3.6 (as W is nongeneric), namely $\cap_i g_i \cdot W = \emptyset$. As each $g_i \in G^{00}$, we have that $g_i \cdot C = C$ for each i and hence $\cap_i g_i \cdot D$ contains C , so again by compactness there is some definable over M subset D' of G such that $C \subset D' \subseteq \cap_i g_i \cdot D$. And note that D' is generic. Now $tp(g_1, \dots, g_n/M')$ is finitely satisfiable in M , hence there are $h_1, \dots, h_n \in G(M)$ such that

- (i) $\cap_i h_i \cdot W = \emptyset$, and
- (ii) $D' \subseteq \cap_i h_i \cdot D$.

As $D = X \cap W$ it follows from (i) and (ii) that $D' \subseteq \cup_{i=1, \dots, n} h_i \cdot X$. But as D' is generic and defined over M , finitely many left translates of D' by elements of $G(M)$ cover G . Hence (as the h_i are in $G(M)$) we see that X is left generic in $G(M)$. The same proof gives right generic. \square

Lemmas 3.5 and 3.7 give part (i) of Theorem 3.4. Parts (ii) and (iii) follow immediately.

We can now conclude the other main result:

Theorem 3.8. (*T has NIP, G is an \emptyset -definable measure-stable group, and M is any model.*) Consider the $G(M)$ -flow, $(S_G(M), G(M))$ (with $G(M)$ acting on the left), and let (S, \cdot) be the enveloping Ellis semigroup. Then for any minimal closed left ideal I of S and idempotent $u \in I$, the group $u \cdot I$ with its quotient topology is isomorphic to the compact group G/G^{00} .

Proof. The proof is just like that of Proposition 4.8 in [9] making use of Theorem 3.4 above in place of Lemma 4.6 of [9]. But for completeness we go through some of the details.

First by Fact 1.6 we know that (S, \cdot) coincides as an Ellis semigroup with the space $(S_{G,M}(\bar{M}), \cdot)$ of global types concentrating on G and finitely satisfiable in M , where $p \cdot q = tp(a \cdot b/\bar{M})$ with b realizing q and a realizing $p|\bar{M}, b$. Also with $S_{G,ext}(M)$ with the corresponding semigroup operation. Moreover, as a $G(M)$ -flow, S coincides with $(S_{G,M}(\bar{M}), G(M))$ (or $(S_{G,ext}(M), G(M))$) with the natural left action of $G(M)$.

Claim (S, \cdot) has a unique (closed) left ideal I , consisting of the global generic types (in $S_{G,M}(\bar{M})$), equivalently by Theorem 3.4 the (externally) generic types in $S_{G,ext}(M)$.

Proof. By virtue of the above identifications, and Fact 1.5 (iii) it suffices to prove that the class I say of generic types in $S_{G,ext}(M)$ is the unique minimal subflow of $(S_{G,ext}(M), G(M))$. First we know I to be closed. Now let I' be any subflow of $(S_{G,ext}(M), G(M))$. Let $p \in (S_{G,ext}(M))$ be generic, and let the externally definable set $Z \subseteq G(M)$ be in p . As Z is generic, for any $q \in I'$, some left translate $g \cdot q$ of q by some $g \in G(M)$ contains Z . Hence as I' is a subflow, Z is contained in some $q' \in I'$. As I' is closed, $p \in I'$. Hence $I \subseteq I'$ and the claim is proved.

Now G is definably amenable, and by Fact 3.6 (i) and relevant definitions of the semigroup operations, the Ellis semigroup (I, \cdot) of global generic types, coincides with the Ellis semigroup of right f -generic types of G considered in section 2 (see Lemma 2.2). Hence applying Proposition 2.5 completes the proof of Theorem 3.8. □

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