Logarithmic derivatives on nonconstant commutative algebraic groups, and transcendence questions (Joint work with D. Bertrand)

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Aims of the talk

- I will talk about a functional/differential algebraic analogue of the Lindemann-Weierstrass (L-W) theorem, for semiabelian varieties $G$ over function fields $K$, whose statement is still moving.

- L-W says that if $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly independent algebraic numbers, then $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent. It is the “exponential side” of Schanuel’s conjecture that $\text{tr.deg}(\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})/\mathbb{Q}) \geq n$ for an arbitrary set $(x_i)_{i}$ of $\mathbb{Q}$-linearly independent complex numbers.

- The novelty, compared with say work of Ax on the function field case, is that we will allow “nonconstant” semiabelian varieties.

- I will always concentrate on the “exponential” side where the $x_i$’s are rational over the base field $K$, even though some methods give information on other cases such as the logarithmic side too.
Let $K$ be an algebraically closed field of transcendence degree 1 over $\mathbb{C}$. We can equip $K$ with a derivation $\partial$ with field of constants $\mathbb{C}$ (e.g. $\partial$ extends $d/dt$).

If $x \in K$, $y = exp(x)$ makes sense, as a point in a larger differential field $F$: $x \in K_0$ for some finitely generated differential subfield of $K$ containing $\mathbb{C}$. So $x$ can be viewed as a rational function on a complex curve $S$, so $exp(x)$ lives in a differential field $F_0$ of meromorphic functions on some small disc in $S$, and can be jointly embedded with $K$ over $K_0$ into suitable $F$.

Moreover the differential relation $\partial y/y = \partial x$ is satisfied by any $(y, x)$ for which $y = exp(x)$. 

Theorem 1.1
(Exponential side of Ax) Suppose $x_1, \ldots, x_n \in K$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$. Then

(i) if $y_1, \ldots, y_n$ are elements of a differential field $F > K$ such that $\partial y_i / y_i = \partial x_i$ for $i = 1, \ldots, n$ then $y_1, \ldots, y_n$ are algebraically independent over $K$.

(ii) In particular if $y_i = \exp(x_i)$ for $i = 1, \ldots, n$ then $y_1, \ldots, y_n$ are algebraically independent over $K$.

Note that in this functional setting, the “modulo $\mathbb{C}$” part of the hypothesis is needed.
Proof. (i)

- If not then we may choose such solutions $y_1, \ldots, y_n$ in $K^{diff}$ with $\text{tr.deg}(K(y_1, \ldots, y_n)/K) < n$.
- Let $a_i = \partial x_i \in K$. So $(y_1, \ldots, y_n)$ is a solution of the system $\partial y_i = a_i y_i$, $i = 1, \ldots, n$ of linear differential equations.
- $L = K(y_1, \ldots, y_n)$ is a Picard-Vessiot extension of $K$.
- In fact if $\sigma \in Aut(L/K)$ then $\sigma(y_i) = y_i \cdot b_i(\sigma)$ for some unique $b_i(\sigma) \in \mathbb{C}^*$, and the map which takes $\sigma$ to $(b_1(\sigma), \ldots, b_n(\sigma))$ is an isomorphism of $Aut(L/K)$ with a proper algebraic subgroup $H$ of $\mathbb{C}^{*n}$.
- $H$ is defined by equations $z_1^{k_1} \cdot \ldots \cdot z_n^{k_n} = 1$ ($k_i \in \mathbb{Z}$, not all 0).
- Hence for some such $k_1, \ldots, k_n$ we have that $b_1(\sigma)^{k_1} \cdot \ldots \cdot b_n(\sigma)^{k_n} = 1$ for all $\sigma \in Aut(L/K)$.
Then check that $\sigma(y) = y$ for all $\sigma \in \text{Aut}(L/K)$, where $y = y_1^{k_1} \cdots y_n^{k_n}$.

But then $y \in K$.

It is clear that $\frac{\partial y}{y} = \partial x$ where $x = k_1x_1 + \ldots + k_nx_n$, and $x \notin \mathbb{C}$ by hypothesis.

So we have reduced the theorem to the case $n = 1$, which states essentially that a rational function $f(z)$ cannot be both a derivative and a logarithmic derivative, unless it is 0. And this is left to the reader.

End of proof.
The functional case for arbitrary semiabelian varieties over \( \mathbb{C} \)

- For \( G \) a commutative connected \( n \)-dimensional algebraic group over \( \mathbb{C} \) and \( LG = \mathbb{G}_a^n \) its Lie algebra, we have \( \exp_G : LG(\mathbb{C}) = \mathbb{C}^n \rightarrow G(\mathbb{C}) \), an analytic surjective homomorphism between the two complex Lie groups, characterized by its differential at 0 being the identity.

- We have Kolchin’s logarithmic derivative \( \partial \ln_G : G \rightarrow LG \). This is a first order differential rational homomorphism, surjective when considering points in a differentially closed field, and with kernel the constants in whichever differential field the map is being evaluated.

- For example if \( G \) is an elliptic curve over \( \mathbb{C} \) in standard form \( \partial \ln_G \) is \( \partial x/y \).

- We just write \( \partial : \mathbb{G}_a^n \rightarrow \mathbb{G}_a^n \) for the map taking \((x_1, .., x_n)\) to \((\partial(x_1), .., \partial(x_n))\).
The functional case for arbitrary semiabelian varieties over $\mathbb{C}$

- If $K$ is as before (tr.deg 1 algebraically closed extension of $\mathbb{C}$ with derivation $\partial$), and $x \in LG(K) = K^n$, then $y = exp_G(x) \in G(F)$ for suitable $F > K$ makes sense, and we have:
  - $\partial \ln_G(y) = \partial(x)$

- We consider a semiabelian variety $G$ defined over $\mathbb{C}$, namely we have an exact sequence $T \to G \to A$ of commutative algebraic groups over $\mathbb{C}$ with $T$ an algebraic torus and $A$ an abelian variety.

- Let $\tilde{G}$ be the “universal vectorial extension” of $G$. Namely $\tilde{G}$ is an extension of $G$ by some vector group $W = \mathbb{G}_a^m$ and for any other such extension $H$ of $G$ there unique $\tilde{G} \to H$ with everything commuting.
Theorem 1.2

(Exponential side of Ax-Kirby-Bertrand) Let $G$ be a semiabelian variety over $\mathbb{C}$, and let $x \in LG(K)$ be such that $x \notin LH(K) + LG(\mathbb{C})$ for any proper algebraic subgroup $H$ of $G$.

(i) Let $y$ be any solution of $\partial \ell n(y) = \partial(x)$ in a differential field $F$ extending $K$. Then $\text{tr.deg}(K(y)/K) = \dim(G)$. In particular $\text{tr.deg}(K(\exp_G(x))/K) = \dim(G)$.

(ii) Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of $x$. Then again for any solution $\tilde{y}$ of $\partial \ell n(\tilde{x}) = \partial(\tilde{x})$ we have that $\text{tr.deg}(K(\tilde{y})/K) = \dim(\tilde{G})$. In particular $\text{tr.deg}(K(\exp_{\tilde{G}}(\tilde{x}))/K) = \dim(\tilde{G})$.

Again this result reduces, via differential Galois theory, to showing that $y \notin G(K)$ in some “irreducible” contexts.
Let $K$ be as before and we will consider commutative connected algebraic groups $G$ defined over $K$.

We call $G$ constant if $G$ is isomorphic as an algebraic group to one defined over $\mathbb{C}$.

$G$ always has a maximal constant algebraic subgroup, denoted by $G_{(0)}$.

There are at least two sources of nonconstant $G$; first nonconstant abelian varieties, such as the elliptic curve $y^2 = x(x-1)(x-t)$ where $t \in K \setminus \mathbb{C}$.

Secondly nonconstant extensions of a constant abelian variety $A$ by an algebraic torus: the extensions of $A$ by $\mathbb{G}_m$ have a moduli space (which is the dual abelian variety $\hat{A}$).
If $A$ is an abelian variety over $K$ then up to isogeny $A = A_0 \times A_1$ where $A_0$ is constant, and $A_1$ of $\mathbb{C}$-trace 0 (totally nonconstant).

If $T \to G \to A$ is a semiabelian variety, let $G_0$ denote the preimage in $G$ of $A_0$ and call it the *semiconstant* part of $G$. So $G(0) \subseteq G_0$. 
For $G$ a commutative connected algebraic group over $K$ and $LG = \mathbb{G}_a^n$ its Lie algebra, and for $x \in LG(K)$ we can speak of $exp_G(x)$, as a point in a larger differential field:

Again $x \in LG(K_0) = \mathbb{C}(S)$ for some complex curve $S$ with all data defined over $K_0$.

$G$ is the “generic fibre” of a fibration $G \to S$ of complex varieties, where the fibres $G_s$ are complex algebraic groups.

Likewise there is a corresponding complex vector bundle $LG \to S$ whose generic fibre is $LG$.

$x \in LG(K_0)$ is then a rational section of $LG \to S$, holomorphic on some small $S_0$.

Applying appropriate $exp$’s in the fibres, gives us a holomorphic section $exp_G(x)$ of $G \to S$ above $S_0$, which we call $exp_G(x)$, and lives in the differential field of meromorphic functions on $S_0$, which extends $K_0$. 
Let now $G$ be a possibly nonconstant semiabelian variety over $K$.

To obtain an appropriate analogue of the differential relation $\partial \ln(y) = \partial(x)$ which was satisfied by the graph of exponentiation in the constant case, we are in general forced to pass to the universal vectorial extension $\tilde{G}$ of $G$.

The point is that $\tilde{G}$ has a (unique) so-called $D$-group structure, namely an extension $\partial'$ of $\partial$ on $K$ to a derivation of the “coordinate ring” of $\tilde{G}$ which respects co-multiplication.

Equivalently, a $D$-group structure on $\tilde{G}$ is given by a $K$-rational homomorphic section $s : \tilde{G} \to T_{\partial}(\tilde{G})$.

Here $T_{\partial}(\tilde{G})$ is the “first prolongation” or “shifted tangent bundle” of $\tilde{G}$, which can be described as follows:
As above view $\tilde{G}$ as the generic fibre of a group scheme $\pi : \tilde{G} \to S$.

We have the induced group scheme $T\pi : T\tilde{G} \to TS$.

View $\partial$ as a vector field on $S$. For $t$ a generic point of $S$, $(t, \partial(t)) \in TS$, and then $T\partial(\tilde{G})$ is precisely $(T\pi)^{-1}(t, \partial(t))$, which is both an algebraic group (over $K$), and a torsor for $TG$.

In any case, the $K$-rational homomorphic section $s$ yields our logarithmic derivative $\partial \ell_n \tilde{G} : \tilde{G} \to LG$ as follows:

For $F$ a differential field extending $K$ and $g \in \tilde{G}(K)$, $\partial \ell_n \tilde{G}(g) = \partial(g) - s(g)$ where $-$ is in the sense of the canonical group structure on $T\partial\tilde{G}$. (The same definition works to give Kolchin’s log.derivative in the constant case, taking $s = 0$.)
The $D$-structure on $\tilde{G}$ gives rise to the “connection” $\partial_{L\tilde{G}}$ on $L\tilde{G}$:

Either by differentiating (in the sense of Kolchin) $\partial\ell n_{\tilde{G}}$ at the identity, or by considering the map from the cotangent space of $\tilde{G}$ at the identity to itself, induced by the derivation $\partial'$ (as in [PZ]).

In any case $\partial_{L\tilde{G}} : L\tilde{G} \to L\tilde{G}$ is additive and satisfies the Leibniz rule with respect to scalar multiplication, namely equips the vector space $L\tilde{G}$ with a $\partial$-module structure, but now possibly nontrivial.

When $A$ is an abelian variety over $K$, then $L\tilde{A}$ identifies with the dual of the de Rham cohomology group $H^{1}_{dR}(A)$, and $\partial_{L\tilde{G}}$ coincides with the dual of the standard Gauss-Manin connection on $H^{1}_{dR}(A)$. 
In any case for \( \tilde{x} \in L\tilde{G}(K) \), and \( \tilde{y} = \exp_{\tilde{G}}(\tilde{x}) \) it is again the case that \( \partial \ell \ln_{\tilde{G}}(\tilde{y}) = \partial_{L\tilde{G}}(\tilde{x}) \), although with our differential algebraic definitions above, this requires some work to verify.

We are now in a position to state the main theorem, of which Theorem 1.2 above is a special case.
Theorem 2.1

Let $G$ be a semiabelian variety over $K$. Let $x \in LG(K)$. Assume that

$\text{Hyp}_x$: $x \notin LH(K) + LG(0)(\mathbb{C})$ for any proper algebraic subgroup $H$ of $G$; moreover for any quotient $G_1$ of $G$, the same holds for the image of $x$ in $L(G_1)$.

Let $\tilde{x} \in L\tilde{G}(K)$ be any lift of $x$. Then

(i) If $\tilde{y}$ is any solution of $\partial \ln \tilde{G}(-) = \partial L\tilde{G}(\tilde{x})$ in a differential field $(F, \partial) \supseteq (K, \partial)$ then $\text{tr.deg}(K(\tilde{y})/K) = \dim(\tilde{G})$.

(ii) In particular $\text{tr.deg}(K(exp_{\tilde{G}}(\tilde{x}))/K) = \dim(\tilde{G})$, and so $\text{tr.deg}(K(exp_G(x)/K)) = \dim(G)$. 
The hypothesis $Hyp_x$ is easily seen to be necessary. But when the semiconstant part $G_0$ of $G$ coincides with the constant part $G_{(0)}$, then the moreover clause in $Hyp_x$ follows from the first clause, so can be dispensed with.

But in the simplest case where the semiconstant part of $G$ is not constant, namely when $G$ is a nonconstant extension of a constant elliptic curve $E$ by $\mathbb{G}_m$, the moreover clause cannot be dropped. Even to see this counterexample requires results around variation of mixed Hodge structure.

Note that when $G = A$ is an abelian variety with $\mathbb{C}$-trace 0 then $Hyp_x$ says simply that $x \notin LB(K)$ for any proper abelian subvariety of $A$, and is a direct translation of the hypothesis on $x_1, .., x_n$ in the number theoretic situation (Theorems 1.1, 1.2).
Applying Theorem 2.1 to the case where $G$ is a power of a nonconstant elliptic curve, one obtains:

If $\wp$ is an elliptic function with nonconstant invariant $j \in \mathbb{C}(z)$ and zeta function $\zeta$, and if $x_1(z), \ldots, x_n(z)$ are $\mathbb{Z}$-linearly independent algebraic functions, then the $2n$ analytic functions defined on some open domain in $\mathbb{C}$ by $\wp(x_1(z)), \ldots, \wp(x_n(z)), \zeta(x_1(z)), \ldots, \zeta(x_n(z))$ are algebraically independent over $\mathbb{C}(z)$. 

"Main theorem and remarks III"
The proof of Theorem 2.1 is inductive in nature and takes us into the category of “almost semiabelian $D$-groups”.

Deligne’s theorem of the fixed part (that the set of $K$-rational solutions of the linear DE $\partial_{L\tilde{A}}(-) = 0$ is trivial when $A$ is abelian and traceless) plays a role.

There are essentially two base cases of the inductive proof. The first can be taken to be the case when $G$ is constant (so Theorem 1.2).

The second is a kind of $n = 1$ case of the other extreme: and says that when $G = A$ is simple and of $\mathbb{C}$-trace 0, $x \in LA(K)$ is nonzero, and $\tilde{x} \in L\tilde{A}(K)$ is an arbitrary lift of $x$, then there is NO $\tilde{y} \in \tilde{A}(K)$ satisfying $\partial_{\ell n \tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x})$.

The latter is precisely Manin’s “theorem of the kernel” in the form discussed by Coleman and proved by Chai.
We call $\tilde{G}$ $K$-large, if working in the differential closure $K^{\text{diff}}$ of $K$, the kernel of $\partial \ln \tilde{G}$ is contained in $\tilde{G}(K)$.

If $\tilde{G}$ is $K$-large, then the reduction to the two special cases above can be effected via (generalized) differential Galois theory, as in our proof of Theorem 1.1 above.

However $K$-largeness of $\tilde{G}$ is a rather restrictive condition. But it holds for example if $G$ is a product of a torus, a constant $A_0$ and a “general” traceless $A_1$.

To effect the inductive proof in general we need the “socle theorem” (from [PZ]): If $G$ is a connected finite-dimensional differential algebraic group and $X$ is an irreducible differential algebraic subvariety of $G$ with trivial stabilizer, then $X$ is contained in a coset of the maximal “split” or “algebraic” connected differential algebraic subgroup of $G$. 
Even in this exponential side of nonconstant $A_x$, our statement is not optimal. One would like for example, for arbitrary $x \in LG(K)$ a geometric object attached to $x$ which governs the relevant transcendence degrees (as in the usual statements of $A_x$).

One would again look for such statements in the logarithmic and mixed cases, although some work on the logarithmic case already appears in Bertrand’s paper in the Newton volume.