Pseudofinite model theory and combinatorics
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Introduction I

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One class of problems is what we call Erdős-Hajnal-type problems.

This means trying to find “large” $V_0 \subseteq V$ and $W_0 \subseteq W$ such that $V_0 \times W_0$ is homogeneous for $R$, namely $V_0 \times W_0 \subseteq R$, or $V_0 \times W_0 \subseteq R^c$ (the complement of $R$). (So Ramsey-type theorems.)
The actual Erdös-Hajnal conjecture, restricts attention to the class of finite graphs \((V, W, R)\) omitting a given induced finite subgraph \(H\), and asks there to be \(\delta > 0\) (depending on \(H\)), such that for all \((V, W, R)\), there is homogeneous \(V_0 \times W_0\) with \(|V_0| \geq |V|^\delta\), and \(|W_0| \geq |W|^\delta\).
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In this most general formulation, \(H\) is an arbitrary finite graph. But we could restrict attention to specific \(H\) and aim for better results (which we do later).

The second class of problems concerns trying to decompose, or partition, \(V\) and \(W\) into a “small” number of sets \(V = V_1 \cup \ldots \cup V_n\), \(W = W_1 \cup \ldots \cup W_m\), such that each induced subgraph \((V_i, W_j, R|_{V_i \times W_j})\) is “regular”. Namely sufficiently large induced subgraphs of \((V_i, W_j, R|_{V_i \times W_j})\) have approximately the same density.
In this general context we have Szemeredi’s regularity theorem, which says that given $\epsilon > 0$, there is $N_\epsilon$ such that for all $(V, W, R)$, we can partition $V, W$ as above, with $n, m \leq N_\epsilon$, and such that outside an “$\epsilon$-small” exceptional set $\Sigma$ of $(i, j)$, each $(V_i, W_j, R_{| (V_i \times W_j)})$ is $\epsilon$-regular. “$\epsilon$-small” means that $| \bigcup_{i,j \in \Sigma} V_i \times W_j | \leq \epsilon | V \times W |$. 
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\]

And \( \epsilon \)-regularity of \((V_i, W_j, R|(V_i \times W_j))\) means that for any induced subgraph \((V', W', R|(V' \times W'))\) of \((V_i, W_j, R|(V_i \times W_j))\), with \( |V'| \geq \epsilon |V_i| \) and \( |W'| \geq \epsilon |W_j| \), the densities of \((V_i, W_j, R|(V_i \times W_j))\) and \((V', W', R|(V' \times W'))\) differ by \( < \epsilon \).
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(The regularity lemma also includes a statement that the $V_i$’s are roughly the same size. Also the $W_j$’s.) Under additional assumptions on the relation $R$ we would like to obtain stronger conclusions, with for example homogeneity replacing regularity, and maybe with no exceptional set.
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But we would nevertheless like to see some version of Szemeredi, which is compatible with the group structure. This problematic falls under the description of “arithmetic regularity theorems”. An important paper of Ben Green deals with the case where $G$ is commutative, and $X$ arbitrary.

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This last topic is really “work in progress”, so I will not say so much about it in these lectures.
I should say that the use of nonstandard methods (essentially nonstandard analysis) to obtain (asymptotic) results in finite combinatorics, was already done by Tao for Szemerédi regularity, where the Radon-Nikodym theorem came into play.
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What is kind of new in the recent applications of model theory is that the nonstandard methods are combined with applying nontrivial structural theorems in the nonstandard (pseudofinite) model.

This point of view was in a sense initiated when model theorists found another proof (valid in all characteristics) of Tao’s algebraic regularity theorem (Tao) for graphs defined in finite fields (Pillay-Starchenko, Hrushovski).
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Actually among the themes of our recent work with Conant and Terry (CPT1, CPT2) and my expository paper “Domination and regularity”, is that certain “domination statements” yield fairly directly, the relevant graph regularity statements, in the infinite setting. Hopefully I will try to explain some of this in these talks.
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However I should also mention that our methods do not, as a rule, give optimal bounds, although the problem of good bounds is an important aspect of the combinatorial conjectures and results.
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For simplicity I will assume we are in a 1-sorted situation (namely just one sort), so the relation and function symbols come with a finite “arity”. We also assume a distinguished binary relation symbol $=$ (for equality). The many-sorted context is an easy generalization, and I may freely work in such a context.
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From these symbols, together with the logical connectives ($\neg$, $\vee$, $\wedge$, $\exists$, $\forall$ and parentheses) as well as a supply of variables $v_i$ or $x_i$ or $y_i$, we build $L$-formulas.
Model theory II

- $L$-formulas are typically denoted $\phi, \psi, \phi(\bar{x}), \psi(\bar{y})$ to witness the free variables. $L$-sentences, namely $L$-formulas with no free variables, are typically denoted $\sigma, \tau, \ldots$. 

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- We have the notion of an $L$-structure $M$, a set equipped with actual relations, functions, distinguished elements, interpreting the symbols of $L$. We often notationally identify an $L$-structure $M$ with its underlying set or universe.
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For $M$ an $L$-structure, $\phi(\bar{x})$ an $L$-formula, and $\bar{a}$ a tuple of the appropriate length from $M$, “$M \models \phi(\bar{a})$” means that the formula is true in the structure $M$ when $\bar{x}$ is interpreted as $\bar{a}$. If $\phi$ is a sentence we also say $M$ is a model of $\phi$. 
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- If $\phi(\bar{x}, \bar{y})$ is an $L$-formula, and $\bar{b}$ a tuple from $M$ then $X = \{ \bar{a} \in M : M \models \phi(\bar{a}, \bar{b}) \}$ is called a set definable in $M$ over $\bar{b}$, or a $\bar{b}$-definable set in $M$. If $B$ is a subset of $M$ containing the tuple $\bar{b}$ we may also say “$B$-definable in $M$”.
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We can also formalize “definability over $B$ in $M$”, by adding new constants to the language $L$ for elements of $B$, to form a language $L_B$, and we just mean definable by an $L_B$ formula in the tautological expansion of $M$ to an $L_B$-structure.
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If $B \subseteq M$ and $\bar{a}$ an $n$-tuple, then $tp_M(\bar{a}/B)$ denotes the collection of $L_B$-formulas $\phi(\bar{x})$ true of $\bar{a}$ in $M$ (equivalently the collection of $B$-definable sets $X$ of $n$-tuples in $M$ such that $\bar{a} \in X$).
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A collection $\Sigma(\bar{x})$ of $L_M$-formulas (with free variables among $\bar{x}$) is said to be consistent (with $M$) if it is finitely satisfiable in $M$, namely for each finite subset $\Sigma'$ of $\Sigma$ there is $\bar{a}$ in $M$ such that $M \models \wedge \Sigma'(\bar{a})$. 
A key notion is “$N$ is an elementary extension of $M$” (or $M$ is an elementary substructure of $N$): $M \subseteq N$ in the obvious sense, and $M, N$ are models of the same $L_M$-sentences.
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The compactness theorem says that a collection $\Sigma$ of $L$-sentences has a model if every finite subset of $\Sigma$ has a model. It implies that any $L$-structure $M$ has an elementary extension $N$ with the property that for every consistent (with $M$) collection $\Sigma(\bar{x})$ of $L_M$-formulas, there is a tuple $\bar{a}$ from $N$ such that $N \models \Sigma(\bar{a})$ (where the latter notation means that $N \models \phi(\bar{a})$ for all $\phi(\bar{x}) \in \Sigma$, and we also say that $\bar{a}$ realizes $\Sigma(\bar{x})$ in $N$.)
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We mention a couple of consequences. First modulo some set theory, for any \(L\)-structure \(M\) and sufficiently large cardinal \(\kappa\), \(M\) has an elementary extension \(N\) which is \(\kappa\)-saturated and is of cardinality \(\kappa\).
\(\kappa\)-saturation of \(N\) means that whenever \(B\) is a subset of \(N\) of cardinality \(<\kappa\) and \(\Sigma(\bar{x})\) is a consistent (with \(N\)) collection of \(L_B\)-formulas then \(\Sigma\) is realized in \(N\).
$\kappa$-saturation of $N$ means that whenever $B$ is a subset of $N$ of cardinality $< \kappa$ and $\Sigma(\bar{x})$ is a consistent (with $N$) collection of $L_B$-formulas then $\Sigma$ is realized in $N$.

Such a $\kappa$-saturated model $N$ of cardinality $\kappa$ is unique up to isomorphism, in the sense that its isomorphism type is determined by its first order theory $T = Th(N)$, the set of $L$-sentences $\sigma$ such that $N \models \sigma$. 
\( \kappa \)-saturation of \( N \) means that whenever \( B \) is a subset of \( N \) of cardinality \( < \kappa \) and \( \Sigma(\bar{x}) \) is a consistent (with \( N \)) collection of \( L_B \)-formulas then \( \Sigma \) is realized in \( N \).

Such a \( \kappa \)-saturated model \( N \) of cardinality \( \kappa \) is unique up to isomorphism, in the sense that its isomorphism type is determined by its first order theory \( T = Th(N) \), the set of \( L \)-sentences \( \sigma \) such that \( N \models \sigma \).

Secondly, fixing \( M \), a subset \( B \) of \( M \), an \( n < \omega \), the Stone space (space of ultrafilters) of the Boolean algebra of formulas \( \phi(\bar{x}) \) in \( L_B \) up to equivalence in \( M \), coincides with \( \{tp_N(\bar{a}/B) : \bar{a} \in N\} \) where \( N \) is some sufficiently saturated elementary extension of \( M \). We call the space \( S_n(B) \) (although it depends on the \( L_B \)-theory of \( M \)).
We have been talking about structures or models so far, but in fact the main objects of study of model theory, are first order theories $T$, where an $L$-theory $T$ is simply a collection of $L$-sentences which has a model. $T$ is often assumed to be complete.
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$T$ identifies, up to bi-interpretability with the category $\text{Def}(T^{eq})$. 
We fix a complete $L$-theory $T$ and typically work in a $\kappa$-saturated model $\bar{M}$ of cardinality $\kappa$ for some large $\kappa$. $M, N$ etc denote small elementary substructures, and $A, B, ..$ small subsets. It is also convenient to let $x, y, ..$ range over finite tuples of variables, rather than individual variables.
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Such a formula \( \phi(x, y) \) can be seen in at least three different ways: (i) as defining a bipartite graph (in some/any) model of \( T \), (ii) as giving rise to a family of definable sets, namely the sets defined in \( \bar{M} \) (or in a model \( M \)) by the formulas \( \phi(x, b) \), as \( b \) varies over tuples of the right length in \( \bar{M} \) (or \( M \)), (iii) as a collection of continuous \( \{0, 1\} \)-valued functions on a suitable compact space.
We fix a complete $L$-theory $T$ and typically work in a $\kappa$-saturated model $\bar{M}$ of cardinality $\kappa$ for some large $\kappa$. $M, N$ etc denote small elementary substructures, and $A, B, ..$ small subsets. It is also convenient to let $x, y, ..$ range over finite tuples of variables, rather than individual variables.

Many model theoretic notions are concerned with $L$-formulas of the form $\phi(x, y)$ where $x, y$ is some fixed partition of the free variables of $\phi$.

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Maybe the third point needs some comments.
Fix a model $M$. Then $S_x(M)$ is a compact space. For each $b \in M$, we obtain a continuous function $f_b$ on $S_x(M)$ where $f(p) = 1$ if $\phi(x,b) \in p$ and $= 0$ otherwise. So we get a (definable) family of functions $f_b$, $b \in M$. This makes a connection of model theory with functional analysis, and in fact some of the basic theorems of stability theory were proved by Grothendieck in his thesis (1951) in this context. (First noticed by Ben-Yaacov).
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**Definition 0.1**

The formula $\phi(x, y)$ is $k$-stable (for $T$) if there do not exist $a_1, \ldots, a_k, b_1, \ldots, b_k$ in some/any model $M$ of $T$ such that $M \models \phi(a_i, b_j)$ iff $i \leq j$. 
Definition 0.2
The formula is $k$-NIP (for $T$), if there do not exist $a_1, \ldots, a_k$ and $b_s$ for $s \subseteq \{1, \ldots, k\}$ in some/any model $M$ of $T$ such that $M \models \phi(a_i, b_s)$ iff $i \in s$. 
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- By compactness $\phi(x, y)$ is $k$-stable (in $T$) for some $k$ iff there do not exist $a_i, b_i \in \bar{M}$ for $i = 1, 2, 3, \ldots$ such that $\bar{M} \models \phi(a_i, b_j)$ iff $i \leq j$. We just say that $\phi(x, y)$ is stable (for $T$).
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- There is a similar statement for $k$-NIP. It is left to the reader. We just say $\phi(x, y)$ is NIP for $T$.
- $T$ is said to be stable if every formula $\phi(x, y)$ is stable (for $T$). Likewise $T$ is said to be NIP if every formula $\phi(x, y)$ is NIP for $T$. In both cases it is enough to consider formulas where $x$ is a single variable, rather than a tuple.
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A connection with Erdös-Hajnal, is the following: Suppose $H$ is a fixed finite graph. Then there is $k$ such that a graph $(V, W, R)$ is $k$-NIP, if it omits $H$ (as an induced subgraph).
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So dealing with the class of $k$-NIP graphs is relevant to studying graphs omitting a fixed finite subgraph $H$. 
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For an $L$-formula $\phi(x, y)$, we also have the notion of a complete $\phi$-type over a set $A$ or model $M$.

This is precisely the restriction of a complete type $p(x)$ over $M$ to the collection of Boolean combinations of formulas $\phi(x, b)$ for $b \in M$. It is “determined” (when $M$ is a model) by the collection of $\phi(x, b)$, $\neg \phi(x, b)$ for $b \in M$, true of a given $a \in \bar{M}$ (realizing $p$).
When $\phi(x, y)$ is stable, a complete $\phi$-type $p$ over $M$ has remarkable properties. First it is *definable.*
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Secondly, if $p'$ is a complete $\phi$-type over $\bar{M}$ extending (or containing $p$) and $p'$ is finitely satisfiable in $M$, then $p$ is precisely the $\phi$-type over $\bar{M}$ obtained from applying the definition mentioned above; namely for $b \in \bar{M}$, $\phi(x, b) \in p'$ iff $\bar{M} \models \psi(b)$. We call this the local (i.e. formula by formula) theory in stability. (References: GST for example, but also done in Grothendieck's thesis.)
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A formula $\psi(x, b)$ (where $b$ witnesses the parameters) divides over a set $A$ of parameters if there is some infinite $A$-indiscernible sequence $(b = b_0, b_1, \ldots)$ such that the set $\{\phi(x, b_i) : i < \omega\}$ is consistent.
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Where $(b_i : i < \omega)$ is $A$-indiscernible means that $tp(b_{i_1},...b_{i_n}/A) = tp(b_{j_1},...,b_{j_n}/A)$ for all $i_1 < .. < i_n$ and $j_1 < ... < j_n$. 

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And $\psi(x,b)$ forks over $A$ if it implies a finite disjunction of formulas each of which divides over $A$. 

Types III

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- And $\psi(x, b)$ forks over $A$ if it implies a finite disjunction of formulas each of which divides over $A$.

- In any case, with the previous assumptions (stability of $\phi(x, y)$ etc.) $p'$ can be characterized also by: $p \subset p'$ and no formula in $p'$ divides (forks) over $A$. 
Types IV

▶ When $T$ is stable (namely every $L$-formula $\phi(x, y)$ is stable for $T$), then the local theories cohere to give a nice theory of “independence”, the characteristic feature of which is uniqueness of free extensions.
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- Again fix a model $M$, and an arbitrary type $p(x) \in S_x(M)$. Then there is a unique $p'(x) \in S_x(M)$ extending $p(x)$ which satisfies each of the following equivalent conditions:
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Moreover we have essentially the same conclusions when $M$ is replaced by an algebraically closed set $A$ (finite equivalence relation theorem).
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Moreover we have essentially the same conclusions when $M$ is replaced by an algebraically closed set $A$ (finite equivalence relation theorem).
Even though the theory $T$ may be unstable, there still may be types $p(x)$ over some model $M$ which satisfy conditions (i), (ii), (iii) above, and we call these \textit{generically stable types}. (We sometimes assumed the theory $T$ to be \textit{NIP}.)
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We give an example. The theory $ACF_0$ of algebraically closed fields of characteristic 0 in the ring language is the archetypal example of an (interesting stable theory.
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Consider the theory of pairs $(F < K)$ of algebraically closed fields (with a predicate $P$ for the bottom model $F$) such that there is moreover some additional structure on $F$, such as adding an additional predicate for a real closed subfield $F_0$ of $F$ such that $F = F_0(i)$).
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Then the theory $T$ of $K$ equipped with all this structure is $NIP$ and unstable.
Fix a model $M$ of $T$. Consider the set of formulas $\Sigma(x)$ over $N$ expressing that $x$ is not in the (field-theoretic) algebraic closure of $M$ and $P(\bar{M})$. 

Then $\Sigma(x)$ determines a complete type $p(x) \in S_1(M)$. $p(x)$ is generically stable, and can be considered to be the "generic type" over $M$. It is also "regular" in the sense of Pillay-Tanovic. Working out the details of all of this is left to the reader. I mainly introduced generically stable types as a motivation for the notion of generically stable measure that will come later.
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Fix again $T$ and $\bar{M}$ a monster model and $M \prec \bar{M}$.

Let $p(x), q(y)$ be complete types over $M$ (in variables $x, y$ respectively). $p(x)$ and $q(y)$ are said to be weakly orthogonal if $p(x) \cup q(y)$ extends to a unique complete type $r(x, y)$ over $M$. 
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Now let $(V, W, R)$ be a (bipartitite graph) definable in $\bar{M}$ with parameters from $M$. So $(V(M), W(M), R(M))$ is a bi-partite graph definable in $M$ (with parameters).
Now let $p(x) \in S_V(M)$ (i.e. $p(x)$ is a complete type over $M$ containing the formula “$x \in V$”). Likewise let $q(y) \in S_W(M)$.
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We can think of $p$ as defining a $\{0,1\}$ valued measure on the Boolean algebra of definable subsets of $V(M)$. (Namely a definable set has measure 1 or is “large” if the formula defining it is in $p$). Similarly for $q(y)$ and $W(M)$.

**Theorem 0.3**

*In this context, suppose $p(x)$ and $q(y)$ are weakly orthogonal. Then there are large definable subsets $V_0$ of $V(M)$ and $W_0$ of $W(M)$ such that $(V_0, W_0)$ is homogeneous for $R(M)$. Namely either $(V_0, W_0, R|_{(V_0 \times W_0)})$ is a complete graph or an empty graph.*
Proof.
Let $r(x, y)$ be the unique complete type over $M$ extending $p(x) \cup q(y)$.

Case (i) $R(x, y) \in r(x, y)$.
So working in $\overline{M}$, $p(x) \cup q(y) \models R(x, y)$. By compactness (i.e.
saturation of $\overline{M}$), there are formulas $\phi(x) \in p(x)$, $\psi(y) \in q(y)$
such that $\overline{M} \models (\forall x)(\forall y)(\phi(x) \land \psi(y) \rightarrow R(x, y))$. So the
sentence $(\forall x)(\forall y)(\phi(x) \land \psi(y) \rightarrow R(x, y))$ is also true in $M$. Let
$V_0$ be the subset of $V$ defined by $\phi(x)$ in $M$, Likewise for $W_0$, and
we see that $(V_0, W_0, R|_{V_0 \times W_0})$ is a complete graph. Both $V_0,
W_0$ are large.

Case (ii), $\neg R(x, y) \in r(x, y)$.
Similarly we obtain large $V_0, W_0$ such that $(V_0, W_0, R(V_0 \times W_0))$ is
the empty graph. \qed
A stronger condition will yield the ultimate “regularity” theorem for the graph \((V, W, R)\) (or \((V(M), W(M), R(M))\)).
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The proof, using compactness as above, is left as an exercise for relative beginners in model theory who are attending the course.
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The proof, using compactness as above, is left as an exercise for relative beginners in model theory who are attending the course.

**Theorem 0.4**

Suppose that \(p(x)\) and \(q(y)\) are weakly orthogonal for all \(p(x) \in S_V(M)\) and \(q(y) \in S_W(M)\). Then we can partition \(V(M)\) into definable sets \(V_0, \ldots, V_n\), and partition \(W(M)\) into definable sets \(W_0, \ldots, W_m\) such that each \((V_i, W_j)\) is homogeneous for \(R\).
At this point it is convenient to introduce pseudofiniteness in a reasonably flexible form.
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**Definition 0.5**

Let $M$ be an $L$-structure, and $A$ a subset of some sort $X$ of $M$ (e.g. if $M$ is 1-sorted then $X$ could be the sort consisting on $n$-tuples from $M$). We will say that “$A$ is pseudofinite in $M$” if whenever $\sigma$ is a sentence in the language $L$ together with an additional predicate symbol for $A$, and $(M, A) \models \sigma$, then there is an $L$-structure $M'$ and subset $A'$ of $X(M')$ such that $(M', A') \models \sigma$. 

Let's make some remarks:

Pseudofiniteness of $A$ in $M$ is a property of $\text{Th}(M, A)$ (in the language $L(P) = L \cup \{P\}$).

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- If $M$ is 1-sorted and $A$ is $M$ itself, we say that $M$ is pseudofinite.
Suppose that $A$ is definable in the $L$-structure $M$ by a formula $\phi(x, b)$. Then pseudofiniteness of $A$ in $M$ is equivalent to: for every $L$-formula $\psi(y)$ in $tp_M(b)$, there is an $L$-structure $M'$ and $b' \in M'$ such that $M \models \psi(b')$ and $\phi(x, b')(M')$ is finite.
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So if $A$ is definable by a formula $\phi(x)$ of $L$ (without parameters), then pseudofiniteness of $A$ in $M$ means that for every $\sigma \in Th(M)$ there is a model $M'$ of $\sigma$ such that $\phi(M')$ is finite.
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We now give some routine equivalences to pseudofiniteness.
Lemma 0.6
For $M$ an $L$-structure and $A$ a subset of a sort $X$ in $M$, the following are equivalent:

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- (i) $A$ is pseudofinite in $M$,
- (ii) $(M, A) \models \Sigma$ where $\Sigma$ be the set of $L(P)$-sentences which are true in every $L(P)$-structure $(M', A')$ where $A'$ is finite,
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- (iii) $(M, A)$ is elementarily equivalent to some ultraprocess of $L(P)$-structures $(M', A')$ where $A'$ is finite.
Lemma 0.6

For $M$ an $L$-structure and $A$ a subset of a sort $X$ in $M$, the following are equivalent:

1. $A$ is pseudofinite in $M$,
2. $(M, A) \models \Sigma$ where $\Sigma$ be the set of $L(P)$-sentences which are true in every $L(P)$-structure $(M', A')$ where $A'$ is finite,
3. $(M, A)$ is elementarily equivalent to some ultrapower of $L(P)$-structures $(M', A')$ where $A'$ is finite.

Proof.

Let $\Sigma$ be as in (ii). Then obviously $(M, A) \models \Sigma$ iff $(M, A)$ is pseudofinite. On the other hand, assuming $(M, A)$ to be pseudofinite, let $I$ be the collection of finite subsets of $Th(M, A)$, for each $i \in I$, Let $(M_i, A_i) \models i$ with $A_i$ finite. Then any nonprincipal ultrapower of the $A_i$ is a model of $Th(M, A)$.
The main use of nonstandard models will be to have available the “nonstandard normalized counting measure” (also called the Loeb measure) on pseudofinite sets.
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So as to avoid being precise about what exactly is included in \(V\), we will just take, notationally, the ground structure to be the (standard) model \((\mathbb{V}, \varepsilon)\) of set theory, and \((\mathbb{V}^*, \varepsilon^*)\) to be a “monster model”, i.e. saturated elementary extension. (Although this doesn’t make such a lot of sense formally.)
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An object in \(\nabla^*\) is said to be *internal* if it is definable (with parameters) in \((\nabla^*, \epsilon^*)\).
In $V^*$ we have the nonstandard versions $\mathbb{N}^*$, $\mathbb{R}^*$ of $\mathbb{N}$ and $\mathbb{R}$, (as well as of cardinals). Moreover any internal object which is a $*$-set, has a (nonstandard) cardinality.
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It is natural to define a \textit{pseudofinite} object as an internal object of $\bar{V}^*$ which is finite in the sense of $V^*$ (i.e. whose cardinality is in $\mathbb{N}^*$ (i.e. a nonstandard finite object).
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First the easy direction: Suppose that $(M, A)$ is an $L(P)$-structure in $V^*$ and $A$ is finite in the sense of $V^*$, and let $\sigma$ be an $L(P)$-sentence true in $(M, A)$. 

This is expressed by the satisfaction of some formula $\chi(x, y, z)$ of set theory by $(M, A, \sigma)$ in $V^*$. So as $V \prec V^*$ we can find $(M', A')$ in $V$ such that $A'$ is finite and $(M', A') \models \sigma$. 


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First the easy direction: Suppose that \((M, A)\) is an \( L(P) \)-structure in \( V^* \) and \( A \) is finite in the sense of \( V^* \), and let \( \sigma \) be an \( L(P) \)-sentence true in \((M, A)\).

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The converse will be stated more precisely.
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Lemma 0.7
Suppose $M$ is an $L$-structure, $A$ a subset of a sort of $M$ and $A$ is pseudofinite in $M$ (in the sense of Definition 0.5). Then there is some appropriate $(M^*, A^*)$ in $\mathbb{V}^*$ such that

(i) $(M^*, A^*)$ is an $L(P)$-structure elementarily equivalent to $(M, A)$,
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(i) $(M^*, A^*)$ is an $L(P)$-structure elementarily equivalent to $(M, A)$,

(ii) $A^*$ is finite in the sense of $\mathbb{V}^*$,

(iii) whenever $\chi(y, z)$ is a formula of set theory true of $(M^*, A^*)$ in $\mathbb{V}^*$ then there is $(M, A) \in \mathbb{V}$ such that $A$ is finite and $\chi(y, z)$ is true of $(M, A)$ (in $\mathbb{V}$).
Proof.

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- This collection of formulas is finitely satisfiable in $V$, so realized in a saturated elementary extension $V^*$, as required.
The following addition to Lemma 0.7 will be useful. The proof is left to the audience.

Lemma 0.8
Suppose in addition that 
\((M, A_n)\) is a model of the common theory of \((M_n, A_n)\) (\(n < \omega\)) where \(A_n\) is finite and of increasing size with \(n\), and \(A\) is infinite. Then \((M^*, A^*)\) can be chosen to also satisfy:

(iii)' Whenever \(\chi(y, z)\) is a formula of set theory true of \((M^*, A^*)\) in \(V^*\), then \(\chi(y, z)\) is true of infinitely many \((M_n, A_n)\) (in \(V\)).

Here are some remarks on the constructions.

If \(V^*\) is \(\kappa\)-saturated, of cardinality \(\kappa\), then so is \((M^*, A^*)\) (as an \(L(P)\)-structure).

So if \((M, A)\) was already \(\kappa\)-saturated of cardinality \(\kappa\), then it will be isomorphic to \((M^*, A^*)\), so can be assumed to live in the nonstandard model with \(A^*\) finite in the sense of the model.

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Nonstandard analysis V
Suppose $A$ is an internal object in $\mathbb{V}^*$ which is finite in the sense of $\mathbb{V}^*$. (In particular $A$ is a set in $\mathbb{V}^*$). So for each internal $Z \subseteq A$ we have $|Z| \in \mathbb{N}^*$, and we define $\mu^*(Z)$ to be $|Z|/|A| \in [0, 1]^*$. This is the nonstandard counting measure on internal subsets of $A$, with value in the nonstandard unit interval. Each element of $[0, 1]^*$ has a unique "standard part" $st(\mu^*(Z))$ gives us a "measure" on internal subsets of $A$ with values in $[0, 1]$. The end result is that if $A$ is pseudofinite in the $L$-structure $M$, and the pair $(M, A)$ is saturated then we have in particular constructed a certain $[0, 1]$-valued "measure" $\mu$ on $L_M$-definable subsets of the ambient sort $X$: $\mu(Z) = st(\mu^*(Z \cap A))$. Our rather roundabout way of constructing this "pseudofinite Keisler measure" is partly to avoid an appeal to ultraproducts, which I am allergic to.
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**Definition 0.9**

Fix a sort $X$ over which variables $x$ range. (So $X$ could be the sort of $n$-tuples.) By a Keisler measure $\mu(x)$ on $X$ over $M$, we mean a finitely additive probability measure on $M$-definable subsets of $X$. 
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This means that $\mu$ has values in $[0, 1]$, $\mu(x = x) = 1$, $\mu(x \neq x) = 0$ and for disjoint $M$-definable $Y, Z$, $\mu(Y \cup Z) = \mu(Y) + \mu(Z)$. 
As with types we can fix an $L$-formula $\phi(x, y)$ and consider the Boolean algebra generated by sets defined by $\phi(x, b)$, for $b \in M$, and by a Keisler $\phi$-measure over $M$, we mean a finitely additive probability measure on this Boolean algebra.
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Sometimes we consider the case where \( M = \bar{M} \) and we talk about a global Keisler (\( \phi \)) measure.
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Sometimes we consider the case where $M = \bar{M}$ and we talk about a global Keisler ($\phi$) measure.

Note that a Keisler measure on $X$ over $M$ coincides with the identically defined finitely additive probability measure on the Boolean algebra of definable (with parameters) subsets of $X(M)$ (i.e. without talking about the monster model $\bar{M}$).
As with types we can fix an $L$-formula $\phi(x, y)$ and consider the Boolean algebra generated by sets defined by $\phi(x, b)$, for $b \in M$, and by a Keisler $\phi$-measure over $M$, we mean a finitely additive probability measure on this Boolean algebra.

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A Keisler measure on $X$ over $M$ is the same thing as a regular Borel probability measure on the Stone space $S_X(M)$. (To be explained.)
Example 0.10
A complete type $p(x) \in S_x(M)$ is a $\{0, 1\}$-valued Keisler measure on $X$ over $M$. From the point of view of the last bullet point, it is a “Dirac”.

Example 0.11
Let $T$ be the theory of an equivalence relation $E$ with two classes, both infinite. $T$ is $\omega$-categorical with quantifier elimination.

Let $M$ be the unique countable model of $T$. Let $\mu$ be the Keisler measure on the universe, over $M$ say, determined by assigning $0$ to each formula $x = a$ and assigning $1/2$ to each equivalence class.

The $\mu$ is the average of the two nonrealized $1$-types $p(x), q(x)$ over $M$.

Note that $\mu$ is $\text{Aut}(M)$-invariant.
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Example 0.12
Let $A$ be a finite subset of $X(M)$, and for $Z$ an $M$-definable subset of $X$ let $\mu_A(Z) = |Z \cap A|/|A|$. $\mu_A$ is a “counting” Keisler measure (on $X$ over $M$).
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Example 0.13
Likewise let $M$ be an $L$-structure living in a nonstandard model $\mathbb{V}^*$ of set theory, and let $A$ be a finite, in the sense of $\mathbb{V}^*$, subset of a sort $X(M)$. For $Z$ a definable subset of $M$, let $\mu_A(Z)$ be as defined earlier ($st(|Z \cap A|/|A|)$). $\mu_A$ is a “pseudofinite counting” Keisler measure on $X$ over $M$. 
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Here the theory will be \( RCF \) (real closed fields in the language of rings with a symbol for the ordering if you wish).
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- As definable subsets of the real line $\mathbb{R}$ are finite unions of points and intervals, they are measurable, so clearly $\lambda_I$ induces a Keisler measure $\mu$ on $x = x$ over $\mathbb{R}$. 
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As definable subsets of the real line $\mathbb{R}$ are finite unions of points and intervals, they are measurable, so clearly $\lambda_I$ induces a Keisler measure $\mu$ on $x = x$ over $\mathbb{R}$.

Let $\bar{M}$ be the saturated elementary extension of $\mathbb{R}$, another real closed ordered field.
Our observation is that $\mu$ has a unique extension to a global Keisler measure on $x = x$. Because if $\mu'$ extends $\mu$ it is forced to assign 0 to infinitesimal intervals.
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The property of unique extension to a larger model is called *smoothness* and shows the difference with types where the only smooth types over a model are realized ones ($tp(a/M)$ for $a \in M$).
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Such measures as well as *generically stable measures* (generalizing generically stable types) will appear later.
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Remember that a graph \((V, W, R)\) is called \(k\)-stable if it omits the \(k\)-half graph (which has vertex sets \(\{a_1, .., a_k\}\) and \(\{b_1, .., b_k\}\) with \(R(a_i, b_j) \iff i \leq j\)
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**Theorem 0.15**

*For every $\epsilon > 0$ there is $N_\epsilon$ such that for every $k$-stable finite graph $(V, W, R)$, there are partitions $V = V_1 \cup \ldots \cup V_n$, $W = W_1 \cup \ldots \cup W_m$ with $m, n \leq N_\epsilon$, and such that for every $i, j$, $(V_i, W_j, R|_{(V_i \times W_j)})$ is $\epsilon$-homogeneous, namely either almost complete ($|(V_i \times W_j) \setminus R| \leq \epsilon|V_i \times W_j|$) or almost empty ($|(V_i \times W_j) \cap R| \leq \epsilon|V_i \times W_j|$).*
So the conclusion improves that of the general Szemeredi regularity lemma, by getting rid of the exceptional pairs (the error) and replacing $\epsilon$-regularity by the much stronger $\epsilon$-homogeneity.
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The original proof by Malliaris-Shelah of (a version of) Theorem 0.15, was not a pseudofinite proof, and gave good bounds (on $N_\epsilon$). I will follow my treatment in “Domination and regularity” which is close to the Malliaris-Pillay account. (See subsequent references.)
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The original proof by Malliaris-Shelah of (a version of) Theorem 0.15, was not a pseudofinite proof, and gave good bounds (on $N_\epsilon$). I will follow my treatment in “Domination and regularity” which is close to the Malliaris-Pillay account. (See subsequent references.)

The general idea is simply to study graphs $(V, W, R)$ definable in an arbitrary structure such that the relation $R$ is defined by a stable formula $\phi(x, y)$, and where the $V$-sort is equipped with a Keisler $\phi$-measure $\mu$, and then apply Lemmas 0.7 and 0.8 where $\mu$ is taken to be the pseudofinite counting measure.
The first key observation is that if $\phi(x, y)$ is stable then any Keisler $\phi$-measure over a model $M$ say, is a weighted average of complete $\phi$-types over $M$. 
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This is actually a basic fact about Borel probability measures on “scattered spaces”; that they are averages of Diracs. But anyway, we give a sketch of what is going on. We work in the usual model-theoretic context.
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This is actually a basic fact about Borel probability measures on “scattered spaces”; that they are averages of Diracs. But anyway, we give a sketch of what is going on. We work in the usual model-theoretic context.

**Lemma 0.16**

*Suppose that $\phi(x, y)$ is a stable formula, and $\mu$ is a Keisler $\phi$-measure over $M$. Then there are $p_i(x) \in S_\phi(M)$, and $\alpha_i \in (0, 1]$ for $i = 1, 2, \ldots$ (maybe finite) such that $\sum_i \alpha_i = 1$ and $\mu = \sum_i \alpha_i p_i$.***
Proof.

It is convenient to assume \( L \) (language of \( T \)) countable, and to define a \( \phi \)-formula over \( M \) to be a Boolean combination of formulas \( \phi(x,b) \) for \( b \in M \) and \( x = a \) for \( a \in M \).

So the relevant type space \( S_{\phi}(M) \) is the collection of complete \( \phi \)-types over \( M \), i.e. which decide every such \( \phi \)-formula.

We have seen in the section on types that from stability of \( \phi(x,y) \) every \( p(x) \in S_{\phi}(M) \) is definable. In particular for any countable \( M_0 \prec M \), \( S_{\phi}(M_0) \) is countable.

It follows that the space \( S_{\phi}(M) \) is scattered, in the sense that it is exhausted by the Cantor-Bendixson analysis.

Where recall that for given a topological space \( S \), the CB analysis is as follows: the points \( p \in S \) of CB-rank 0 are the isolated points.
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Where recall that for given a topological space $S$, the $CB$ analysis is as follows: the points $p \in S$ of $CB$-rank 0 are the isolated points.
The points of $CB$ rank $\alpha + 1$ are the ones that are isolated after throwing away from $S$ the (open set of) points of $CB$-rank $\leq \alpha$ etc.
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We will also freely use that the $\phi$-measure $\mu$ can be identified with a (regular) Borel probability measure on the space $S_\phi(M)$. 
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Now let $p_1, \ldots, p_k$ be the elements in $S_\phi(M)$ of $CB$-rank $\gamma$.

Without loss of generality $p_1, \ldots, p_r$ have positive $\mu$-measure (say $\alpha_1, \ldots, \alpha_r$) and $p_{r+1}, \ldots, p_k$ have $\mu$-measure 0. (Maybe $r = 0$).
Let $U$ be the complement of $\{p_1, \ldots, p_k\}$ in $S_\phi(M)$, so $U$ is open and has $\mu$-measure $\beta = 1 - (\alpha_1 + \ldots + \alpha_r)$. 
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Now we can find clopens $U_1 \subset U_2 \subset \ldots \subset U$, and positive reals $\beta_1 < \beta_2 < \ldots$ such that $\lim_{i \to \infty} \beta_i = \beta$ and $\mu(U_i) = \beta_i$ (using regularity of $\mu$).
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Now $U_1$ and each $U_{i+1} \setminus U_i$ correspond to $\phi$-definable sets (over $M$), each of which has positive $\mu$-measure, as well as $CB$-rank $< \gamma$ (explain).
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So we can apply induction to write each of $\mu|U_1$ and $\mu|(U_{i+1} \setminus U_i)$ as a suitable $\sum_j \delta_j q_j$, and put them together with $\alpha_1 p_1 + \ldots + \alpha_r p_r$ to find the required expression for $\mu$. 

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We will only be working with a Keisler measure \(\mu\) on \(V\) over \(M\) (without worrying about \(W\)).
With this notation, here is the result:

Lemma 0.17

\[(V, W, R)\] is definable in \(M\), \(\mu\) is a Keisler measure on \(V\) over \(M\), and we assume that the \(L\)-formula \(\phi(x, y)\) defining the edge relation \(R\) is stable (with respect to \(T = Th(M)\)). Then for any \(\epsilon > 0\), there are partitions \(V_1 \cup \ldots \cup V_n\) of \(V\) and \(W_1 \cup \ldots \cup W_m\) of \(W\), such that for each \(i, j\), either for all \(b \in W_j\), \(\mu(V_i \setminus R(x, b)) \leq \epsilon \mu(V_i)\), or for all \(b \in W_j\), \(\mu(V_i \cap R(x, b)) \leq \epsilon \mu(V_i)\).

Moreover, each \(V_i\) can be defined by a \(\phi\)-formula (over \(M\)), and each \(W_j\) by a \(\phi^*\)-formula (over \(M\)).
With this notation, here is the result:

**Lemma 0.17**

So $(V, W, R)$ is definable in $M$, $\mu$ is a Keisler measure on $V$ over $M$, and we assume that the $L$-formula $\phi(x, y)$ defining the edge relation $R$ is stable (with respect to $T = Th(M)$). Then for any $\varepsilon > 0$, there are partitions $V_1 \cup \ldots \cup V_n$ of $V$ and $W_1 \cup \ldots \cup W_m$ of $W$, such that for each $i, j$, either for all $b \in W_j$, $\mu(V_i \setminus R(x, b)) \leq \varepsilon \mu(V_i)$, or for all $b \in W_j$, $\mu(V_i \cap R(x, b)) \leq \varepsilon \mu(V_i)$. 
With this notation, here is the result:

**Lemma 0.17**

So \((V, W, R)\) is definable in \(M\), \(\mu\) is a Keisler measure on \(V\) over \(M\), and we assume that the \(L\)-formula \(\phi(x, y)\) defining the edge relation \(R\) is stable (with respect to \(T = Th(M)\)). Then for any \(\epsilon > 0\), there are partitions \(V_1 \cup \ldots \cup V_n\) of \(V\) and \(W_1 \cup \ldots \cup W_m\) of \(W\), such that for each \(i, j\), either for all \(b \in W_j\),

\[\mu(V_i \setminus R(x, b)) \leq \epsilon \mu(V_i),\]

or for all \(b \in W_j\),

\[\mu(V_i \cap R(x, b)) \leq \epsilon \mu(V_i).\]

Moreover, each \(V_i\) can be defined by a \(\phi\)-formula (over \(M\)), and each \(W_j\) by a \(\phi^*\)-formula (over \(M\)).
Proof.

We will only have to consider the restriction of the measure $\mu$ to the Boolean algebra of $\phi$-formulas over $M$, equivalently to the space $S_{\phi}(M)$, so we let $\mu$ denote this restriction.

By Lemma 0.16, $\mu = \sum_{i \in I} \alpha_i p_i$ for some $p_i \in S_{\phi}(M)$ and $\alpha_i \in (0,1]$ where $\sum_{i} \alpha_i = 1$ and where we assume $I$ to be either $\omega$ or a finite initial segment of $\omega$.

Note that $\mu(p_i) = \alpha_i$ for $i \in I$.

Fix small $\epsilon > 0$.

For each $i \in I$, let $V_i$ be a formula in $p_i$ (equivalently a clopen containing $p_i$) such that $\mu(V_i) < \alpha_i/(1 - \epsilon)$.

Let $B = S_{\phi}(M) \setminus \{p_i : i \in I\}$. So $B$ is Borel and $\mu(B) = 0$. 
Proof.

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Note that $\mu(p_i) = \alpha_i$ for $i \in I$.

Fix small $\epsilon > 0$. 

stable graphs III
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We will only have to consider the restriction of the measure \( \mu \) to the Boolean algebra of \( \phi \)-formulas over \( M \), equivalently to the space \( S_\phi(M) \), so we let \( \mu \) denote this restriction.

By Lemma 0.16, \( \mu = \sum_{i \in I} \alpha_i p_i \) for some \( p_i \in S_\phi(M) \) and \( \alpha_i \in (0, 1] \) where \( \sum_i \alpha_i = 1 \) and where we assume \( I \) to be either \( \omega \) or a finite initial segment of \( \omega \).

Note that \( \mu(p_i) = \alpha_i \) for \( i \in I \).

Fix small \( \epsilon > 0 \).

For each \( i \in I \), let \( V_i \) be a formula in \( p_i \) (equivalently a clopen containing \( p_i \)) such that \( \mu(V_i) < \alpha_i/(1 - \epsilon) \).
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We will only have to consider the restriction of the measure $\mu$ to the Boolean algebra of $\phi$-formulas over $M$, equivalently to the space $S_\phi(M)$, so we let $\mu$ denote this restriction.

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Let $B = S_\phi(M) \setminus \{p_i : i \in I\}$. So $B$ is Borel and $\mu(B) = 0$. 

Stable graphs III
Let $\delta = (\alpha_0 / (1 - \epsilon)) - \mu(V_0)$, so $\delta > 0$, and we can find open $U \supseteq B$ such that $\mu(U) < \delta$. 
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So, let $U, V_0, ..., V_n$ form a finite subcover. It is not hard to refine the $V_j$ so that they are disjoint and we still have $V_j \in p_j$ and $\mu(V_j) < \alpha_j/(1 - \epsilon)$ (for $j = 0, .., n$).
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Let $V_0'$ be the complement of $V_1 \cup .. \cup V_n$ in $S_{\phi}(M)$. 
Let $\delta = (\alpha_0/1 - \epsilon) - \mu(V_0)$, so $\delta > 0$, and we can find open $U \supseteq B$ such that $\mu(U) < \delta$.

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Let $V'_0$ be the complement of $V_1 \cup \ldots \cup V_n$ in $S_\phi(M)$.

So $V'_0$ is clopen, and $p_0 \in V'_0 \subseteq U \cup V_0$. 
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Let us now replace $V_0$ by $V'_0$ (i.e. $V'_0$ is the new $V_0$).
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Let us now replace $V_0$ by $V'_0$ (i.e. $V'_0$ is the new $V_0$).
The aim and end result of the manipulations so far is to obtain clopen sets $V_0,..,V_n$ partitioning $S_\phi(M)$ (equivalently $\phi$-formulas $V_0,..,V_n$ which partition $V$) such that $V_i \in p_i$ and $\mu(V_i \setminus p_i) < \epsilon \mu(V_i)$, for $i = 0,..,n$. 

For each $i = 0,..,n$ let $\psi_i(y)$ be the definition of $p_i$. Namely $\psi_i(y)$ is a $\phi^*$-formula over $M$ such that for all $b \in M$, $\phi(x,b) \in p_i$ iff $M |= \psi_i(y)$.

For each subset $J$ of $\{0,..,n\}$, let $W_J$ be the subset of $W$ defined by $\wedge_{i \in J} \psi_i(y) \wedge \wedge_{i \not\in J} \neg \psi_i(y)$.

So the $W_J$ partition $W$ (ignoring those that are empty) into sets defined by $\phi^*$-formulas. 

$V = V_0 \cup .. \cup V_n$ and $W = \cup W_J$ will be the desired partitions.
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Namely $\psi(y)$ is a $\phi^*$-formula over $M$ such that for all $b \in M$, $\phi(x, b) \in p_i$ iff $M \models \psi_i(y)$.
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For each \( i = 0, \ldots, n \) let \( \psi_i(y) \) be the definition of \( p_i(x) \). Namely \( \psi(y) \) is a \( \phi^* \)-formula over \( M \) such that for all \( b \in M \), \( \phi(x, b) \in p_i \) iff \( M \models \psi_i(y) \).

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Note that for each $i \in \{0, \ldots, n\}$ and $J \subseteq \{0, \ldots, n\}$, we have either

(a) $\phi(x, b) \in p_i$ for all $b \in W_J$, or

(b) $\neg \phi(x, b) \notin p_i$ for all $b \in W_J$.

In case (a) $\mu(V_i \setminus R(x, b)) \leq \mu(V_i \setminus p_i) < \epsilon \mu(V_i)$ for all $b \in W_J$.

In case (b) $\mu(V_i \cap R(x, b)) \leq \mu(V_i \setminus p_i) < \epsilon \mu(V_i)$, for all $b \in W_J$.

End of proof of Lemma 0.17 (which one sees is almost tautological).
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In case (a) \( \mu(V_i \setminus R(x, b)) \leq \mu(V_i \setminus p_i) < \epsilon \mu(V_i) \) for all \( b \in W_J \).
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In case (b) \( \mu(V_i \cap R(x, b)) \leq \mu(V_i \setminus p_i) < \epsilon \mu(V_i) \), for all \( b \in W_J \).
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Proof of stable regularity lemma I

- We put things together to prove Theorem 0.15.
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- We put things together to prove Theorem 0.15.
- The proof, using Lemma 0.17 (as well as Lemmas 07 and 0.8), is a model for all later proofs deducing facts about all suitable finite graphs from results about single suitable infinite graphs.
Proof of stable regularity lemma I

- We put things together to prove Theorem 0.15.
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- So suppose for a contradiction that Theorem 0.15 fails.
Proof of stable regularity lemma I

- We put things together to prove Theorem 0.15.
- The proof, using Lemma 0.17 (as well as Lemmas 07 and 0.8), is a model for all later proofs deducing facts about all suitable finite graphs from results about single suitable infinite graphs.
- So suppose for a contradiction that Theorem 0.15 fails.
- So there is an \( \epsilon > 0 \), such that for any \( N \), there is a finite \( k \)-stable graph \((V_N, W_N, R_N)\) such that there is no partition of each of the vertex sets into at most \( N \) subsets, such that for each \( V', W' \) in the partition, \((V', W', R|(V' \times W'))\) is \( \epsilon \)-homogeneous.
We put things together to prove Theorem 0.15.

The proof, using Lemma 0.17 (as well as Lemmas 07 and 0.8), is a model for all later proofs deducing facts about all suitable finite graphs from results about single suitable infinite graphs.

So suppose for a contradiction that Theorem 0.15 fails.

So there is an $\epsilon > 0$, such that for any $N$ there is a finite $k$-stable graph $(V_N, W_N, R_N)$ such that there is no partition of each of the vertex sets into at most $N$ subsets, such that for each $V', W'$ in the partition, $(V', W', R|_{(V' \times W')})$ is $\epsilon$-homogeneous.

The sizes of the $V_N$ can be assumed to be growing (by thinning the sequence).
We put things together to prove Theorem 0.15. The proof, using Lemma 0.17 (as well as Lemmas 07 and 0.8), is a model for all later proofs deducing facts about all suitable finite graphs from results about single suitable infinite graphs. So suppose for a contradiction that Theorem 0.15 fails. So there is an $\epsilon > 0$, such that for any $N$ there is a finite $k$-stable graph $(V_N, W_N, R_N)$ such that there is no partition of each of the vertex sets into at most $N$ subsets, such that for each $V', W'$ in the partition, $(V', W', R|(V' \times W'))$ is $\epsilon$-homogeneous. The sizes of the $V_N$ can be assumed to be growing (by thinning the sequence). So we can find a saturated model $(V, W, R)$ of the common theory of the $(V_N, W_N, R)$ such that $V$ is infinite, and clearly pseudofinite in the structure $(V, W, R)$. 
By Lemmas 0.7 and 0.8, and the Remarks following it, we may assume $(V, W, R)$ to be in $\mathbb{V}^*$, with $V$ finite in the sense of $\mathbb{V}^*$, equipping $V$ with the nonstandard Keisler counting measure $\mu$.
By Lemmas 0.7 and 0.8, and the Remarks following it, we may assume \((V, W, R)\) to be in \(\mathbb{V}^*\), with \(V\) finite in the sense of \(\mathbb{V}^*\), equipping \(V\) with the nonstandard Keisler counting measure \(\mu\).

Moreover clause (iii)' in Lemma 0.8, holds, namely any formula of set theory true of \((V, W, R)\) in \(\mathbb{V}^*\) is true of infinitely many \((V_N, W_N, R_N)\) in \(\mathbb{V}\).
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As \(k\)-stability is expressed by a sentence (in the language of bipartite graphs), it follows that \((V, W, R)\) is \(k\)-stable, in particular stable.
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As \(k\)-stability is expressed by a sentence (in the language of bipartite graphs), it follows that \((V, W, R)\) is \(k\)-stable, in particular stable.

So Lemma 0.17 can be applied, for \(\epsilon/2\), yielding some partitions \(V = V_1 \cup \ldots \cup V_n\), and \(W = W_1 \cup \ldots \cup W_m\) (into definable sets, so sets internal in \(\mathbb{V}^*\)) such that for each \(i, j\), either for all \(b \in W_j\), \(\mu(V_i \setminus R(x, b)) \leq (\epsilon/2)\mu(V_i)\), or \(\mu(V_i \cap R(x, b)) \leq (\epsilon/2)\mu(V_i)\).
Proof of stable regularity lemma III

- Remember that $\mu(Z)$ for $Z$ any definable subset of $V$, is the standard part of $|Z|/|V|$ (where cardinality is computed in $\mathbb{V}^*$).
Proof of stable regularity lemma III

- Remember that \( \mu(Z) \) for \( Z \) any definable subset of \( V \), is the standard part of \( |Z|/|V| \) (where cardinality is computed in \( V^* \)).

- It follows easily that, (*) for each \( i, j \), either (a) for all \( b \in W_j \), \( |V_i \setminus R(x, b)| \leq \epsilon |V_i| \), or (b) for all \( b \in W_j \), \( |V_i \cap R(x, b)| \leq \epsilon |V_i| \).
Remember that $\mu(Z)$ for $Z$ any definable subset of $V$, is the standard part of $|Z|/|V|$ (where cardinality is computed in $\mathbb{V}^*$).

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(*) for each $i, j$, either (a) for all $b \in W_j$, $|V_i \setminus R(x, b)| \leq \epsilon|V_i|$, or (b) for all $b \in W_j$, $|V_i \cap R(x, b)| \leq \epsilon|V_i|$.

All this (the partitions, and the property (*) in the last item) is expressed by a formula of set theory, true in $\mathbb{V}^*$ of the data $(V, W, R)$. 
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- All this (the partitions, and the property (*) in the last item) is expressed by a formula of set theory, true in $\mathbb{V}^*$ of the data $(V, W, R)$.

- By clause (iii)' of Lemma 0.8, mentioned above, the formula is true of infinitely many of the $(V_N, W_N, R_N)$ in $\mathbb{V}$. 
Proof of stable regularity lemma III

- Remember that $\mu(Z)$ for $Z$ any definable subset of $V$, is the standard part of $|Z|/|V|$ (where cardinality is computed in $\mathbb{V}^*$).

- It follows easily that, (*) for each $i, j$, either (a) for all $b \in W_j$, $|V_i \setminus R(x, b)| \leq \epsilon|V_i|$, or (b) for all $b \in W_j$, $|V_i \cap R(x, b)| \leq \epsilon|V_i|$.

- All this (the partitions, and the property (*) in the last item) is expressed by a formula of set theory, true in $\mathbb{V}^*$ of the data $(V, W, R)$.

- By clause (iii)' of Lemma 0.8, mentioned above, the formula is true of infinitely many of the $(V_N, W_N, R_N)$ in $\mathbb{V}$.

- Now choose $N \geq n, m$. So there are partitions $V_{N,1} \cup .. \cup V_{N,n}$ of $V_N$ and $W_{N,1} \cup .. \cup W_{N,m}$ of $W_N$ with the property (*).
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So in fact we have decompositions of the vertices of $(V_N, W_N, R_N)$ into $\leq N$ pieces such that each of the induced subgraphs is $\epsilon$-homogeneous, which is a contradiction to our assumption about $(V_N, W_N, R_N)$.

This contradiction ends the proof of Theorem 0.15.

The proof can also be modified slightly to yield that in Theorem 0.15 the $V_i$ can be defined by $\phi$-formulas and the $W_j$ by $\phi^*$-formulas.
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Proof of stable regularity lemma IV

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Distal regularity I

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Among the motivations was to place existing results of combinatoricists (Jacob Fox et al ...) in a general model theoretic context, so not exactly a really new contribution to combinatorics.

We still focus on the bipartitite case although a lot of work goes on in the unipartitite case. The context studied by combinatorics people was *semialgebraic graphs*, namely graphs $G = (V, W, R)$ definable in the structure $(\mathbb{R}, +, \times)$.

For such a fixed such semialgebraic graph $G$, one can consider the family of finite graphs $(V', W', R|(V' \times W'))$ as $V'$, $W'$ range over finite subsets of $V, W$ respectively.
Strong Erdös-Hajnal (which is a theorem in this situation) says that there is $\delta$ depending on $G$ such that for each such finite $V', W'$ there are $V_0 \subseteq V'$ and $W_0 \subseteq W'$, with $|V_0| \geq \delta|V'|$ and $|W_0| \geq \delta|W'|$, such that $V_0, W_0$ is homogeneous for $R$. 

The closely related strong regularity theorem, provides, given $\epsilon > 0$ some $N_\epsilon$ such that for every finite $V', W'$ there is a decomposition $V' = V_1 \cup \ldots \cup V_n, W' = W_1 \cup \ldots \cup W_m$ with $m, n < N\epsilon$ such that outside a small exceptional set $\Sigma$ of pairs $(i, j)$, each $V_i, W_j$ is outright homogeneous for $R$.

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The distal theorems give the same results but replacing the structure $(\mathbb{R}, +, \times)$ by any first order structure $M$ such that $Th(M)$ is distal (and our pseudofinitite formalism adapts well to this set-up).
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Examples of distal first order theories are $RCF$ (more generally $o$-minimal theories), $Th(\mathbb{Q}_p, +, \times)$, $Th(\mathbb{Z}, +, <)$, $RCVF$ (real closed valued fields).

The theory of algebraically closed valued fields is an important unstable $NIP$ theory, but is not distal because the residue field is stable (in the correct sense of a sort or definable set in an ambient theory being stable).
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The theory of dense pairs of real closed fields is unstable, $NIP$, but not distal (for subtle reasons that I have forgotten).

A characterization of distality which is convenient for our purposes is:

**Definition 0.18**

A (complete) theory is distal if $T$ is $NIP$ and every generically stable Keisler measure is smooth.
(This is stuff from more than 10 years ago ...) We have already alluded to smooth Keisler measures, but let us repeat the formal definition. As usual the context is a complete theory $T$ etc.
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**Definition 0.19**

Let $\mu(x)$ be a Keisler measure over a model $M$. $\mu$ is said to be **smooth** if $\mu(x)$ has a unique extension to a Keisler measure $\mu'(x)$ over $\bar{M}$ (equivalently over any elementary extension of $M$).
Smooth and generically stable measures I

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- Before defining generically stable measures, let us remark on how established notions for types generalize to measures.
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- We could also restrict the notion of smoothness to Keisler $\phi$-measures, in the obvious way.
- Before defining generically stable measures, let us remark on how established notions for types generalize to measures.
- For some of these definitions a global assumption that $T$ has $NIP$ may be useful.
Let $\mu(x)$ be a Keisler measure over a model $M$ and let $A \subseteq M$, $M_0 \prec M$. 

We say that $\mu(x)$ does not fork (divide) over $A$ if whenever $\phi(x,b)$ is over $M$, and $\mu(\phi(x,b)) > 0$ then $\phi(x,b)$ does not fork (divide) over $A$.

We say that $\mu(x)$ is finitely satisfiable in $M_0$, if whenever $\phi(x,b)$ is over $M$ and $\mu(\phi(x,b)) > 0$, then $\phi(x,b)$ is realized by an element (tuple) of $M_0$.

Assume $M$ is $|A|$-saturated. We say that $\mu$ is definable over $A$ if for every $L$-formula $\phi(x,y)$, and closed set $C \subseteq [0,1]$, \{ $b \in M$: $\mu(\phi(x,b)) \in C$ \} is "type-definable" over $A$.
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Note that these definitions agree with the usual ones when $\mu(x)$ is a complete type.
Let \( \mu(x) \) be a Keisler measure over a model \( M \) and let \( A \subseteq M, M_0 \prec M \).

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Assume $M$ is $|A|^+$-saturated. We say that $\mu$ is definable over $A$ if for every $L$-formula $\phi(x, y)$, and closed set $C \subseteq [0, 1]$, $\{b \in M : \mu(\phi(x, b)) \in C\}$ is “type-definable” over $A$. (explain..).
Let $\mu(x)$ be a Keisler measure over a model $M$ and let $A \subseteq M$, $M_0 \prec M$.

We say that $\mu(x)$ does not fork (divide) over $A$ if whenever $\phi(x,b)$ is over $M$, and $\mu(\phi(x),b)) > 0$ then $\phi(x,b)$ does not fork (divide) over $A$.

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Note that these definitions agree with the usual ones when $\mu(x)$ is a complete type.
Let us remark for interested members of the audience that measures behave similarly to types with respect to forking if \( T \) is \( NIP \).

Namely, assume \( T \) is \( NIP \), and \( \mu \) is a Keisler measure over \( \bar{M} \). Then \( \mu \) does not fork over \( M_0 \) iff \( \mu \) is \( Aut(\bar{M}/M_0) \)-invariant.

Definition 0.20
(Assume \( T \) is \( NIP \)). Let \( \mu(x) \) be a Keisler measure over a model \( M \). We say that \( \mu \) is **generically stable** if \( \mu \) has an extension \( \mu'(x) \) over \( \bar{M} \) which is both definable over \( M \) and finitely satisfiable in \( M \) (and in fact \( \mu' \) turns out to be the unique global nonforking extension of \( \mu \)).
We have a nice alternative characterization of generically stable measures; a strong form of the VC-theorem.
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**Lemma 0.21**

(Assume $T$ NIP.) Let $\mu(x)$ be a Keisler measure over $M$. The following are equivalent:

(i) $\mu$ is generically stable,

(ii) For any $L$-formula $\phi(x, y)$, and $\epsilon > 0$, there are $a_1, \ldots, a_n$ in $M$ such that for any $b \in M$, $\mu(\phi(x, b))$ is within $\epsilon$ of the proportion of $a_i$ which satisfy $\phi(x, b)$. 
One source of generically stable measures (in an \textit{NIP} theory) is so-called average measures: let \( I = (a_i : i \in [0, 1]) \) be an indiscernible “segment” in a model \( M \) and for \( \phi(x) \) over \( M \), define \( \mu_I(\phi(x)) \) to be the Lebesgue measure of \( \{ i : M \models \phi(a_i) \} \). This makes sense, because \( \phi(x, y) \) being \textit{NIP}, the set of \( \{ i \in [0, 1] : M \models \phi(a_i) \} \) is a finite union of points and convex sets, hence finite unions of points and intervals, so measurable.
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For an $NIP$ formula $\phi(x, y)$, there should be (and maybe already is) a good theory of generically stable $\phi$-types (as well as a notion of $\phi$-distality), which would help place subsequent results and proofs in a formula-by-formula context.
Distality theorems I

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- As expected the proofs involve proving theorems about single bipartitite graphs definable in a model of a distal theory, which will be almost tautological, and then applying the pseudofinite stuff.
- We first give our version of distal regularity.
Theorem 0.22
Given $\mathcal{G}$, suppose that one of the following happens:

- (i) The graphs in $\mathcal{G}$ are uniformly definable in some model $M$ of a distal theory,

- (ii) For some model $M$ of a distal theory $\mathcal{T}$, there is a graph $(V,W,R)$ definable in $M$ such that every graph in $\mathcal{G}$ is a finite (induced) subgraph of $(V,W,R)$,

- (iii) Every model of the common theory of the $\mathcal{G}_i$'s (in the language of bipartite graphs) is definable in some model of some distal theory.

Then for any $\epsilon > 0$ there is $N_\epsilon$ such that for every $(V,W,R) \in \mathcal{G}$ there are partitions $V_1,..,V_n$ of $V$ and $W_1,..,W_m$ of $W$ with $n,m \leq N_\epsilon$ such that outside a small exceptional set of pairs $(i,j)$, each pair $V_i,W_j$ is homogeneous for $R$. 
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1. The graphs in \( \mathcal{G} \) are uniformly definable in some model \( M \) of a distal theory,
2. For some model \( M \) of a distal theory \( T \), there is a graph \( (V, W, R) \) definable in \( M \) such that every graph in \( \mathcal{G} \) is a finite (induced) subgraph of \( (V, W, R) \), or
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THEN for any $\epsilon > 0$ there is $N_\epsilon$ such that for every $(V,W,R) \in \mathcal{G}$ there are partitions $V_1, \ldots, V_n$ of $V$ and $W_1, \ldots, W_m$ of $W$ with $n, m \leq N_\epsilon$ such that outside a small exceptional set of pairs $(i,j)$, each pair $V_i, W_j$ is homogeneous for $R$. 
Distality theorems III

- So in comparison with the conclusion of Szemeredi regularity, Theorem 0.22 has the improved conclusion of outright homogeneity in place of $\epsilon$-regularity, but the small error (exceptional set) is still there (and cannot be done without).

Note that with assumption (ii), 0.22 recovers the Fox et al results.

Our strong Erdos-Hajnal theorem has the same assumptions as in Theorem 0.22, but the conclusion is that there is $\delta > 0$ such that for each $(V, W, R)$ in $G$ there are $V_0 \subseteq V$, $W_0 \subseteq W$ with $|V_0| \geq \delta |V|$ and $|W_0| \geq \delta |W|$ such that $V_0, W_0$ is homogeneous for $R$. This clearly follows from Theorem 0.22.
So in comparison with the conclusion of Szemerédi regularity, Theorem 0.22 has the improved conclusion of outright homogeneity in place of $\epsilon$-regularity, but the small error (exceptional set) is still there (and cannot be done without).

In comparison with the conclusion of the stable regularity lemma, we have the improvement of homogeneity instead of $\epsilon$-homogeneity, but on the other hand the small error (exceptional set), in place of no exceptional set.
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In comparison with the conclusion of the stable regularity lemma, we have the improvement of homogeneity instead of $\epsilon$-homogeneity, but on the other hand the small error (exceptional set), in place of no exceptional set.

Note that with assumption (ii), 0.22 recovers the Fox et al results.

Our strong Erdos-Hajnal theorem has the same assumptions as in Theorem 0.22, but the conclusion is that there is $\delta > 0$ such that for each $(V, W, R)$ in $G$ there are $V_0 \subseteq V$, $W_0 \subseteq W$ with $|V_0| \geq \delta|V|$ and $|W_0| \geq \delta|W|$ such that $V_0, W_0$ is homogeneous for $R$. This clearly follows from Theorem 0.22.
Our proof of Theorem 0.22 will use a couple of results, first a regularity theorem for arbitrary definable graphs \((V, W, R)\) equipped with Keisler measures on \(V, W\), at least one of which is smooth, which we do in this section. The other, discussed later is the fact that in the \(NIP\) environment the pseudofinite counting measure is generically stable (which follows from the Vapnik-Chervonenkis theorem).
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We start with a basically immediate “domination” statement for smooth measures in arbitrary theories.
Lemma 0.23

(T an arbitrary theory.) Let $\mu(x)$ be a Keisler measure over a model $M_0$ on the sort $X$. Suppose $\mu$ to be smooth. Let $\mu$ also denote the induced (Borel probability) measure on $S_X(M_0)$. And let $\pi : X = X(\bar{M}) \to S_X(M_0)$ be the tautological map $\pi(a) = tp(a/M_0)$. Then for every definable (with parameters from $\bar{M}$) subset $Y$ of $X$, there is a closed subset $E$ of $S_X(M_0)$ of $\mu$-measure 0, such that for all $p \in S_X(M_0)$ such that $p \notin E$, either $\pi^{-1}(p) \subset Y$ or $\pi^{-1}(p) \cap Y = \emptyset$. 
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Then for every definable (with parameters from $\bar{M}$) subset $Y$ of $X$, there is a closed subset $E$ of $S_X(M_0)$ of $\mu$-measure 0, such that for all $p \in S_X(M_0)$ such that $p \notin E$, either $\pi^{-1}(p) \subset Y$ or $\pi^{-1}(p) \cap Y = \emptyset$. 
Proof.

- We make use of some basic manipulations around extending measures.
Regularity theorem for smooth measures III

Proof.

▶ We make use of some basic manipulations around extending measures.

▶ Let $E$ be the (closed) subset of $S_X(M_0)$ consisting of those $p$ which are consistent with both $x \in Y$ and $x \notin Y$. 

▶ Suppose, for a contradiction, that $\mu(E) > 0$. Then let $(\mu_E)$ denote the localization of $\mu$ at $E$, namely as a measure on $S_X(M_0)$, $(\mu_E)(B) = \mu(B \cap E)/\mu(E)$ for Borel $B$.

▶ Then $(\mu_E)$ has two different extensions to a Keisler measure over $\overline{M}$, one giving $Y$ measure 1 and one giving $Y$ measure 0.

▶ From which it follows that $\mu$ itself has two different extensions to $\overline{M}$, contradicting smoothness.
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Let $(V, W, R)$ be a graph definable in a structure $M$. Let $\mu, \nu$ be Keisler measures over $M$ on $V$, $W$, respectively, and assume that $\mu$ is smooth.
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Let $\epsilon > 0$. 

Then there are partitions $V = V_1 \cup \ldots \cup V_n$, $W = W_1 \cup \ldots \cup W_m$ into definable sets, and an “exceptional set” $\Sigma$ of indices $(i, j)$ such that

- $(\mu \times \nu)(\bigcup \{(i, j) \in \Sigma : V_i \times W_j\}) < \epsilon$, and
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For any $b \in W$, let $E_b$ be the closed $\mu$-measure 0 subset of $S_V(M)$ outside of which each fibre of $\pi$ is contained in or disjoint from $E_b$.

$E_b$ clearly only depends on $tp(b/M)$, so we write $E_b$ as $E_q$ where $q = tp(b/M)$. Let $Z_q$ be an $M$-definable set containing $E_q$ with $\mu(Z_q)$ with $\mu$-measure $< \epsilon$. 
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By compactness we can partition $V \setminus Z_q$ into $M$-definable sets $V_{q,1}, ..., V_{q,n_q}$ such that for each $i$, $\pi^{-1}(V_{q,i})$ is either contained in $R(x, b)$ for some/all $b$ realizing $q$, or is disjoint from $R(x, b)$ for some/all $b$ realizing $q$. 

By compactness we can replace $q$ by a formula (or $M$-definable set) $W_q$ in $q$ such that for all $i = 1, \ldots, n_q$, either $V_{q,i}$ is contained in $R(x, b)$ for all $b \in W_q$, or $V_{q,i}$ is disjoint from $R(x, b)$ for all $b \in W_q$. 

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Doing this for each $q \in S_W(M)$, applying compactness and possibly refining some $W_q$’s gives us $q_1, .., q_m \in S_W(M)$, and a partition $W = W_{q_1}, .., W_{q_m}$ into $M$-definable sets (with $W_{q_j} \in q_j$), and
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It has the following consequence for Keisler measures:

Suppose $\mu(x)$ is a Keisler measure over $M$. Let $\phi(x, y)$ be an $L$-formula which has $k$-$NIP$.

Then for any $\epsilon$, there is $N = N_{k, \epsilon}$ depending only on $k$ and $\epsilon$, such that there are $p_1(x), \ldots, p_N(x) \in S_x(M)$, such that for all $b \in M$, $\mu(\phi(x, b))$ is within $\epsilon$ of the proportion of the $p_1, \ldots, p_N$ which contain $\phi(x, b)$. 
In the special case when $A$ is a finite set of tuples from $M$ of the appropriate length, and $\mu = \mu_A$ is the counting measure with respect to $A$ (which we could recall), then this says that there are $a_1, \ldots, a_N \in A$ such that for all $b \in M$, $\mu(\phi(x, b))$ is within $\epsilon$ of the proportion of the $a_1, \ldots, a_N$ which satisfy $\phi(x, b)$. 

We conclude the following:

Lemma 0.25
Suppose $M$ is a model of an $NIP$ theory, $A$ is a subset of $X(M)$ for some sort $X$, $A$ is pseudofinite in $M$, $(M, A)$ is saturated (?), and $\mu(x)$ is a pseudofinite counting measure on $X(M)$ (over $M$) given after Lemma 0.8. Then $\mu$ is generically stable.
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Proof of Lemma 0.25.

So we know (from Lemma 0.7 and the construction) that $\mu(Z)$ is the standard part of $|Z \cap A|/|A|$ for $Z$ a definable subset of $X(M)$, and where $|\cdot|$ denotes cardinality in $V^*$ (which is finite in the sense of $V^*$ for $A$ and its internal subsets).

On the other hand, every sentence of set theory true of $(M,A)$ in $V^*$ is true of some $(M',A')$ in $V$ with $A'$ finite.

Fix a formula $\phi(x,y)$ of $L$ which we know has $k$-NIP in $M$, for some $k$, so we may assume that in every relevant $(M',A')$ with $A'$ finite, $\phi(x,y)$ has $k$-NIP in $M'$.

So fixing $\epsilon > 0$ and letting $N = N_{k,\epsilon}/2$ be as above, it follows that there are $a_1, \ldots, a_N$ in $A$ such that for any $b \in M$, $|\phi(x,b)(M) \cap A|/|A|$ is within $\epsilon/2$ of the proportion of the $a_i$ which satisfy $\phi(x,b)$ in $M$. 
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So we know (from Lemma 0.7 and the construction) that \( \mu(Z) \) is the standard part of \( |Z \cap A|/|A| \) for \( Z \) a definable subset of \( X(M) \), and where \( |.| \) denotes cardinality in \( \mathbb{V}^* \) (which is finite in the sense of \( \mathbb{V}^* \) for \( A \) and its internal subsets).
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So fixing \( \epsilon > 0 \) and letting \( N = N_{k, \epsilon/2} \) be as above, it follows that there are \( a_1,..,a_N \) in \( A \) such that for any \( b \in M \), 
\[
|\phi(x, b)(M) \cap A| / |A| \text{ is within } \epsilon/2 \text{ of the proportion of the } a_i \text{ which satisfy } \phi(x, b) \text{ in } M.
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So for each $b \in M$, $\mu(\phi(x, b))$ is within $\epsilon$ of the proportion of the $a_i$ which satisfy $\phi(x, b)$ in $M$. 

By Lemma 0.21, $\mu$ is generically stable, completing the proof of Lemma 0.25.

Assuming that we have a good notion of generically stable $\phi$-measure where $\phi(x, y)$ is a NIP formula, then the proof above will show that a pseudofinite counting measure, restricted to a NIP formula $\phi(x, y)$, will be generically stable.
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By Lemma 0.21, $\mu$ is generically stable, completing the proof of Lemma 0.25.

Assuming that we have a good notion of generically stable $\phi$-measure where $\phi(x, y)$ is a $NIP$-formula, then the proof above will show that a pseudofinite counting measure, restricted to a $NIP$-formula $\phi(x, y)$, will be generically stable.
We prove Theorem 0.22. We will give the proof under assumption (ii) which is the context of the combinatoricists results on semialgebraic graphs, as well as the Chernikov-Starchenko theorem.

But let us note in passing that assumption (i) would be vacuous in distal theories such as $\mathbf{RCF}$ or Presburger, as there are uniform bounds on the cardinality of finite uniformly definable sets. But not vacuous for the theory of the $p$-adics.

As in the proof of stable regularity, assume the conclusion fails. So there is $\epsilon > 0$ such that for every $N$ there is a finite induced subgraph $(V'_N, W'_N, R_N)$ for which there is no suitable partition (into at most $N$ sets).

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Proof of distal regularity I

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- We may assume that at least the cardinalities of the $V_N$ are strictly increasing.
Add new predicates $P$ and $Q$ for the distinguished finite subsets of $V, W$, to get a family of $L(P, Q)$ structures, and as usual take a saturated model of the common $L(P, Q)$-theory of the $(M, V_N, W_N)$. 
Proof of distal regularity II

- Add new predicates $P$ and $Q$ for the distinguished finite subsets of $V, W$, to get a family of $L(P, Q)$ structures, and as usual take a saturated model of the common $L(P, Q)$-theory of the $(M, V_N, W_N)$.

- Call this model $(M^*, V^*, W^*)$ (where $V^*$, $W^*$ are pseudofinite subsets of $V(M^*), W(M^*)$).

- Both $V^*$, and $W^*$ induce the pseudofinite counting measures $\mu$, $\nu$, on $V(M^*), W(M^*)$ respectively.
Proof of distal regularity II

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- Fix $\epsilon$ and apply Lemma 0.24 with $\epsilon/2$ to $(V(M^*), W(M^*), R(M^*))$ equipped with $\mu$ and $\nu$, to get a partitions of size $n, m$ of the vertex sets with the appropriate properties.
Apply Lemma 0.8 to obtain \((M, V_N, W_N)\) satisfying the appropriate formulas of set theory in \(\mathbb{V}\), to get a contradiction, as in the proof of the stable regularity lemma.
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Note that there is a difference with the stable proof, as the \(V_N, W_N\) etc are not in the language \(L\).
Remarks on the $NIP$ case I

- There is an almost identical version of Theorem 0.22 for $NIP$ theories.

- The assumptions are weakened by replacing "distal theory" everywhere by "$NIP$ theory".

- And the conclusion is weakened by replacing "homogeneous" by $\epsilon$-homogeneous.

- The analogue of the regularity theorem for smooth measures (Lemma 0.24) is a regularity theorem for generically stable measures (in an ambient $NIP$ theory) where (ii) in the conclusion is replaced by an $\epsilon$-homogeneity statement (but involving additional machinery including nonforking products of measures).

- And the "compact domination" statement for smooth measures (Lemma 0.23) on which 0.24 depends is replaced by a "generic compact domination" statement for generically stable measures.
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- First, given a Keisler measure $\mu(x)$ over a model $M$, and a partial type $\Sigma(x)$ over $M$, we say that $\Sigma(x)$ is $\mu$-random (the expression $\mu$-wide is also used), if every finite conjunction of formulas in $\Sigma$ has positive $\mu$-measure.
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**Lemma 0.26**

*Suppose $T$ is NIP and $\mu(x)$ is a Keisler measure on a sort $X$ over a model $M_0$, such that $\mu|\bar{M_0}$ is generically stable. Let $\pi: X = X(\bar{M}) \rightarrow S_X(M_0)$ be as before. Let $Y \subseteq X$ be definable over $\bar{M}$. Then there is closed set $E \subseteq S_X(M_0)$ of $\mu$-measure 0 such that all $p(x) \in S_X(M_0) \setminus E$, exactly one of $p(x) \cup \{x \in Y\}$ and $p(x) \cup \{x \not\in Y\}$ is $\mu$-random.*
Finally there is a regularity lemma just for finite bipartite graphs \((V, W, R)\) for which the edge relation \(R\) is \(k\)-NIP, or equivalently, as we have mentioned earlier, which omit a fixed induced subgraph. This is again proved by the combinatoricists, and in fact is a celebrated theorem of Lovasz-Szegedy, if I am not mistaken, and implies the results above.

Also proved later by Chernikov and Starchenko with model-theoretic methods.

This could be obtained by our methods, given a generic compact domination theorem for generically stable \(\phi\) measures where \(\phi(x,y)\) is NIP.

In any case the regularity lemma alluded to above, still has the exceptional pairs, but has \(\epsilon\)-homogeneity rather than \(\epsilon\)-regularity.
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In contrast, in the group case, under the assumption $k$-$NIP$ of the relation $xy \in X$, we obtain new theorems. The methods involve structural results in local $fsg$-group theory (and local stable group theory in the $k$-stable case).
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First what can be said in general?

In all the work by combinatoricists on this problem, there is a blanket assumption that $G$ is commutative, probably so as to be able to use Fourier analytic methods.
As mentioned in the introduction from \((G, X)\) we obtain a bipartitite graph \((G, G, R)\) where \(R(x, y)\) iff \(xy \in X\), so one would expect some improved statement of Szemeredi regularity in which the group structure is respected in some sense.
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However when restricted to the class of finite-dimensional vector spaces over \(\mathbb{F}_2\) (equipped with a distinguished subset \(X\)), it yields the following:
Theorem 0.27

For every $\epsilon$ there is $N$ such that for all $(G, X)$ (where $G = \mathbb{F}_2^n$ for some $n$), there is a partition of $G$ into cosets $H + 0, H + g_1, \ldots, H + g_k$ with respect to a subgroup (vector subspace) $H$ of $G$ of index at most $N$, such that outside a small exceptional set of pairs, each graph $(H + g_i, H + g_j, R|((H + g_i) \times (H + g_j)))$ is $\epsilon$-regular. (where remember the graph relation $R(x, y)$ is $x + y \in X$).
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Terry and Wolf (TW) in 2017 considered the case where $G = \mathbb{F}_p^n$ for $p$ fixed, AND where the relation $x + y \in X$ is $k$-stable, obtaining stronger structural results; $X$ is almost a union of cosets of a subspace of small index.
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▶ Alon, Fox, and Zhao, subsequently considered the case where $G$ is (finite) abelian and $x + y \in X$ is $k$-NIP.
With Conant and Terry, we considered first arbitrary \((G, X)\) where \(G\) is arbitrary (not necessarily abelian) and \(xy \in X\) is \(k\)-stable, and then the more general case where \(xy \in X\) is \(k\)-NIP.

The thrust is that \(X\) is close to being a union of translates of a nice subobject (a subgroup or a “Bohr neighbourhood”).

One cannot expect such kinds of results in general, even when \(G = \mathbb{F}_n^2\).

However, we do have a general rather soft “coset regularity” statement (for arbitrary \((G, X)\)), which we may give later.

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We will now use $A$ rather than $X$ to denote the distinguished subset of $G$. 

And we will define $A$ to be $k$-stable if the relation $xy \in A$ is. Likewise for $k$-$NIP$.

The stable case yields a strong and transparent statement:

**Theorem 0.28**

Fix $k$. For any $\epsilon > 0$ there is $N$ depending on $\epsilon$ (and $k$) such that for any pair $(G,A)$ where $G$ is a finite group and $A$ is a $k$-stable subset, there is a normal subgroup $H$ of $G$ of index at most $N$, such that:

1. For each coset $C$ of $H$ in $G$, either $|C \setminus A| \leq |H|$ (or $|C \cap A| \leq |H|$.

Moreover:

2. There is a union $Y$ of cosets of $H$ such that $A = Y$ up to a set of cardinality $\leq \epsilon |H|$. 


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2. *(ii) There is a union \( Y \) of cosets of \( H \) such that \( A = Y \) up to a set of cardinality \( \leq \epsilon|H| \).*
When $A$ is $k$-NIP, and $G$ is of bounded exponent, we obtain the same conclusion, but now with an exceptional set of cosets of $H$.

**Theorem 0.29**

*Fix $k$ and $r$. Then for any $\epsilon > 0$ there is $N$ such that for any pair $(G, A)$ where $G$ is a finite group of exponent $\leq r$ and $A$ is a $k$-NIP subset of $A$, there is a normal subgroup $H$ of $G$ of index at most $N$, and a union $Z$ of cosets of $H$ (the exceptional set) with $|Z| \leq \epsilon|G|$ such that*
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In general a Bohr neighbourhood of an (abstract) group $G$ is the preimage of a neighbourhood of the identity $U$ of a compact group $L$ with respect to a homomorphism $\pi : G \to L$ (and sometimes $\pi$ is assumed to have dense image in $L$, although this only makes sense when $G$ is infinite).
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For certain reasons to be discussed later we will be interested only in homomorphisms from $H$ to $\mathbb{T}^n$, where $\mathbb{T}^n$ is the $n$-dimensional torus, i.e. the $n$-fold product of the circle group.
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In fact the $\mathbb{T}^n$’s are precisely the compact connected commutative Lie groups.

So we define an $(\epsilon, n)$-Bohr neighbourhood of a (possibly finite) group $H$ to be the preimage of the open ball of radius $\epsilon$ around the identity under a homomorphism $\pi : H \to \mathbb{T}^n$. 
Theorem 0.30

Fix $k$. Then for any $\epsilon > 0$, there is $N$ (depending on $\epsilon$ and $k$) such that for any pair $(G, A)$ where $G$ is a finite group and $A$ is a $k$-NIP subset of $G$, there are

- a normal subgroup $H$ of $G$ of index at most $N$,
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- a normal subgroup \( H \) of \( G \) of index at most \( N \),
- a \((\delta, r)\)-Bohr neighbourhood \( B \) in \( H \) for some \( r \leq N \) and \( \delta \geq 1/N \), and
- a subset \( Z \subseteq G \) with \( |Z| \leq \epsilon |G| \) (exceptional set), such that
  - (i) for any \( g \in G \setminus Z \), either \( |gB \setminus A| \leq \epsilon |B| \) or \( |gB \cap A| \leq \epsilon |B| \), and moreover
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(ii) there is a union $Y$ of translates of $B$ such that $A$ is equal to $Y$ up to a set of cardinality $\leq \epsilon |B|$, after throwing away $Z$. 
This is a recent (2019) observation by us, which is relatively soft, but yields Green’s Theorem 0.27 for example.

Let $G$ be a finite group, and $A$ a subset. Let $H$ be a subgroup of $G$ and $C$ a coset of $H$ in $G$. We say that $C$ is $\epsilon$-coset-regular for $A$, if for sufficiently large subgroups $K$ of $H$ and coset $D$ of $K$ in $G$ such that $D \subseteq C$, the density $|A \cap C|/|C|$ is within $\epsilon$ of the density $|A \cap D|/|D|$.

Sufficiently large means that $|K| \geq \epsilon |H|$.

This is a natural notion of regularity of a coset $C$ of a subgroup $H$ of $G$ with respect to $A$, but where we only consider the densities with respect to large subsets of $C$ which are themselves cosets of subgroups.
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Theorem 0.31

For any $\epsilon$ there is $N$, such that if $(G, A)$ is any pair consisting of a finite group $G$ and a subset $A$, then there is a normal subgroup $H$ of index at most $N$, and a union $Z$ of cosets of $H$ (the exceptional set) with $|Z| \leq \epsilon|G|$, such that for any coset $C$ of $H$ in $G$ such that $C$ is not contained in $Z$, then $C$ is $\epsilon$-coset-regular with respect to $A$.
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Note that when $G$ is simple (noncommutative), Theorem 0.31 says that $G$ is itself $\epsilon$-coset regular. But anyway Theorem 0.31 is only meaningful when $G$ has a reasonable supply of subgroups.
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We fix a group $G$ definable in a model $M$ (to work in some degree of generality), as well as a $L$-formula $\delta(x, y)$, $x$ ranging over $G$ and $y$ over some other sort (maybe tuples from $G$).
The stable case I

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- We assume that $\delta(x, y)$ is “left invariant” meaning that for any $b \in M$, and $g \in G$, the left translate by $g$ of the subset of $G$ defined by $\delta(x, b)$ is defined by $\delta(x, c)$ for some $c$. 

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As before a $\delta$-formula (over $M$) is a Boolean combination of formulas $\delta(x,b)$ for $b \in M$, and the subset of $G$ it defines is called a $\delta$-definable set. (We treat $x = x$, $x \neq x$ as degenerate $\delta$-formulas, and sometimes we may want to include Boolean combinations of $x = g$ etc. too....)
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A $\delta$-definable subset $X$ of $G$ is said to be (left) generic, if finitely many left translates of $X$ (by elements of $G$ of course) cover $G$. 
The stable case II

- Note that by our assumptions on \( \delta \) the class of \( \delta \)-definable subsets of \( G \) is closed under left translation by elements of \( G \).
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- A type \( p(x) \in S_\delta(M) \) is called generic (or a \( \delta \)-generic type of \( G \)) if it only contains generic formulas.
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With this notation and assumptions, here is the fundamental theorem of local stable group theory.
The stable case III

Theorem 0.32

$G$ has a smallest $\delta$-definable subgroup of finite index which we call $G_\delta^0$. 

(ii) The $\delta$-generic types of $G$ are in one-one-correspondence with the left cosets of $G_\delta^0$, namely each left coset of $G_\delta^0$ is (as a $\delta$-formula) contained in a unique $\delta$-generic type of $G$.

(iii) There is a unique left-invariant (Keisler) $\delta$-measure on $G$, $\mu$ say, and moreover

(iv) for any $\delta$-definable set $X$, $\mu(X) > 0$ iff $X$ is generic.
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- (iv) for any \( \delta \)-definable set \( X \), \( \mu(X) > 0 \) iff \( X \) is generic.
Corollary 0.33

(In the same context as that of Theorem 0.32, and the same notation.)

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Let $X$ be a $\delta$-definable subset of $G$. Then

1. For each left coset $C$ of $G^0_\delta$, either $\mu(C \setminus X) = 0$, or $\mu(C \cap X) = 0$. 

2. $X$ is a union of left cosets of $G^0_\delta$ up to a set of $\mu$-measure 0.
Corollary 0.33

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- (i) For each left coset $C$ of $G^0_\delta$, either $\mu(C \setminus X) = 0$, or $\mu(C \cap X) = 0$.

- (ii) $X$ is a union of left cosets of $G^0_\delta$ up to a set of $\mu$-measure 0.
To go from Theorem 0.32 and Corollary 0.33 to Theorem 0.28, we take $\delta(x, y)$ to be the formula $yx \in A$ which is by assumption $k$-stable and left invariant in the finite $(G, A)$. Passing to the limit, i.e. taking some saturated infinite model $(G, A)$ of some collection of the finite $(G_i, A_i)$ will then preserve $k$-stability, so stability of $\delta(x, y)$, whereby 0.32 and 0.33 can be applied. But the crucial point is that the pseudofinite Keisler measure on $G$ (coming from $V^*$) will be left invariant, hence by the uniqueness aspect of Theorem 0.32(iii), must coincide on $\delta$-definable sets with the $\delta$-measure $\mu$ from Theorem 0.32. This allows us to pull Theorem 0.32 and Corollary 0.33 to the finite (of course using some approximations) and obtain Theorem 0.28. Note that Theorem 0.28 also implies that $k$-stable sets in finite simple groups better be (asymptotically) either almost everything or almost nothing.
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Note that Theorem 0.28 also implies that $k$-stable sets in finite simple groups better be (asymptotically) either almost everything or almost nothing.
We saw in the discussion at the end of the last section that up to small cardinality, suitable subsets of the finite groups $G$ are controlled by bounded index subgroups, i.e. all the action is going on in $G/H$ for some bounded index subgroup $H$. A slight variant of this will actually be the case in general. That is, roughly speaking, for an infinite pseudofinite group $G$ with its pseudofinite Keisler measure $\mu$, internal sets of positive measure will be controlled in a sense by a compact (rather than finite) quotient $G/G^{00}$. And passing to approximations, this will be reflected in various ways in the finite. It is a rather surprisingly important role for those compact group, although variants are also behind the classification of approximate subgroups.
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Bounded index means of index at most $\leq 2^{|M|+|L|}$, which can be shown to be equivalent to $< \kappa$ where $\kappa$ is the degree of saturation of $\bar{M}$. 
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In fact, because of the bounded index assumption, the coset of $g$ modulo $(G^*)_0^0_M$ depends only on $tp(g/M)$, whereby the canonical homomorphism from $G^*$ to $(G^*)_0^0_M$ factors through the tautological map to the type space $S_G(M)$, and this equips $G^*/(G^*)_0^0_M$ with its compact Hausdorff topology. It is a definable groups analog of the so-called KP Galois group.
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Likewise we could consider a collection $\Delta$ of $L$-formulas $\delta(x,y)$ (or even a single such formula), and consider $(G^*)^0_{M,\Delta}$, the smallest subgroup of $G^*$ of “bounded index” defined by a collection of $\Delta$-formulas over $M$. (Not necessarily normal any more.)
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If $H$ is a compact (Hausdorff) topological group then $H$ is an inverse (or projective) limit of compact Lie groups.

In particular we have an exact sequence
$$1 \to H^0 \to H \to H/H^0 \to 1,$$
where $H^0$ denotes the connected component of the identity of $H$ as a topological group;
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So for arbitrary $M$, and $G$ definable in $M$ we also call $G^*/(G^*)^{00}_M$ the “definable Bohr compactification” of $G = G(M)$. 

Lemma 0.34
Suppose $G$ is a pseudofinite group, considered as definable in the structure $M = \mathbb{V}^*$. Then the definable Bohr compactification of $G$ is profinite-by-commutative, that is the connected component of $G^*/(G^*)^{00}_M$ (as a topological group) is an inverse limit of connected commutative compact Lie groups.
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We work in a saturated model \( \bar{M} \), and one of the main results is “generic compact domination” of \( G \) by \( G/G^{00} \).
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What this means (at least one form), is that given a definable (with parameters from $\bar{M}$) subset $Y$ of $G$, there is a closed $E_Y \subset G/G^{00}$ of (normalized) Haar measure 0, such that for all cosets $C$ of $G^{00}$ outside $E_Y$, not both “$x \in C \land x \in Y$” and “$x \in C \land x \notin Y$” are $\mu$-random.
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This implies in particular that \( \mu \) is the unique translation invariant measure on \( G \).
The desired local theory means working just with a single translation invariant $NIP$-formula $\delta(x, y)$. Together with Lemma 0.34, which explains where the Bohr neighbourhoods come from, this will suffice to prove Theorem 0.30.
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We assume for simplicity that $G_{00}^\delta$ (which exists by discussions in the previous section) is normal in $G$. Then we have:
Local generic compact domination III

- The assumptions are that $G$ (with some additional structure in a language $L$) is saturated, pseudofinite, and a group.
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- We assume for simplicity that $G_0^{00}$ (which exists by discussions in the previous section) is normal in $G$. Then we have:

**Theorem 0.35**

- (i) There is a unique left invariant Keisler $\delta$-measure $\mu$ on $G$. 

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We assume for simplicity that $G^0_{\delta}$ (which exists by discussions in the previous section) is normal in $G$. Then we have:

**Theorem 0.35**

(i) There is a unique left invariant Keisler $\delta$-measure $\mu$ on $G$.

(ii) The $\delta$-definable sets of positive $\mu$-measure are precisely the (left) generic $\delta$-definable sets.
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**Theorem 0.35**

- (i) There is a unique left invariant Keisler $\delta$-measure $\mu$ on $G$.
- (ii) The $\delta$-definable sets of positive $\mu$-measure are precisely the (left) generic $\delta$-definable sets.
- (iii) Given a $\delta$-definable (over $\bar{M}$) set $Y \subseteq G$, there is a closed subset $E_Y \subseteq G/G^{00}_\delta$, of $\mu$-measure 0 such that for $C \in G/G^{00}_\delta$, $C \notin E_Y$, exactly one of $x \in C \cup x \in Y$, $x \in C \cup x \notin Y$ is $\mu$-random (equivalently by (ii) extends to a global generic type).
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