In this note we point out that any strongly minimal pseudofinite structure (or set) is unimodular in the sense of [1], [5], [2], and hence measurable in the sense of Macpherson and Steinhorn [3], [2] as well as 1-based. The argument, involving nonstandard finite cardinalities, is straightforward. A few people asked about this issue in private conversations and communications, in particular Martin Bays - Pierre Simon, Dugald Macpherson - Charles Steinhorn (in MSRI, spring 2014), and more recently Alex Kruckman. So we thought it worthwhile to clarify the situation with a quick proof. Thanks to all the above people for discussions.

Recall the basic notions. A structure \( M \) is in language \( L \) is pseudofinite if every sentence true in \( M \) is true in some finite \( L \)-structure. Equivalently \( M \) is elementarily equivalent to an ultraproduct of finite \( L \)-structures. If \( M \) is pseudofinite and saturated say, then every definable set \( X \) in \( M \) has a “nonstandard finite cardinality” \( |X| \) which is an element of a saturated elementary extension of \((N, +, \times, <, ....)\), and the map taking \( X \) to \( |X| \) satisfies the usual properties inherited from the finite setting.

Suppose \( D = M \) is strongly minimal and saturated. \( D \) is said to be unimodular if whenever \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) are each independent \( n \)-tuples from \( D \) and \( a \in acl(b) \) (so also \( b \in acl(a) \)) then \( mlt(a/b) = mlt(b/a) \).

Definable means possibly with parameters. We refer to [5] for basics of stability, Morley rank \((RM(-))\) etc.

**Lemma 0.1.** Suppose \( D \) is strongly minimal, saturated and pseudofinite. Let \( X \) be a definable set in \( D \). Let \( b = |D| \). Then there is a polynomial \( P_X(x) \) in one variable \( x \) with (standard) integer coefficients and positive leading coefficient, such that \( |X| = P(b) \). Moreover \( RM(X) \) equals the degree of \( P_X(x) \).
Proof. This is the main point and has maybe been observed before, although I have not seen anything. We prove the Lemma by induction on $RM(X)$ also using the fact that $D^n$ has Morley rank $n$ and Morley degree 1. If $X$ is finite, then $|X| = |X|$. Suppose $RM(X) = n$ and $X \subseteq D^m$ (for some $m \geq n$). After writing $X$ as a finite disjoint union of suitable definable sets, we may assume (using the induction hypothesis) that for some projection $\pi : D^m \rightarrow D^n$, and some positive integer $t$, $\pi(X)$ has Morley rank $n$ and $\pi|X$ is $t$-to-one. So $|X| = t|\pi(X)|$. And $|\pi(X)| = |D^n| - |D^n \setminus \pi(X)|$. Now $|D^n| = b^n$, and $RM(D^n) \setminus \pi(X)$ has Morley rank $< n$. So we can apply the induction hypothesis to get the desired $P_X(x)$ and note that the leading coefficient of $P_X$ is $t > 0$. 

Now there are a few ways to proceed. We could use the pair $(RM(X), t_X)$ where $t_X$ is the leading coefficient of $P_X$ to show directly $MS$-measurability of $D$. Or directly obtain unimodularity. We will do the latter.

**Corollary 0.2.** Suppose $D$ is strongly minimal and pseudofinite. Then $D$ is unimodular.

**Proof.** We may assume $D$ is saturated. Let $a, b \in D^n$ each be generic over $\emptyset$ with $acl(a) = acl(b)$. Let $k = mlt(b/a)$ and $\ell = mlt(a/b)$. We have to prove that $k = \ell$. Let $\psi(x, y)$ be an $L$-formula such that $|= \phi(a, b), \psi(a, y)$ isolates $tp(b/a)$ and $\psi(x, b)$ isolates $tp(a/b)$. Let $\phi_1(x)$ be $\exists \exists k y(\psi(x, y))$ and $\phi_2(y)$ be $\exists \exists x(\psi(x, y))$. Let $\chi(x, y)$ be the formula $\phi(x, y) \land \phi_1(x) \land \phi_2(y)$. So $\chi(x, y)$ is true of $(a, b)$ in $D$. Let $Z \subseteq D^{2n}$ be the set defined by $\chi(x, y)$. We compute $|Z|$ in two ways. Let $X$ be the projection of $Z$ on the first $n$-coordinates, and $Y$ the projection of $Z$ on the last $n$ coordinates. Then $|Z| = k|X| = \ell|Y|$. Note that other $X$ and $Y$ have Morley rank $n$ hence by Lemma 0.1, there are polynomials $P(x)$, $Q(x)$ over $Z$ of degree $< n$ such that $|X| = b^n - P(b)$ and $|Y| = b^n - Q(b)$. If by way of contradiction $k > \ell$ we have $(k - \ell)(b^n) = kP(b) - \ell Q(b)$. This is impossible, as the right hand side is an integral polynomial of degree $< n$ in $b$, for example by considering sufficiently large standard natural numbers $b$. So the Corollary is proved.

**Remark 0.3.** (i) One can deduce by standard means that any pseudofinite theory of finite $U$-rank (i.e. every complete type has finite $U$-rank) is $1$-based. See the proof of Proposition 3.5 in [2] for example. In particular all definable groups in such a theory are abelian-by-finite.
(ii) There are examples of $\omega$-stable non abelian-by-finite pseudofinite groups in [4].

(iii) We would tentatively conjecture that any regular type in a stable pseudofinite theory is locally modular???

References


