The Helmholtz Resonator

Consider a container of volume V open to the atmosphere through a duct of length ℓ and cross-section S as shown in the figure. We assume the pressure and density inside the container to be uniform and depend only on time, i.e., $p'_{in}(t)$ and $\rho'_{in}(t)$, and we denote the pressure in the duct by $p'(x,t)$. We linearize the pressure and the density with respect to their mean atmospheric values, p_0 and ρ_0, respectively.

$$p_{in}(t) = p_0 + p'_{in}(t) \tag{1}$$

$$\rho_{in}(t) = \rho_0 + \rho'_{in}(t) \tag{2}$$

$$p(x,t) = p_0 + p'(x,t) \tag{3}$$

Air may be moving in or out of the duct at a velocity u. As air moves in(out) the fluid inside the container is compressed (expanded). We assume this process to be adiabatic and hence $p'_{in} = c_0^2 \rho'_{in}$, where c_0 is the speed of sound

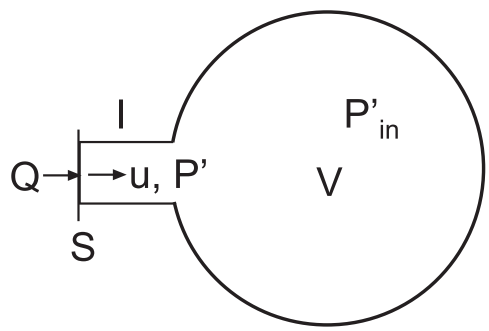

Figure 1: Helmholtz Resonator

The mass flow rate entering the container is given by

$$\dot{m} = \rho_0 S u. \tag{4}$$

As a result the density ρ'_{in} increases at the rate

$$\frac{d\rho'_{in}}{dt} = \frac{\dot{m}}{V}. \tag{5}$$

Integrating (5),

$$\rho'_{in} = \frac{1}{V} \int^t \dot{m} dt. \tag{6}$$

Expressing the pressure in terms of the density,

$$p'_{in} = c_0^2 \rho'_{in} = \frac{c_0^2}{V} \int^t \dot{m} \text{ d}t. \tag{7}$$

The velocity u and pressure p' are related by the momentum equation

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x} - F, \tag{8}$$

where F represents a resistance force due to friction or radiation effects. It is reasonable to assume that

$$F = \frac{Ru}{l}. \tag{9}$$

We now assume that the flow quantities have harmonic dependence on time, i.e., $p' = \bar{p}' e^{i\omega t}$. Equation (8) can then be written as

$$i\rho_0 \omega u + \frac{Ru}{l} = -\frac{\partial p'}{\partial x}. \tag{10}$$

The conservation of mass tells us that the air velocity in the duct is independent of x, therefore we can integrate (10) and get

$$p'_{in} = p'_0 - i\rho_0 \omega l u - Ru, \tag{11}$$

where p'_0 denotes the pressure at the duct inlet. Using (4, 7), we get

$$p'_0 = p'_{in} + i\rho_0 \omega l u + Ru = \frac{c_0^2}{V} \int^t \dot{m} \text{ dt} + i\omega l \frac{\dot{m}}{S} + \frac{R}{\rho_0 S} \dot{m}. \tag{12}$$

Differentiating (12)

$$\frac{dp'_0}{dt} = i\omega p'_0 = \dot{m}\left(\frac{c_0^2}{V} - \frac{\omega^2 l}{S} + i\frac{\omega R}{\rho_0 S}\right) \tag{13}$$

Introducing the impedance

$$Z = \frac{p'}{u} = R - i\rho_0 c_0 \left(\frac{c_0 S}{\omega V} - \frac{\omega l}{c_0}\right). \tag{14}$$

Or in non-dimensional form,

$$\zeta = \frac{Z}{\rho_0 c_0} = \tilde{R} - i\left(\frac{c_0 S}{\omega V} - \frac{\omega l}{c_0}\right), \tag{15}$$

where we have set $\tilde{R} = R/(\rho_0 c_0)$. If we introduce

$$\omega_r = c_0 \sqrt{\frac{S}{Vl}}, \tag{16}$$

we get

$$\zeta = \tilde{R} + i\frac{\omega l}{c_0}\left(1 - \left(\frac{\omega_r}{\omega}\right)^2\right). \tag{17}$$

Resonance occurs when the reactance $Im(\zeta) = 0$, i.e., $\omega = \omega_r$. The resonant frequency in Hertz is given by

$$f_r = \frac{c_0}{2\pi} \sqrt{\frac{S}{Vl}} \tag{18}$$