Rate of Change of Circulation

**Kelvin Theorem:** In an inviscid barotropic fluid, the circulation is constant along a circuit moving with the fluid.

The circulation $\Gamma$ along a circuit $C$ is defined by the integral

$$\Gamma = \int_C V \cdot ds. \quad (1)$$

Taking the material derivative of (1),

$$\frac{D}{Dt} \Gamma = \int_C \frac{D}{Dt}(V \cdot ds) \quad (2)$$

$$= \int_C a \cdot ds + \int_C V \cdot \frac{D}{Dt} ds, \quad (3)$$

where $a$ represents the acceleration of a fluid particle given in terms of the material derivative

$$a = \frac{D}{Dt} V = \frac{\partial V}{\partial t} + \zeta \times V + \frac{1}{2} V^2. \quad (4)$$

Note that

$$\frac{D}{Dt} ds = d\frac{D}{Dt} s = dV.$$

Hence the last integral in (3) is a total differential, whose integral along a closed circuit vanishes,

$$\int_C V \cdot dV = 0.$$

Thus

$$\frac{D}{Dt} \Gamma = \int_C a \cdot ds. \quad (5)$$

If $\Sigma$ denotes a surface having $C$ for boundary, then using Stokes theorem gives

$$\frac{D}{Dt} \Gamma = \int_\Sigma (B \cdot n) d\sigma, \quad (6)$$

where we have put $B = \nabla \times a$. From Crocco’s equation,

$$\frac{\partial V}{\partial t} + \zeta \times V = -\nabla h_0 + T \nabla S. \quad (7)$$

Substituting the expression of $\frac{\partial V}{\partial t} + \zeta \times V$ given in (7) into (4) and taking the curl of (4), gives

$$B = \frac{\partial \zeta}{\partial t} + \nabla \times (\zeta \times V) = \nabla T \times \nabla S. \quad (8)$$

Therefore,

$$\frac{D}{Dt} \Gamma = \int_C [(\nabla T \times \nabla S) \cdot n] d\sigma, \quad (9)$$

Euler’s equations give
\[ a = f - \frac{1}{\rho} \nabla p \] (10)

whose curl gives
\[ \mathbf{B} = \nabla \times f - \nabla \left( \frac{1}{\rho} \right) \nabla p. \] (11)

If \( f \) is conservative, \( f = -\nabla \Omega \) and \( \nabla \times f = 0 \). If in addition, \( p = f(\rho) \), i.e., the fluid is barotropic, then \( \nabla \left( \frac{1}{\rho} \right) \times \nabla p = 0 \). Hence \( \mathbf{B} = 0 \), and
\[ \frac{D}{Dt} \Gamma = 0. \] (12)