Consider a wing in a uniform upstream flow, $V$ and let the $y_0$-axis be the axis along the span centered at the wing root. and let $c(y_0)$ be the chord length. We define the lift per unit span, $L'(y_0)$, as that of an infinite span wing whose geometry and angle of attack to the mean flow are those of the wing at $y_0$. The corresponding lift coefficient is

$$c_\ell = \frac{L'(y_0)}{\frac{1}{2} \rho V^2 c(y_0)}, \quad (1)$$

where $c(y_0)$ is the wing chord length at $y_0$. Using the theorem of Kutta-Joukowski, $L'(y_0) = \rho V \Gamma(y_0)$, we rewrite (1) as

$$c_\ell = \frac{2 \Gamma(y_0)}{V c(y_0)}. \quad (2)$$

The expression for $c_\ell$ can also be written in terms of the effective angle of attack $\alpha_{eff} = \alpha - \alpha_i$,

$$c_\ell = a_0 (\alpha - \alpha_{L=0} - \alpha_i), \quad (3)$$

where the induced angle of attack $\alpha_i$ is calculated using the Biot-Savart law,

$$\alpha_i = \frac{1}{4 \pi V} \int_{-b/2}^{b/2} \frac{dx}{y_0 - y} dy. \quad (4)$$

$a_0$ is a constant. For a thin airfoil, $a_0 = 2\pi$.

At every position $y_0$ along the span, we can then write

$$\alpha(y_0) - \alpha_{L=0}(y_0) - \alpha_i(y_0) = \frac{2 \Gamma(y_0)}{a_0 V c(y_0)}. \quad (5)$$

Note that $\alpha(y_0) = \alpha(y_0) - \alpha_{L=0}(y_0)$ is determined by the wing geometry and angle of attack. Substituting the expression (4) for $\alpha_i$ in (5) gives the fundamental equation of the finite wing theory,

$$\alpha(y_0) = \frac{2 \Gamma(y_0)}{a_0 V c(y_0)} + \frac{1}{4 \pi V} \int_{-b/2}^{b/2} \frac{dx}{y_0 - y} dy. \quad (6)$$

The integral in (6) should be understood as a Cauchy principal value.

We note that wings are symmetric, i.e., $b = \frac{b}{2}$.

We introduce the transformation

$$y_0 = -\frac{b}{2} \cos \theta_0$$

$$y = -\frac{b}{2} \cos \theta. \quad (7)$$
Equation (6) can then be rewritten as

$$\bar{\alpha}(\theta_0) = \frac{2\Gamma(\theta_0)}{a_0 V_\infty c(\theta_0)} + \frac{1}{2\pi V_\infty b} \int_0^\pi \frac{d\Gamma}{d\theta} \cos\theta - \cos\theta_0 d\theta.$$  \hspace{1cm} (9)

We note that $\Gamma(y_0)$ vanishes at both ends of the wing. Moreover, we assume the wing to be symmetric, i.e., $\Gamma(-y_0) = \Gamma(y_0)$. This suggests the following expansion for $\Gamma$:

$$\Gamma(\theta) = 2bV \sum_{1}^{N} A_n \sin n\theta$$ \hspace{1cm} (10)

$A_1, A_2, \ldots, A_N$ are constants to be determined. The condition of wing symmetry, $\Gamma(\pi - \theta) = \Gamma(\theta)$, implies $A_n = 0$ for even $n$.

We note that

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0} d\theta = \pi \frac{\sin n\theta_0}{\sin\theta_0}$$ \hspace{1cm} (11)

Substituting (10) into (4 and 9) and using (11), we obtain the following expressions for the induced angle of attack

$$\alpha_i(\theta_0) = \sum_{1}^{N} nA_n \frac{\sin n\theta_0}{\sin\theta_0},$$ \hspace{1cm} (12)

and the fundamental equation (9) for the finite wing becomes

$$\bar{\alpha}(\theta_0) = \frac{4b}{a_0 c(\theta_0)} \sum_{1}^{N} A_n \sin n\theta_0 + \sum_{1}^{N} nA_n \frac{\sin n\theta_0}{\sin\theta_0}$$ \hspace{1cm} (13)

Equation (5) must be satisfied at $N$ locations of the span. This gives $N$ equations for determining $A_1, A_3, \ldots, A_N$. The expressions for the wing lift, $L$, and induced drag, $D_i$, are readily obtained in terms of $\Gamma$,

$$L = \rho V \int_{-b}^{b} \Gamma(y_0) dy_0,$$ \hspace{1cm} (14)

$$D_i = \rho V \int_{-b}^{b} \alpha_i \Gamma(y_0) dy_0.$$ \hspace{1cm} (15)

We define the wing lift and induced drag coefficients as follows

$$C_L = \frac{L}{\frac{1}{2} \rho V^2 S},$$ \hspace{1cm} (16)

$$C_{D,i} = \frac{D_i}{\frac{1}{2} \rho V^2 S}.$$ \hspace{1cm} (17)
This gives:

\[ C_L = \pi \mathcal{A} \mathcal{R} A_1, \quad (18) \]

\[ C_{D,i} = \pi \mathcal{A} \mathcal{R} A_1^2 \left[ 1 + \sum_{n=2}^{N} n \left( \frac{A_n}{A_1} \right)^2 \right], \quad (19) \]

which is commonly cast as

\[ C_{D,i} = \frac{C_L^2}{\pi \mathcal{A} \mathcal{R}} (1 + \delta). \quad (20) \]

For a wing with no geometric twist

\[ C_L = a(\alpha - \alpha_{L=0}) \]

\[ a = \frac{a_0}{1 + \left( \frac{a_0}{\pi \mathcal{A} \mathcal{R}} \right)(1 + \tau)} \]

For a thin airfoil, \( a_0 = 2\pi \).

**ELLIPTIC WING**

For a wing of uniform cross-section and no geometric twist, \( \bar{\alpha}(\theta) \) is constant. We further assume the wing to have an elliptic planform, i.e.,

\[ c = c_0 \sqrt{1 - \left( \frac{2y}{b} \right)^2} \quad \text{or} \quad c(\theta) = c_0 \sin \theta \]

Substituting (11) into (5), we find the following solution

\[ A_1 = \frac{\bar{\alpha}}{1 + \frac{4b}{\pi a_0} A_1} = \frac{\alpha}{1 + \frac{\pi \mathcal{A} \mathcal{R}}{a_0}} \]

\[ A_2 = A_3, \ldots, = A_N = 0. \]

All aerodynamic quantities can now be calculated:

\[ \Gamma(\theta) = 2b V_\infty \frac{\bar{\alpha}}{1 + \frac{\pi \mathcal{A} \mathcal{R}}{a_0}} \sin \theta \]

\[ \alpha_i = A_1 = \frac{\bar{\alpha}}{1 + \frac{\pi \mathcal{A} \mathcal{R}}{a_0}} \]

3
\[ C_L = \pi AR\alpha_i = \frac{a_0\alpha}{1 + \frac{a_0}{\pi AR}} \]

\[ C_{D,i} = \frac{C_L^2}{\pi AR} \]

\[ a = \frac{a_0}{1 + \frac{a_0}{\pi AR}} \]