Flow Induced by Vorticity

1 Free-Space Green Function for Laplace Equation

The Green function, \( g(\vec{x}, \vec{y}) \), is a solution of the equation

\[
\nabla^2 g(\vec{x}, \vec{y}) = -4\pi \delta(\vec{x} - \vec{y}),
\]

where \( \delta(\vec{x}) \) is the Dirac function. Let \( r = |\vec{x} - \vec{y}| \). Then, since \( g \) depends only on \( r \),

\[
\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}.
\]

Equation (1) reduces to

\[
\frac{d^2 g}{dr^2} + \frac{2}{r} \frac{dg}{dr} = 0, \quad \text{for} \quad r \neq 0.
\]

A solution to (2) is

\[
g(r) = \frac{K}{r},
\]

where \( K \) is a constant. We use the divergence theorem to determine \( K \),

\[
\int_{\mathcal{V}} \nabla^2 \left( \frac{K}{r} \right) d\mathcal{V} = \int_{\Sigma} \nabla \left( \frac{K}{r} \right) \cdot \vec{n} d\Sigma = -K \int_{\Sigma} \frac{(\vec{x} - \vec{y}) \cdot \vec{n}}{|\vec{x} - \vec{y}|^3} d\Sigma = -4K\pi,
\]

where \( \Sigma \) is a sphere of radius \( R \) centered on \( \vec{y} \), and \( \vec{n} \) is the outward unit vector normal to \( \Sigma \). The volume integral of the right hand side of (1) is \(-4\pi\). Therefore, \( K = 1 \), and we have

\[
\nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(\vec{x} - \vec{y})
\]
2 Poisson’s Equation

Consider the inhomogeneous equation,
\[ \nabla^2 \vec{V} = -4\pi \vec{f}. \] (6)

We note that
\[ \vec{f}(\vec{x}) = \int_{V} \vec{f}(\vec{y}) \delta(\vec{y} - \vec{x}) d\vec{y} \] (7)
and using (5) for every component of \( \vec{f} \), we get
\[ V(\vec{x}) = \int_{V} \frac{\vec{f}(\vec{y})}{|\vec{x} - \vec{y}|} d\vec{y}. \] (8)

Taking the curl of (8)
\[ \nabla \times V(\vec{x}) = \int_{V} \nabla \left( \frac{1}{r} \right) \times \vec{f}(\vec{y}) d\vec{y}. \] (9)

3 Velocity in Terms of the Vorticity

Consider the solution to Poisson’s equation
\[ \nabla^2 \vec{A} = -\vec{V}. \] (10)

Taking the curl of both sides of (10), we get
\[ \nabla^2 (\nabla \times \vec{A}) = -\vec{\zeta}, \] (11)
where \( \vec{\zeta} = \nabla \times \vec{V} \). A solution to (11) is
\[ (\nabla \times \vec{A}) = \frac{1}{4\pi} \int_{V} \frac{\vec{\zeta}(\vec{y})}{|\vec{x} - \vec{y}|} d\vec{y}. \] (12)

Taking the curl of both sides of (12), we get
\[ \nabla \times (\nabla \times \vec{A}) = \frac{1}{4\pi} \int_{V} \nabla \cdot \left( \frac{1}{|\vec{x} - \vec{y}|} \right) \times \vec{\zeta} d\vec{y}. \] (13)

We recall the mathematical identity,
\[ \nabla \times (\nabla \times \vec{A}) \equiv \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}. \] (14)

If we further assume that \( \vec{A} \) is solenoidal, i.e., \( \nabla \cdot \vec{A} = 0 \), then \( \vec{V} \) is also solenoidal, i.e., \( \nabla \cdot \vec{V} = 0 \). In this case (14) reduces to
\[ \nabla \times (\nabla \times \vec{A}) \equiv -\nabla^2 \vec{A} = \vec{V}. \] (15)
Substituting this result into (13), we get

\[ \vec{V} = \frac{1}{4\pi} \int_V \nabla_x (\frac{1}{|x - y|}) \times \vec{\zeta} d\vec{y}. \]  

(16)

or

\[ \vec{V} = \frac{1}{4\pi} \int_V \vec{\zeta} \times (\frac{x - y}{|x - y|^3}) d\vec{y}. \]  

(17)

This formula for the induced velocity corresponds exactly to the formula of Biot and Savart for the magnetic field induced by a current. The elementary velocity induced by the vorticity in the element of volume \(d\vec{y}\) is

\[ d\vec{V} = \frac{1}{4\pi} \frac{(\vec{\zeta} d\vec{y}) \times (x - y)}{|x - y|^3}. \]  

(18)

4 Vorticity Concentrated in a Vortex Filament

We now consider a vortex filament \(C\). Let \(\sigma\) be the infinitesimal cross-section of the filament orthogonal to the vorticity \(\vec{\zeta}\). Since \(\nabla \cdot \vec{\zeta} = 0\), \(|\vec{\zeta}|\sigma = \text{constant}\) along the filament. Moreover, Stokes theorem states that the circulation, \(\Gamma\), around a circuit surrounding the filament is equal to the flux of the vorticity, i.e.,

\[ \Gamma = \vec{\zeta}\sigma. \]  

(19)

Let \(d\vec{s}\) be the elemental arc in the \(\vec{\zeta}\) direction, then (18) becomes

\[ d\vec{V} = \frac{\sigma}{4\pi} \frac{d\vec{s} \times (x - y)}{|x - y|^3}. \]  

(20)

and the total induced velocity

\[ \vec{V} = \frac{\sigma}{4\pi} \int_C d\vec{s} \times (\vec{x} - \vec{y}). \]  

(21)

If \(\vec{\tau} = \vec{\zeta}/|\vec{\zeta}|\), then \(d\vec{s} = \vec{\tau} ds\), and we have

\[ \vec{V} = \frac{\sigma}{4\pi} \int_C \vec{\tau} \times (\vec{x} - \vec{y}) ds. \]  

(22)

or

\[ \vec{V} = \frac{\sigma}{4\pi} \int_C \vec{\tau} \times \nabla_y (\frac{1}{|\vec{x} - \vec{y}|}) ds. \]  

(23)