THE BIOT-SAVART LAW

Consider a vortex filament with a circulation $\Gamma$ as shown in Figure 1. An elemental segment $d\vec{l}$ centered at the point $M$ of the vortex filament induces an elemental velocity

$$d\vec{V} = -\frac{\Gamma}{4\pi} \frac{d\vec{l} \times \vec{r}}{|r^3|},$$

(1)

where $\vec{r} = \overrightarrow{MP}$, and $r = |\vec{r}|$.

We now apply the Biot-Savart law (1) to a straight vortex filament of infinite length as sketched in Figure 2. The velocity $d\vec{V}$ induced at point $P$ by any elemental segment of the vortex filament $d\vec{l}$ is given by (1). Because the filament is a straight line, $d\vec{V}$ is perpendicular to the plane defined by the filament and the point $P$. The velocity induced at point $P$ by the entire vortex filament is

$$\vec{V} = -\frac{\Gamma}{4\pi} \int_{-\infty}^{+\infty} \frac{d\vec{l} \times \vec{r}}{|r^3|}.$$
The direction of the induced velocity can be obtained from the right-hand screw rule. Its magnitude, $V = |\vec{V}|$, is given by

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{+\infty} \frac{\sin\theta}{r^2} dl.$$ 

From the geometry shown in Figure 2

$$r = \frac{h}{\sin \theta}$$

$$l = -\frac{h}{\tan \theta}$$

$$dl = \frac{h}{\sin^2 \theta} d\theta$$

Substituting in equation (2), we have

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{+\infty} \frac{\sin\theta}{r^2} dl = \frac{\Gamma}{4\pi h} \int_0^{\pi} \sin \theta d\theta$$

Or

$$V = \frac{\Gamma}{2\pi h}$$

Figure 2: Velocity induced at point P by an infinite straight vortex filament.

Consider the semi-infinite vortex filament shown in Figure 3. The filament extends from $O$ to $\infty$. 
\[ V = \frac{\Gamma}{4\pi} \int_0^{+\infty} \frac{\sin\theta}{r^2} \, dl = \frac{\Gamma}{4\pi h} \int_{\frac{\pi}{2}}^{\pi} \sin\theta \, d\theta \]

Or

\[ V = \frac{\Gamma}{4\pi h} \]

The velocity induced at \( P \) by the semi-infinite vortex filament is half that induced by an infinite vortex filament.

Figure 3: Velocity induced at point \( P \) by a semi-infinite straight vortex filament.