THE CONTINUITY EQUATION AND THE STREAM FUNCTION

1. The mathematical expression for the conservation of mass in flows is known as *the continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0. \tag{1}$$

2. For an incompressible flow, $\rho = constant$ and (1) reduces to

$$\nabla \cdot \vec{V} = 0. \tag{2}$$

3. For a two-dimensional incompressible flow in Cartesian coordinates, if $\{u, v\}$ are the x and y-components of the velocity $\vec{V}$, then (2) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

4. This leads to the definition of the stream function ψ,

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}. \tag{4}$$

5. For a two-dimensional incompressible flow in polar coordinates, if $\{u_r, u_\theta\}$ are the radial and circumferential -components of the velocity $\vec{V}$, i.e., $\vec{V} = u_r \vec{e}_r + u_\theta \vec{e}_\theta$, then (2) reduces to

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_\theta}{r \partial \theta} = 0. \tag{5}$$

6. This leads to the definition of the stream function ψ,

$$u_r = \frac{\partial \Psi}{r \partial \theta}, \quad u_\theta = -\frac{\partial \Psi}{\partial r}. \tag{6}$$

7. A flow is said to be irrotational if the vorticity $\zeta = \nabla \times \vec{V} = 0$. For a two-dimensional flow, the vorticity is in the direction z perpendicular to the flow plane and its magnitude is, in Cartesian coordinates,

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{7}$$

In polar coordinates,

$$\zeta_z = \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{\partial u_r}{r \partial \theta}. \tag{8}$$

8. A potential function ϕ can be defined if the flow is irrotational. In cartesian coordinates:

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}$$

In polar coordinates:

$$V_r = \frac{\partial \Phi}{\partial r}, \quad V_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta}. \tag{9}$$

TWO-DIMENSIONAL IRROTATIONAL INCOMPRESSIBLE FLOWS

1. Uniform Flow V_∞

$$\begin{aligned}\Phi &= V_\infty x \\ \Psi &= V_\infty y\end{aligned}$$

2. Source at Origin

$$\begin{aligned}\Phi &= \frac{Q}{2\pi} lnr \\ \Psi &= \frac{Q}{2\pi}\theta\end{aligned}$$

3. Vortex Flow at Origin

$$\begin{aligned}\Phi &= -\frac{\Gamma}{2\pi}\theta \\ \Psi &= \frac{\Gamma}{2\pi} lnr\end{aligned}$$

4. Doublet at Origin

$$\begin{aligned}\Phi &= \frac{k}{2\pi}\frac{cos\theta}{r} \\ \Psi &= -\frac{k}{2\pi}\frac{sin\theta}{r}\end{aligned}$$

5. Source at Point $\vec{x}_0 = (x_0, y_0)$

$$\begin{aligned}\Phi &= \frac{Q}{2\pi} ln|\vec{x} - \vec{x}_0| \\ \Psi &= \frac{Q}{2\pi} tan^{-1}(\frac{y - y_0}{x - x_0})\end{aligned}$$

6. Vortex Flow at Point $\vec{x}_0 = (x_0, y_0)$

$$\begin{aligned}\Phi &= -\frac{\Gamma}{2\pi} tan^{-1}(\frac{y - y_0}{x - x_0}) \\ \Psi &= \frac{\Gamma}{2\pi} ln|\vec{x} - \vec{x}_0|\end{aligned}$$

7. Doublet at point $\vec{x}_0 = (x_0, y_0)$

$$\begin{aligned}\Phi &= \frac{k}{2\pi}\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} \\ \Psi &= -\frac{k}{2\pi}\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}\end{aligned}$$