Acoustic Waves in a Circular Duct

Consider a circular duct of radius $a$. We take a cylindrical coordinate system $\{x, r, \theta\}$, where the $x$ axis is along the duct axis. The acoustic pressure is governed by the wave equation

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0. \quad (1)$$

The pressure must satisfy an initial condition at $x = x_0$ and a wall boundary condition at $r = a$. We use the method of separation of variables and assume

$$p(x, r, \theta, t) = X(x)R(r)\Theta(\theta)T(t). \quad (2)$$

Substituting (2) into (1) and dividing by $X(x)R(r)\Theta(\theta)T(t)$, gives

$$\frac{X''}{X} + \frac{R'' + R'/r}{R} + \frac{\Theta''}{r^2\Theta} - \frac{1}{c^2} \frac{T''}{T} = 0. \quad (3)$$

If we take

$$\frac{\Theta''}{\Theta} = -m^2, \quad (4)$$
$$\frac{X''}{X} = -k^2, \quad (5)$$
$$\frac{T''}{T} = -\omega^2, \quad (6)$$

where $m$ is an integer. This implies a solution of the form

$$p_{mk\omega} = R_m(r)e^{i(kx+m\theta-\omega t)}. \quad (7)$$

The function $R_m$ satisfies the equation

$$r^2 R''_m + r R'_m + (\mu^2 r^2 - m^2) R_m = 0, \quad (8)$$

where we have introduced the eigenvalue $\mu^2 = \omega^2/c^2 - k^2$. For a rigid duct, this equation must satisfy an impermeability condition

$$\frac{dR_m}{dr}_{r=a} = 0. \quad (9)$$

Introducing the non-dimensional variable $\tilde{r} = \mu r$, equation (9) becomes

$$\tilde{r}^2 \frac{d^2 R_m}{d\tilde{r}^2} + \tilde{r} \frac{dR_m}{d\tilde{r}} + (\tilde{r}^2 - m^2) R_m = 0. \quad (10)$$

We recognize the Bessel equation and since the pressure is finite along the axis $R_m = J_m(\tilde{r})$. The wall condition (9) implies

$$J'_m(\mu a) = 0. \quad (11)$$
The boundary-value problem (10, 11) is a Sturm-Liouville problem whose solutions form a complete set. The derivative of the Bessel function has an infinite number of zeros which we denote as \( \{ \alpha_{mn} \} \):

\[
J'_m(\alpha_{mn}) = 0, \quad m = 0, 1, \cdots.
\]  

(12)

Hence, the eigenvalues are

\[
\mu_{mn} = \frac{\alpha_{mn}}{a}.
\]  

(13)

This defines the axial wave number as

\[
k_{mn} = \sqrt{(\frac{\omega}{c})^2 - \mu_{mn}^2}.
\]  

(14)

The eigenfunction

\[
p_{mn} = J_m(\alpha_{mn} \frac{r}{a})e^{i(k_{mn} x + m \theta - \omega t)}
\]  

(15)

is called the \( \{ mn \} \) mode. For every frequency \( \omega \), the solution is then

\[
p_{ \omega} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c_{mn} p_{mn}.
\]  

(16)

The expression for the coefficients \( c_{mn} \) is determined using the initial condition

\[
p_{ \omega}(0, r, \theta, t) = f_{\omega}(r, \theta) e^{-i\omega t}
\]  

(17)

and the orthogonality of the Bessel functions,

\[
c_{mn} = \frac{1}{\pi a^2 (\alpha_{mn}^2 - m^2)} \int_0^{2\pi} \int_0^a f_{\omega}(r, \theta) J_m(\alpha_{mn} \frac{r}{a}) e^{-im \theta} r dr d\theta,
\]  

(18)

where we have used (see Hildebrand, p. 229)

\[
\int_0^a r J_m^2(\alpha_{mn} \frac{r}{a}) dr = \frac{a^2 (\alpha_{mn}^2 - m^2)}{2\alpha_{mn}^2} J_m^2(\alpha_{mn}).
\]  

(19)

Note the condition for propagation of an acoustic mode is that the wave number \( k_{mn} \) must be real. Otherwise the wave will decay exponentially and is known as an evanescent wave. Therefore an \( \{ mn \} \) mode propagates if

\[
\frac{\omega a}{c} > \alpha_{mn}.
\]  

(20)

At low frequencies, only the fundamental mode

\[
p_{00} = e^{i(\omega/c)x - \omega t}
\]  

(21)

propagates. As \( \omega \) increases an additional mode propagates. The frequency at which a mode \( \{ mn \} \) begins to propagate is known as the cutoff frequency of the mode. As the frequency increases (decreases) and is equal to the cutoff frequency of a mode \( \{ mn \} \), the mode \( \{ mn \} \) is said to cut on (cut off).

As an example, consider a duct of radius \( a = 0.5 m \), \( c = 340 m/s \), and the sound frequency is 3000 rpm. \( \omega/c = 0.462 \). From the tables of zeros of Bessel functions, the lowest zero is \( \alpha_{11} = 1.8412 \). hence only the fundamental mode will propagate.
Bessel Function Zeros

The first $k$ roots $x_1, \ldots, x_k$ of the Bessel function $J_n(x)$ are given in the following table. They can be found in Mathematica using the command `BesselJZeros[n, k]` in the Mathematica add-on package `NumericalMath` `BesselZeros` (which can be loaded with the command `<<NumericalMath`).

<table>
<thead>
<tr>
<th>zero $n$</th>
<th>$J_0(x)$</th>
<th>$J_1(x)$</th>
<th>$J_2(x)$</th>
<th>$J_3(x)$</th>
<th>$J_4(x)$</th>
<th>$J_5(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4048</td>
<td>3.8317</td>
<td>5.1356</td>
<td>6.3802</td>
<td>7.5883</td>
<td>8.7715</td>
</tr>
<tr>
<td>2</td>
<td>5.5201</td>
<td>7.0156</td>
<td>8.4172</td>
<td>9.7610</td>
<td>11.0647</td>
<td>12.3386</td>
</tr>
</tbody>
</table>

The first $k$ roots $x_1, \ldots, x_k$ of the derivative of the Bessel function $J'_n(x)$ can be found in Mathematica using the command `BesselJPrimeZeros[n, k]` in the Mathematica add-on package `NumericalMath` `BesselZeros` (which can be loaded with the command `<<NumericalMath`). The first few such roots are given in the following table.

<table>
<thead>
<tr>
<th>zero $n$</th>
<th>$J'_0(x)$</th>
<th>$J'_1(x)$</th>
<th>$J'_2(x)$</th>
<th>$J'_3(x)$</th>
<th>$J'_4(x)$</th>
<th>$J'_5(x)$</th>
</tr>
</thead>
<tbody>
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<td>1.8412</td>
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<td>9.2824</td>
<td>10.5199</td>
</tr>
</tbody>
</table>

SEE ALSO: Bessel Function, Bessel Function of the First Kind. [Pages Linking Here]

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