Expansion of a Function in a Series of Eigenfunctions

and

The Gibbs' Phenomenon

We have shown that for a function \( f(x) \) defined in an interval \([a, b]\) we can define an expansion in a series of eigenfunctions

\[
F(x) = \sum_{n=1}^{\infty} a_n \phi_n
\]

where \( \{\phi_n\} \) are the eigenfunctions of a Sturm-Liouville problem and \( a_n = (\phi_n, rf) \), where \( r \) is the weight function. Furthermore, if \( f(x_0^-) \) and \( f(x_0^+) \) represent, respectively, the limits from the left and from the right of \( f(x_0) \), then

\[
F(x_0) = \frac{1}{2} [f(x_0^-) + f(x_0^+)]
\]

Thus if \( f(x) \) is continuous at \( x_0 \), then \( F(x_0) = f(x_0) \), and if \( f(x_0) \) has a discontinuity at \( x_0 \), then \( F(x_0) \) will converge to the mean value of the two limits \( f(x_0^-) \) and \( f(x_0^+) \).

Certain peculiarities of this fit near the discontinuities should be pointed out. The difficulties arise from the fact that we are trying to represent a discontinuous function at \( x=x_0 \) in terms of continuous functions \( \phi_n \). Let us define the sum of the \( N \) first terms of (1)

\[
S_N = \sum_{n=1}^{N} a_n \phi_n
\]
Note that $S_N$ cannot fit the discontinuity at $x_0$ because a finite number of terms of a convergent series cannot have the infinite slope required by the discontinuity. As the figure shows $S_N$ tries to achieve that infinite slope at $x=x_0$, but in the attempt it overshoots the discontinuity by a certain amount. Even in the limit of $N$ infinite this overshooting persists and the complete series has flanges $D_+$ and $D_-$ on the ends of the discontinuity as shown in the second figure. This is known as the Gibbs' phenomenon.

The following is a specific example of the Gibbs' phenomenon using a Fourier series. Let $f(x)$ be defined in $[0, 2\pi]$ as

$$f(x) = \begin{cases} 1; & 0 < x < \pi \\ +1; & \pi < x < 2\pi \end{cases}$$

(4)

1. Find a Fourier series of period $2\pi$ for $f(x)$.

2. Calculate the limits of $F(\pi^-)$ and $F(\pi^+)$ by computing $S_N(\pi^-)$ and $S_N(\pi^+)$ for large values of $N$ (100, 200, 500, 1000). Find the overshoots $D_+$ and $D_-$. 

3. Calculate the limits of $F(\pi^-)$ and $F(\pi^+)$ by computing $S_N(\pi^-)$ and $S_N(\pi^+)$ where

$$\pi^- = \pi - \pi/(2N+1) \quad \text{and} \quad \pi^+ = \pi + \pi/(2N+1)$$

(5)

for large values of $N$ (100, 200, 500, 1000). Find the overshoots $D_+$ and $D_-$. Compare the results of 2 and 3.

4. Give the expression of $S_N$ in terms of an integral and show that if $\pi^-$ and $\pi^+$ are chosen as in (5), $S_N$ goes to 1.179 as $N$ goes to infinity. The series may overshoots by about 18 per cent over a region of vanishingly small width before settling down to zero, the average value of $f(x)$ at $x=\pi$. 