Scalar and Vector Fields:

Let \( \phi (\mathbf{r}, t) \) and \( \mathbf{V} (\mathbf{r}, t) \)

\( \mathbf{r} = [x_1, x_2, x_3] \)
Positive vector

1. **Gradient:** \( \nabla \phi = \left[ \frac{\partial \phi}{\partial x_1} \right] \)

**Interpretation:** \( \phi (M_2) - \phi (M_1) \), where \( M_1, M_2 \) are two close points

\( \phi (M_2) - \phi (M_1) = (\mathbf{r}_2 - \mathbf{r}_1) \cdot \frac{\partial \phi}{\partial x_i} + \ldots \)

\( = \nabla \phi \cdot d\mathbf{M} \)

**Directional Derivative:**

\( \frac{d\phi}{ds} = \nabla \phi \cdot \mathbf{n} \)

A scalar function has its maximum directional derivative in the direction of its gradient.
A scalar function has zero directional derivative normal to its gradient.

**Application:**

\( \phi = c \) is a surface
\( \nabla \phi \) is orthogonal to the surface.

\( d\phi = \nabla \phi \cdot d\mathbf{M} = 0 \Rightarrow \nabla \phi \perp d\mathbf{M} \).

**Example:**

\( x^2 + y^2 - R^2 = 0 \quad \nabla \phi = [2x, 2y] \)

\( (x-x_0)^2 + (y-y_0)^2 - R^2 = 0 \quad \nabla \phi = [2(x-x_0), 2(y-y_0)] \)
Divergence:
\[ \nabla \cdot \mathbf{V} = \frac{\partial V_i}{\partial x_i} \]

Interpretation:

Consider a one-dimensional fluid \( V(x) \) and a volume of cross-section \( A \perp V \) and width \( dx \) moving with \( V(x) \).

Initially, the volume is in a box, after a time \( \Delta t \), the volume has expanded along the dashed lines to

\[ A \Delta x = V(x) \Delta t + V(x+\Delta x) \Delta t \]

The change in volume is \( A \Delta x \) \( \Delta x \) \( \Delta t \).

The rate of change in volume is \( \frac{\partial V(x+\Delta x)}{\partial x} \frac{\Delta x}{\Delta x} \Delta t \).

The relative rate of volume change is \( \frac{\Delta V}{\Delta x} \).

Hence, divergence represents the relative volume rate of change of a volume moving with the field \( V \).

If \( \mathbf{V} \) is the velocity field of a liquid, then since liquid density is constant, conservation of mass implies conservation of volume, hence \( \nabla \cdot \mathbf{V} = 0 \) for a liquid or incompressible fluid.

If \( \mathbf{V} \) is the velocity field of a gas, then \( \nabla \cdot \mathbf{V} \mathbf{V} \) represents the mass flux outward, percent volume...
\[ \nabla \times \mathbf{F} \]

\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}
\end{vmatrix}
\]

\[
= \frac{\partial}{\partial y} \left( \frac{\partial B_3}{\partial z} - \frac{\partial B_2}{\partial x} \right) - \frac{\partial}{\partial z} \left( \frac{\partial B_1}{\partial x} - \frac{\partial B_3}{\partial y} \right) + \frac{\partial}{\partial x} \left( \frac{\partial B_2}{\partial y} - \frac{\partial B_1}{\partial z} \right)
\]

\[
= \varepsilon_{ijk} \frac{\partial B_j}{\partial x_i}
\]

\[ E_{ijk} \text{ is the permutation index } \neq 1 \]

**Significance:**

1. **RBR**
   \[
   \begin{align*}
   u &= -y \omega \\
   v &= x \omega \\
   w &= 0
   \end{align*}
   \]
   \[
   \nabla \times \mathbf{v} = 2 \mathbf{\omega} \times \mathbf{r}
   \]
   The curl is twice the angular velocity of a rotating rigid body.

2. **Vortex Field**
   \[
   \begin{align*}
   u &= -\frac{\rho y}{x^2 + y^2} \\
   v &= \frac{\rho x}{x^2 + y^2} \\
   \end{align*}
   \]
   \[
   \frac{\partial v}{\partial x} = \frac{\rho - x^2 + y^2}{(x^2 + y^2)^2}, \quad \frac{\partial u}{\partial y} = -\rho \frac{x^2 y}{(x^2 + y^2)^2} \implies \nabla \times \mathbf{v} = 0
   \]
Results:

1. If \( \nabla = \nabla \phi \) then \( \nabla \times \nabla = 0 \)

\[ \nabla \times (\nabla \phi) = 0 \]

and the reciprocal is true, if \( \nabla \times \nabla = 0 \)

there \( \exists \phi \) such that \( \nabla^2 = \nabla \phi \) is potential a corotational.

2. If \( \nabla \cdot \nabla = 0 \), the field is said to be solenoidal or divergence-free.

3. If \( \nabla \cdot \nabla = 0 \) and \( \nabla \times \nabla = 0 \), then \( \exists \phi \)

\[ \nabla = \nabla \phi \quad \nabla \cdot \nabla = \nabla \phi = 0 \quad \text{Laplace Equation} \]

Solutions to Laplace equation are called harmonic functions.

Concept of a circulation

Let \( \nabla \) be a field and \( C \) be a simply connected curve, then

\[ T = \int_C \nabla \cdot ds \] is the circulation of \( \nabla \) along \( C \)

\[ ds = dz \] where \( z \) is the unit tangent to \( C \), and \( ds \) is the elemental length of the arc along \( C \). The line integral is calculated by moving along \( C \) in a given direction. If \( C \) is a closed curve, the positive direction is determined by the right-hand screw rule.
General Theorems:

1. Divergence Theorem:

Consider a volume \( V \) surrounded by a surface \( \Sigma \). Let \( \mathbf{n} \) be the unit outward normal to \( \Sigma \), then

\[
\int_V \nabla \cdot \mathbf{V} \, dV = \int_{\Sigma} \mathbf{V} \cdot \mathbf{n} \, d\Sigma
\]

Concept of a flux = rate of flow/unit time

\[ A(\mathbf{V} \cdot \mathbf{n}) \]

Generalize to

\[ \int_{\Sigma} \mathbf{V} \cdot \mathbf{n} \, d\Sigma \]

Physical interpretation of the divergence theorem

\[ \int_{\Sigma} \mathbf{V} \cdot \mathbf{n} \, d\Sigma = \text{rate of expansion of } V \]

\[ (\mathbf{V} \cdot \mathbf{n}) d\Sigma = \text{rate of expansion of } V \]

For a gas, \( \mathbf{V} \cdot \mathbf{n} \) represents the mass flux per unit volume. This is equal to the change in time of the density. Hence,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \] expresses the conservation of mass

where \( q \) is a source distribution.
2 Green's Theorem

\[ \nabla \cdot \phi = \phi_1 \nabla \cdot \phi_2 \]

\[ \int \nabla \cdot \phi \, d\mathbf{a} = \int \nabla \cdot \phi \, d\Sigma \]

\[ \int \nabla \cdot (\phi_1 \nabla \phi_2) \, d\mathbf{a} = \int \phi_1 \nabla \cdot \phi_2 \, d\Sigma \]

\[ \nabla \cdot (\phi_1 \nabla \phi_2) = \phi_1 \nabla \cdot \phi_2 + \nabla \phi_1 \cdot \nabla \phi_2 \]

*First Green's theorem*

\[ \int \nabla \cdot (\phi_1 \nabla^2 \phi_2 + \nabla \phi_1 \cdot \nabla \phi_2) \, d\mathbf{a} = \int \phi_1 \, \nabla \cdot \phi_2 \, d\Sigma \]

*Second Green's theorem*

\[ \int (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla \cdot \phi_1) \, d\mathbf{a} = \int \nabla \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \, d\Sigma \]

*Special Cases:*

1. \( \phi_1 = \phi_2 = \phi \)

\[ \int [\nabla \cdot (\nabla \phi)^2] \, d\mathbf{a} = \int \phi \, \nabla \cdot \phi \, d\Sigma \]

2. \( \phi_1 = \phi, \phi_2 = 1 \)

\[ \int \phi \, \nabla \phi \, d\mathbf{a} = \int \frac{\partial \phi}{\partial n} \, d\Sigma \]
\[ \phi_1 = 1, \quad \phi_2 = 1 \]

\[ \int_{\partial S} \mathbf{V} \cdot \mathbf{d} \sigma = \int_{S} \mathbf{V} \cdot \mathbf{n} \, d\Sigma \]

**Stokes' Theorem:**

Consider a surface \( S \) having a closed curve \( C \) as its boundary, then

\[ \int_{S} \mathbf{n} \cdot (\mathbf{V} \times \mathbf{n}) \, d\sigma = \int_{C} \mathbf{V} \cdot d\mathbf{r} \]

**Example:**

\[ \int_{\partial R} (\mathbf{V} \times \mathbf{n}) \, d\sigma = \int_{R} \nabla \cdot (\mathbf{V} \times \mathbf{n}) \, d\Sigma \]

\[ \nabla \times \mathbf{n} = 0 \]

\[ \nabla \cdot (\mathbf{V} \times \mathbf{n}) = \mathbf{n} \cdot \nabla \times \mathbf{V} \]