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Mathematical Methods II

Analytic Functions of a Complex Variable

1 Definitions and Theorems

1.1 Definition 1

A function $f(z)$ is said to be analytic in a region $\mathcal{R}$ of the complex plane if $f(z)$ has a derivative at each point of $\mathcal{R}$ and if $f(z)$ is single valued.

1.2 Definition 2

A function $f(z)$ is said to be analytic at a point z if z is an interior point of some region where $f(z)$ is analytic.

Hence the concept of analytic function at a point implies that the function is analytic in some circle with center at this point.

1.3 Theorem

If $f(z)$ is analytic at a point z, then the derivative $f'(z)$ is *continuous* at z.

1.4 Corollary

If $f(z)$ is analytic at a point z, then $f(z)$ has continuous derivatives of *all* order at the point z.

2 Conditions for a Complex Function to be Analytic

2.1 A necessary condition for a complex function to be analytic

Let

$$f(x, y) = u(x, y) + iv(x, y)$$

be a complex function. Since $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$, substituting for x and y gives

$$f(z, \overline{z}) = u(x, y) + iv(x, y)$$

.

A necessary condition for $f(z, \overline{z})$ to be analytic is

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{1}$$

Therefore a necessary condition for $f = u + iv$ to be analytic is that f depends *only* on z. In terms of the of the real and imaginary parts u, v of f, condition (1) is equivalent to

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

Equations (2, 3) are known as the Cauchy-Riemann equations. They are a necessary condition for $f = u + iv$ to be analytic.

2.2 Necessary and sufficient conditions for a function to be analytic

The necessary and sufficient conditions for a function $f = u + iv$ to be analytic are that:

1. The four partial derivatives of its real and imaginary parts $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ satisfy the Cauchy-Riemann equations (2, 3).
2. The four partial derivatives of its real and imaginary parts $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are *continuous*.

2.3 Theorem

If $f(z)$ is analytic, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{4}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{5}$$

The real and imaginary parts of an analytic function are harmonic conjugate functions, i.e., solutions to Laplace equation and satisfy the Cauchy Riemann equations (2, 3).

3 Singularities of Analytic Functions

Points at which a function $f(z)$ is *not* analytic are called *singular points or singularities* of $f(z)$. There are two different types of singular points:

3.1 Isolated Singular Points

If $f(z)$ is analytic everywhere throughout some neighborhood of a point $z = a$, say inside a circle $\mathcal{C} : |z - a| = R$, except at the point $z = a$ itself, then $z = a$ is called an *isolated singular point* of $f(z)$. $f(z)$ cannot be bounded near an *isolated singular point*.

3.1.1 Poles

If $f(z)$ has an isolated singular point at $z = a$, i.e., $f(z)$ is not finite at $z = a$, and if in addition there exists an integer n such that the product

$$(z - a)^n f(z)$$

is analytic at $z = a$, then $f(z)$ has a *pole of order* n at $z = a$, if n is the smallest such integer. Note that because $(z - a)^n f(z)$ is analytic at $z = a$, such a singularity is called a *removable singularity*. Example: $f(z) = 1/z^2$ has a pole of order 2 at $z = 0$.

3.1.2 Essential Singularities

An isolated singular point which is *not* a pole (removable singularity) is called an *essential singular point*. Example: $f(z) = sin(1/z)$ has an essential singularity at $z = 0$.

3.2 Branch Points

When $f(z)$ is a multivalued function, any point which cannot be an interior point of the region of definition of a single-valued branch of $f(z)$ is a *singular branch point*. Example: $f(z) = \sqrt{z - a}$ has a branch point at $z = a$.