Orthogonal Curvilinear Coordinates

1 Definitions

Let \( \mathbf{x} = (x_1, x_2, x_3) \) be the Cartesian coordinates of a point \( M \) with respect to a frame of reference defined by the unit vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \). We introduce three functions defined by

\[
  u_j = u_j(x_1, x_2, x_3), \quad j = 1, 3, \quad (1)
\]

in a region \( \mathcal{R} \). The equation \( u_j = c_j \), where \( c_j \) is a constant, represents a surface. The system of two equations \( u_2 = c_2 \) and \( u_3 = c_3 \) represent a line \( \gamma_1 \) where the two surfaces intersect. Along \( \gamma_1 \), only \( u_1 \) varies. The system of three equations \( u_1 = c_1, u_2 = c_2 \) and \( u_3 = c_3 \) represent a point where the three surfaces intersect. At every point \( M \in \mathcal{R} \), there are three lines \( \gamma_i(u_i) \). For Cartesian coordinates, these surfaces are planes. For cylindrical coordinates, we define

\[
  u_1 = r = (x_1^2 + x_2^2)^{1/2}, \quad (2)
\]

\[
  u_2 = \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right), \quad (3)
\]

\[
  u_3 = x_3. \quad (4)
\]

Here \( r = c_1 \) represents a circular cylinder of radius \( c_1 \), \( \theta = c_2 \) represents a vertical plane, and \( x_3 = c_3 \) represents a horizontal plane. The two equations \( r = c_1 \) and \( x_3 = c_3 \) represent a circle in a horizontal plane, only \( \theta \) varies as we move along the circle.

The position vector of a point \( M \) can be expressed in the Cartesian system as

\[
  \overrightarrow{OM} = \mathbf{x} = x_i \mathbf{e}_i, \quad (6)
\]
where the repeated index implies summation, i.e., \( x_i e_i = x_1 e_1 + x_2 e_2 + x_3 e_3 \). Note that
\[
\frac{\partial x}{\partial x_i} = e_i. \tag{7}
\]
We now want to use \( u_j \) as a new coordinate system. We assume that the Cartesian coordinates \( x_i \) are given in terms of the new coordinates \( u_j \),
\[
x_i = x_i(u_1, u_2, u_3), \quad i = 1, 3.. \tag{8}
\]
Differentiating \( x \) with respect to \( u_j \), we get
\[
dx = \frac{\partial x}{\partial u_j} du_j = e_i \frac{\partial x_i}{\partial u_j} du_j. \tag{9}
\]
The vector
\[
\hat{E}_j = \frac{\partial x}{\partial u_j} = e_i \frac{\partial x_i}{\partial u_j} \tag{10}
\]
is tangent to \( \gamma_j \). Note that
\[
\hat{E}_j = \frac{\partial x}{\partial s_j} \frac{\partial s_j}{\partial u_j}, \tag{11}
\]
where \( \partial s_j \) is the elementary arc length along \( \gamma_j \). We also note that the vector \( E_j = \frac{\partial x}{\partial s_j} \) is a unit vector. Thus if \( h_j = \frac{\partial s_j}{\partial u_j} \), along \( \gamma_j \), \( ds_j = h_j du_j \) and \( \hat{E}_j = h_j E_j \). Hence, using (10), we get
\[
h_j E_j = e_i \frac{\partial x_i}{\partial u_j}. \tag{12}
\]
Since both \( e_i \) and \( E_j \) are orthonormal vectors,
\[
h_j^2 = \sum_{i=1}^{3} \left( \frac{\partial x_i}{\partial u_j} \right)^2. \tag{13}
\]
Equation (13) defines the three scales associated with the new coordinates system.

2 Elementary Quantities

2.1 Elementary Arc Length

The elementary arc length of a line, not coinciding with the three lines defining the coordinate system at a point \( M \), is obtained by taking the magnitude of (9),
\[
(ds)^2 = h_j^2 (du_j)^2. \tag{14}
\]
2.2 Elementary Surface

The elementary surface of \( u_1 = c_1 \) which contains \( \gamma_2 \) and \( \gamma_3 \) is

\[
d\sigma_1 = ds_2 ds_3 = h_2 h_3 du_2 du_3.
\] (15)

2.3 Elementary Volume

The elementary volume

\[
dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 du_1 du_2 du_3
\] (16)

3 Differential Operators

3.1 Gradient

The gradient is defined by

\[
df = \nabla f \cdot dx.
\] (17)

We can also express \( df \) as

\[
df = \frac{\partial f}{\partial u_j} du_j.
\] (18)

Using (9), we get

\[
\nabla f = \mathbf{E}_j \frac{\partial f}{h_j \partial u_j}.
\] (19)

Or

\[
\nabla = \frac{\mathbf{E}_j}{h_j} \frac{\partial}{\partial u_j}.
\] (20)

3.1.1 Useful Results

1. \( \nabla u_j = \frac{\mathbf{E}_j}{h_j} \) (21)

2. Equation(21) implies that

\[
\nabla \times \frac{\mathbf{E}_j}{h_j} = 0
\] (22)

Since

\[
\nabla \times (f \mathbf{a}) \equiv f \nabla \times \mathbf{a} + \nabla f \times \mathbf{a},
\] (23)

then,

\[
\nabla \times \frac{\mathbf{E}_j}{h_j} \equiv \frac{1}{h_j} \nabla \times \mathbf{E}_j + \nabla \frac{1}{h_j} \times \mathbf{E}_j,
\] (24)
we deduce

$$\nabla \times \mathbf{E}_j = \frac{\nabla h_j \times \mathbf{E}_j}{h_j}$$  \hspace{1cm} (25)

### 3.2 Divergence

Note that

$$\frac{\mathbf{E}_1}{h_2 h_3} = \frac{\mathbf{E}_2}{h_2} \times \frac{\mathbf{E}_3}{h_3}.$$

Using (21),

$$\frac{\mathbf{E}_1}{h_2 h_3} = \nabla u_2 \times \nabla u_3.$$

Taking the divergence of both sides and noting that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \equiv \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B},$$

we arrive at

$$\nabla \cdot \frac{\mathbf{E}_1}{h_2 h_3} = 0.$$  \hspace{1cm} (26)

Or

$$\nabla \cdot \frac{\mathbf{E}_i}{h_j h_k} = 0,$$  \hspace{1cm} (27)

where $i \neq j \neq k$.

$$\nabla \cdot \mathbf{F} = \nabla \cdot (F_i \mathbf{E}_i)$$

$$= \nabla \cdot \left( \frac{\mathbf{E}_i}{h_j h_k} (h_j h_k F_i) \right)$$

$$= \frac{\mathbf{E}_i}{h_j h_k} \cdot \nabla (h_j h_k F_i).$$  \hspace{1cm} (28)

This gives the expression for the divergence

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} (h_j h_k F_i),$$  \hspace{1cm} (29)

where $i \neq j \neq k$. 
3.3 curl

Using (25, 23), it is readily shown that

$$\nabla \times F_k E_k = \frac{1}{h_k} \nabla (h_k F_k) \times E_k,$$

(30)

or

$$\nabla \times F_k E_k = \frac{1}{h_j h_k} \frac{\partial (h_k F_k)}{\partial u_j} (E_j \times E_k).$$

(31)

Noting that $E_j \times E_k = \epsilon_{ijk} E_i$, where the permutation symbol $\epsilon_{ijk} = 1$ for $i, j, k$ in order but $i \neq j \neq k$, $\epsilon_{ijk} = -1$ for $i, j, k$ not in order but $i \neq j \neq k$, and $\epsilon_{ijk} = 0$ when two indices are equal, we obtain,

$$\nabla \times F = \epsilon_{ijk} \frac{h_i E_i}{h_1 h_2 h_3} \frac{\partial}{\partial u_j} (h_k F_k).$$

(32)

The expression (32) for the curl can be cast in the familiar form,

$$\nabla \times F = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 E_1 & h_2 E_2 & h_3 E_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

(33)