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Mathematical Methods II

Orthogonal Curvilinear Coordinates

1 Definitions

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the Cartesian coordinates of a point M with respect to a frame of reference defined by the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3,$. We introduce three functions defined by

$$u_j = u_j(x_1, x_2, x_3),\ \ j = 1, 3, \tag{1}$$

in a region $\mathcal{R}$. The equation $u_j = c_j$, where c_j is a constant, represents a surface. The system of two equations $u_2 = c_2$ and $u_3 = c_3$ represent a line γ_1 where the two surfaces intersect. Along γ_1, only u_1 varies. The system of three equations $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ represent a point where the three surfaces intersect. At every point $M \in \mathcal{R}$, there are three lines $\gamma_i(u_i)$. For Cartesian coordinates, these surfaces are planes. For cylindrical coordinates, we define

$$u_1 = \ \ r = \ \ (x_1^2 + x_2^2)^{\frac{1}{2}}, \tag{2}$$

$$u_2 = \ \ \theta = \ \ tan^{-1}(\frac{x_2}{x_1}), \tag{3}$$

$$u_3 = \ \ x_3. \tag{4}$$

$$\tag{5}$$

Here $r = c_1$ represents a circular cylinder of radius c_1, $\theta = c_2$ represents a vertical plane, and $x_3 = c_3$ represents a horizontal plane. The two equations $r = c_1$ and $x_3 = c_3$ represent a circle in a horizontal plane, only θ varies as we move along the circle.

The position vector of a point M can be expressed in the Cartesian system as

$$\overrightarrow{OM} = \mathbf{x} = x_i \mathbf{e}_i, \tag{6}$$

where the repeated index implies summation, i.e., $x_i\mathbf{e}_i = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. Note that

$$\frac{\partial \mathbf{x}}{\partial x_i} = \mathbf{e}_i. \tag{7}$$

We now want to use u_j as a new coordinate system. We assume that the Cartesian coordinates x_i are given in terms of the new coordinates u_j,

$$x_i = x_i(u_1, u_2, u_3),\ \ i = 1, 3.. \tag{8}$$

Differentiating $\mathbf{x}$ with respect to u_j, we get

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_j} du_j = \mathbf{e}_i \frac{\partial x_i}{\partial u_j} du_j. \tag{9}$$

The vector

$$\hat{\mathbf{E}}_j = \frac{\partial \mathbf{x}}{\partial u_j} = \mathbf{e}_i \frac{\partial x_i}{\partial u_j} \tag{10}$$

is tangent to γ_j. Note that

$$\hat{\mathbf{E}}_j = \frac{\partial \mathbf{x}}{\partial s_j} \frac{\partial s_j}{du_j}, \tag{11}$$

where ∂s_j is the elementary arc length along γ_j. We also note that the vector $\mathbf{E}_j = \frac{\partial \mathbf{x}}{\partial s_j}$ is a unit vector. Thus if $h_j = \frac{\partial s_j}{\partial u_j}$, along γ_j, $ds_j = h_j du_j$ and $\hat{\mathbf{E}}_j = h_j \mathbf{E}_j$. Hence, using (10), we get

$$h_j \mathbf{E}_j = \mathbf{e}_i \frac{\partial x_i}{\partial u_j}. \tag{12}$$

Since both $\mathbf{e}_i$ and $\mathbf{E}_j$ are orthonormal vectors,

$$h_j^2 = \sum_{i=1}^{i=3} (\frac{\partial x_i}{\partial u_j})^2. \tag{13}$$

Equation(13) defines the three scales associated with the new coordinates system.

2 Elementary Quantities

2.1 Elementary Arc Length

The elementary arc length of a line, not coinciding with the three lines defining the coordinate system at a point M, is obtained by taking the magnitude of (9),

$$(ds)^2 = h_j^2 (du_j)^2. \tag{14}$$

2.2 Elementary Surface

The elementary surface of $u_1 = c_1$ which contains γ_2 and γ_3 is

$$d\sigma_1 = ds_2 ds_3 = h_2 h_3 du_2 du_3. \tag{15}$$

2.3 Elementary Volume

The elementary volume

$$dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 du_1 du_2 du_3 \tag{16}$$

3 Differential Operators

3.1 Gradient

The gradient is defined by

$$df = \nabla f \cdot d\mathbf{x}. \tag{17}$$

We can also express df as

$$df = \frac{\partial f}{\partial u_j} du_j. \tag{18}$$

Using (9), we get

$$\nabla f = \frac{\mathbf{E}_j}{h_j}\frac{\partial f}{\partial u_j}. \tag{19}$$

Or

$$\nabla = \frac{\mathbf{E}_j}{h_j}\frac{\partial}{\partial u_j}. \tag{20}$$

3.1.1 Useful Results

1. $$\nabla u_j = \frac{\mathbf{E}_j}{h_j} \tag{21}$$

2. Equation(21) implies that

$$\nabla \times \frac{\mathbf{E}_j}{h_j} = 0 \tag{22}$$

Since

$$\nabla \times (f\mathbf{a}) \equiv f\nabla \times \mathbf{a} + \nabla f \times \mathbf{a}, \tag{23}$$

then,

$$\nabla \times \frac{\mathbf{E}_j}{h_j} \equiv \frac{1}{h_j}\nabla \times \mathbf{E}_j + \nabla\frac{1}{h_j} \times \mathbf{E}_j, \tag{24}$$

we deduce

$$\nabla \times \mathbf{E}_j = \frac{\nabla h_j \times \mathbf{E}_j}{h_j} \tag{25}$$

3.2 Divergence

Note that

$$\frac{\mathbf{E}_1}{h_2 h_3} = \frac{\mathbf{E}_2}{h_2} \times \frac{\mathbf{E}_3}{h_3}.$$

Using (21),

$$\frac{\mathbf{E}_1}{h_2 h_3} = \nabla u_2 \times \nabla u_3.$$

Taking the divergence of both sides and noting that

$$\nabla \cdot \mathbf{A} \times \mathbf{B} = \equiv \mathbf{B} \cdot \nabla \mathbf{A} - \mathbf{A} \cdot \nabla \mathbf{B},$$

we arrive at

$$\nabla \cdot \frac{\mathbf{E}_1}{h_2 h_3} = 0. \tag{26}$$

Or

$$\nabla \cdot \frac{\mathbf{E}_i}{h_j h_k} = 0, \tag{27}$$

where $i \neq j \neq k$.

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \nabla \cdot (F_i \mathbf{E}_i) \\ &= \nabla \cdot (\frac{\mathbf{E}_i}{h_j h_k} (h_j h_k F_i)) \\ &= \frac{\mathbf{E}_i}{h_j h_k} \cdot \nabla (h_j h_k F_i). \end{aligned} \tag{28}$$

This gives the expression for the divergence

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_i} (h_j h_k F_i), \tag{29}$$

where $i \neq j \neq k$.

3.3 curl

Using (25, 23), it is readily shown that

$$\nabla \times F_k \mathbf{E}_k = \frac{1}{h_k} \nabla (h_k F_k) \times \mathbf{E}_k, \tag{30}$$

or

$$\nabla \times F_k \mathbf{E}_k = \frac{1}{h_j h_k} \frac{\partial (h_k F_k)}{\partial u_j} (\mathbf{E_j} \times \mathbf{E}_k). \tag{31}$$

Noting that $\mathbf{E}_j \times \mathbf{E}_k = \epsilon_{ijk}\, \mathbf{E}_i$, where the permutation symbol $\epsilon_{ijk} = 1$ for i, j, k in order but $i \neq j \neq k$, $\epsilon_{ijk} = -1$ for i, j, k not in order but $i \neq j \neq k$, and $\epsilon_{ijk} = 0$ when two indices are equal, we obtain,

$$\nabla \times \mathbf{F} = \epsilon_{ijk} \frac{h_i \mathbf{E}_i}{h_1 h_2 h_3} \frac{\partial}{\partial u_j} (h_k F_k). \tag{32}$$

The expression (32) for the curl can be cast in the familiar form,

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{E}_1 & h_2 \mathbf{E}_2 & h_3 \mathbf{E}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \tag{33}$$