Linear Operators and Linear Equations

1 Linear Equations

Let $E$ be an $n$-dimensional space and let $\vec{x} = \{x_1, x_2, \cdots, x_n\} \in E$. We define the inner product

$$ (\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i y_i. \quad (1) $$

and the norm

$$ ||\vec{x}|| = (\vec{x}, \vec{x})^{\frac{1}{2}}. \quad (2) $$

A linear operator $\mathcal{L}$ on $E$ is denoted $\mathcal{L} : E \to F \subseteq E$. In linear algebra, the operator can be represented by a matrix $A$. An $n \times n$ matrix $A$ can be represented by its column vectors, $A = \{c_1, c_2, \cdots, c_n\}$ or by its row vectors $A = \{r_1, r_2, \cdots, r_n\}^t$. The rank of $A$ is the number of independent column vectors or the number of independent row vectors. If the rank of $A$ is $< n$, $A$ is said to be singular.

A system of $n$ linear equations can be written as

$$ A\vec{x} = \vec{b}. \quad (3) $$

Or

$$ \sum_{i=1}^{n} x_i \vec{c}_i = \vec{b}. \quad (4) $$

1.1 $A$ is of rank $n$

The column vectors $\{\vec{c}_i\}$ are independent and span $E$. The associated homogenous equation, $A\vec{x} = 0$, has no nontrivial solutions. Equation (3) has a unique solution

$$ \vec{x} = A^{-1}\vec{b}. \quad (5) $$

The matrix $A$, or the operator $\mathcal{L}$, is said to be invertible.
1.2 $A$ is singular and of rank $n - k < n$

The column vectors span a subspace $S_{n-k}$ of dimension $n-k$. The homogeneous equation associated with (3), $A\vec{x} = 0$, implies that $\vec{r}_i \cdot \vec{x} = 0$ for $i = 1, 2, \cdots, n$. As a result the Null space of $A$, denoted $\mathcal{N}(A)$, is orthogonal to all $\vec{r}_i$ for $i = 1, 2, \cdots, n$. Similarly, $\mathcal{N}(A^t)$ is orthogonal to all $\vec{c}_i$ for $i = 1, 2, \cdots, n$.

1. If $\vec{b} \notin S_{n-k}$, equation 3 has no solution.

2. If $\vec{b} \in S_{n-k}$, then $\vec{b}$ is orthogonal to $\mathcal{N}(A^t)$. The solution of equation 3 is

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

where $\vec{x}_p$ is a particular solution of 3 and $\vec{x}_h$ is a solution of the associated homogeneous equation given by

$$\vec{x}_h = \sum_{i=1}^{i=k} a_i \vec{e}_i$$

where $\vec{e}_i$ are independent vectors which span $\mathcal{N}(A)$ and $a_i$ are arbitrary constants.

2 Hilbert Space

Let $E$ be an infinite dimensional vector space and let $\vec{x} = \{x_1, x_2, \cdots, x_n, \cdots\} \in E$. For example, consider the linear vector space of all functions $f(t)$ continuous on the closed interval $[a, b]$. We denote such a vector space by $C(a, b)$.

1. **Normed Space**: $E$ is said to be a normed vector space if a norm $||\vec{x}||$ is defined in $E$. For example, in $C(a, b)$, we define the norm as the maximum value of the function in the interval $(a, b)$.

2. **Convergence and Complete Space**: A sequence of vectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n, \cdots,$ in $E$ is said to converge to a vector $\vec{x}$ in $E$ if, given $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$||\vec{x} - \vec{x}_n|| < \epsilon,$$  \hspace{1cm} (8)

for all $n > N$. It follows that for a given $\epsilon > 0$, there exists an integer $N_1(\epsilon)$ such that

$$||\vec{x}_m - \vec{x}_n|| < \epsilon,$$  \hspace{1cm} (9)

for all $n$ and $m$ greater than $N_1$. Such a sequence is known as a Cauchy sequence. The vector $\vec{x}$ is called the limit of the sequence, and we write

$$\vec{x} = \lim_{n \to \infty} \vec{x}_n.$$  \hspace{1cm} (10)

A vector space $E$ is said to be complete if for every Cauchy sequence $\{\vec{x}_n\}$ in $E$ there exists a vector $\vec{x} \in E$ such that $\vec{x} = \lim_{n \to \infty} \vec{x}_n$.
3. **Inner Product**: For finite dimensional spaces we have defined the inner product of two vectors by (1). However, for an infinite dimensional space such a product will tend to infinity and thus is meaningless. Various inner products are usually used depending on the spaces. For example, for two functions \( u \) and \( v \) in \( C(0, 1) \), we use the inner product

\[
(u, v) = \int_0^1 u(t)v(t)dt, \tag{11}
\]

and the norm \( ||u|| = (u, u)^{\frac{1}{2}} \), where \( u \) and \( v \in C(0, 1) \).

**Definition**: A complete normed linear space with an inner product is called a *Hilbert space*.

### 3 Sturm-Liouville Theory

Consider the second order differential operator in self-adjoint form

\[
\mathcal{L} \equiv \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x), \tag{12}
\]

where \( p(x) \in C^1(a, b) \) and \( q(x) \in C(a, b) \). Consider the homogeneous equation

\[
\mathcal{L}u = \lambda r(x)u \tag{13}
\]

where \( r(x) \in C(a, b) \) and \( \lambda \) is a constant. The constant \( \lambda \) for which a nontrivial solution to (13) exists is called an *eigenvalue* of \( \mathcal{L} \), and the solution \( u(x) \in C^2(a, b) \) corresponding to \( \lambda \) is called an *eigenfunction*.

**Definition**: If \((\mathcal{L}u, v) = (u, \mathcal{L}^*v)\) for all \( u \) and \( v \in C^2(a, b) \), then \( \mathcal{L}^* \) is called the *adjoint operator* of \( \mathcal{L} \).

**Definition**: An operator \( \mathcal{L} \) is said to be self-adjoint if \( \mathcal{L}^* = \mathcal{L} \).

**Theorem**: Every pair of eigenfunctions belonging to distinct eigenvalues of a self-adjoint operator \( \mathcal{L} : S \subseteq C^2(a, b) \to C(a, b) \) are orthogonal with respect to the weight function \( r(x) \).

**Proof**: \( \mathcal{L}u = \lambda ru \) and \( \mathcal{L}v = \mu rv \). Consider the inner products, \((v, \mathcal{L}u) = \lambda (ru, v)\) and \((u, \mathcal{L}v) = \mu (rv, u)\). Since the operator is self-adjoint, the two inner products are equal. Therefore, \( \lambda (ru, v) = \mu (rv, u) \). Or \((\lambda - \mu)(ru, v) = 0\). Since \( \lambda \neq \mu \), \((ru, v) = 0\).
3.1 Conditions for $L$ to be Self–Adjoint

If $L$ is self–adjoint, we have

$$(L u, v) = (u, L v)$$ (14)

for all functions $u$ and $v \in S$. Substituting the expression for $L$ from 13, we get

$$\int_a^b \left[ \frac{d}{dx} (p u') + q u \right] v dx = \int_a^b u \left[ \frac{d}{dx} (p v') + q v \right] dx$$ (15)

which, after integration by parts, yields the condition

$$[p(uv' - u'v)]_a^b = 0.$$ (16)

where the expression $[X]_a^b = X(b) - X(a)$. This condition shows that the condition for an operator to be self–adjoint depends on the property of the function space $S$ as well as on the operator through the function $p(x)$. The following shows typical conditions to be satisfied for self–adjointness.

1. $u(a) = 0, u(b) = 0$, for all $u \in S$.
2. $u'(a) = 0, u'(b) = 0$, for all $u \in S$.
3. $u'(a) - \sigma_1 u(a) = 0, u'(b) - \sigma_2 u(b) = 0$, for all $u \in S$.
4. $u(a) = u(b)$, and, $p(a)u'(a) = p(b)u'(b)$.
5. $u(a)$ and $u'(a)$ are finite and $p(a) = 0$, and $u(b)$, and, $u'(b)$ are finite and $p(b) = 0$.

Note that the first three conditions are homogeneous boundary conditions which defines the function space $S$ and are independent of the coefficients of the operator. Conditions (4) are periodic and impose conditions on the coefficient $p(x)$ of the operator. Finally, conditions (5) is an example of homogeneous boundary conditions satisfied by the coefficient $p(x)$ of the operator.

These examples clearly shows that self–adjointness is a property of the boundary value problem rather than that of the operator.

**Theorem:** Eigenfunctions corresponding to distinct eigenvalues of a self-adjoint operator $L : S \subseteq C^2(a, b) \to C(a, b)$ are orthogonal with respect to the weight function $r(x)$ and, for homogeneous boundary conditions eigenfunctions corresponding to the same eigenvalue are not independent.

**Proof:** We only give proof to the last statement. If $L u = \lambda r u$ and $L v = \lambda r v$, then

$$v L u - u L v = [p(vu' - uv')]'.$$
Integrating gives, \( p(vu' - uv') = \text{Constant}. \) For homogeneous conditions the constant is zero, and we have, \( vu' - uv' = 0 \), or

\[
\frac{u'}{u} = \frac{v'}{v},
\]

which after integration gives, \( u = K v \), where \( K \) is a constant.

**Definition:** A set of functions \( \{ \varphi_0, \varphi_1, \cdots, \varphi_n, \cdots \} \) is a basis for \( C[a, b] \) if the set is linearly independent and if every element \( f \in C[a, b] \) may be written as a linear combination of the set. Such a set of functions is called a complete set of functions.

**Theorem:** The eigenfunctions \( \{ \varphi_0, \varphi_1, \cdots, \varphi_n, \cdots \} \) of a self-adjoint operator \( \mathcal{L} : S \subseteq C^2(a, b) \rightarrow C(a, b) \) form a complete set.

**Corollary:** If a set of functions is complete in a space then any function in the space can be written as a linear combination of the elements of the complete set. Thus for any square integrable continuous function \( f(x) \) we can write

\[
 f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x),
\]

where \( \varphi_n(x) \) are the eigenfunctions of the adjoint operator.

**Theorem:** Any regular self-adjoint boundary–value problem has an infinite sequence of real eigenvalues \( \lambda_0 < \lambda_1 < \lambda_2 \cdots \) with

\[
 \lim_{n \to \infty} \lambda_n = \infty.
\]

The eigenfunction \( \varphi_n(x) \) corresponding to the eigenvalue \( \lambda_n \) has exactly \( n \) zeros in the interval \([a, b]\).

Example: Consider the boundary–value problem

\[
 \frac{d^2y}{dx^2} + \lambda y = 0, \ y(0) = 0, \ y(\pi) = 0,
\]

whose solution is \( \lambda_n = (n + 1)^2, n = 1, 2, \cdots \) and \( \varphi_n = \sin(n + 1)x \). Then \( \varphi_0 = \sin x \) has no zeros between 0 and \( \pi \), while \( \varphi_n = \sin(n + 1)x \) has \( n \) zeros between 0 and \( \pi \).

**Theorem:** For a regular self–adjoint boundary–value problem, the eigenvalues \( \lambda_n \) are given by the asymptotic formula

\[
 \sqrt{\lambda_n} = \frac{n\pi}{b-a} + \frac{O(1)}{n} \text{for } n = 1, 2, \cdots.
\]
4 Existence and Uniqueness of the Solution of $\mathcal{L}y = f$

**Theorem:** If the homogeneous equation $\mathcal{L}y = 0$ has a non-trivial solution, the solution of the corresponding non-homogeneous equation is not unique. Conversely, if the solution of the non-homogeneous equation is not unique, there exists a non-trivial solution of the homogeneous equation.

**Theorem:** The non-homogeneous equation

$$\mathcal{L}y = f$$  \hfill (18)

has a solution for a given function $f$ if, and only if, $f$ is orthogonal to the null space of the adjoint homogeneous equation

$$\mathcal{L}^*z = 0.$$  \hfill (19)

That is if

$$(f, z) = 0$$  \hfill (20)