Math 60850, Final Test

Dec 7, 2016. Due Friday Dec 16, 2016 at 10:00 am

Instructor: Bei Hu Name: Answer

There is a total of 150 points on this exam. There is a total of 8 problems. *Hand in your exam paper to either the instructor at 174 HURL or to Kathy Phillips at 153 HURL when completed.*

Rules: You may consult books and class notes, go to libraries and search online, use calculators and softwares. *The work should be your own.* You cannot ask for help from anyone other than yourself. If you need clarification on a problem, consult your instructor.

When you quote a theorem or a result, give the page number and identify it. *e.g., (3) Theorem on page 256. you must show your work.* No credit will given if no work is shown even if the answer is correct.

1 (20 points). A family of 4 children, each of which is equally likely to be a boy or a girl, independent of others.

(a) What is the probability that the first one is boy, and third one is a girl?

Independent

$$\frac{1}{2} \times \frac{1}{2} = \boxed{\frac{1}{4}}$$

(b) What is the probability of having at least two girls?

$$P(2 \text{ girls}) + P(3 \text{ girls}) + P(4 \text{ girls})$$
$$= \binom{4}{2}\left(\frac{1}{2}\right)^4 + \binom{4}{3}\left(\frac{1}{2}\right)^4 + \binom{4}{4}\left(\frac{1}{2}\right)^4 = \boxed{\frac{11}{16}}$$

(c) What is the probability of having at least two boys, given that the family has at least one girl?

A = at least two boys, B = at least one girl

$$P(B) = 1 - P(\text{no girls}) = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$
$$P(AB) = \binom{4}{2}\left(\frac{1}{2}\right)^4 + \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{10}{16}$$
$$P(A|B) = \frac{P(AB)}{P(B)} = \boxed{\frac{2}{3}}$$

(d) Which of the following **pairs** AB, BC, AC are independent events?

$A = \{$all the children are of the same sex$\}$,
$B = \{$the family includes both boys and girls$\}$,
$C = \{$there is at least one girl$\}$.

$$P(A) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{2}{16}$$
$$P(B) = 1 - P(A) = \frac{14}{16}$$
$$P(C) = 1 - P(\text{no girls}) = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$

$$P(AB) = 0 \neq P(A)P(B)$$
$$P(AC) = P(\text{all girls}) = \frac{1}{16} \neq P(A)P(C)$$
$$P(BC) = P(B) = \frac{14}{16} \neq P(B)P(C)$$

$\boxed{\text{None of the pairs are independent}}$

2 (18 points). Suppose X and Y are independent discrete random variables taking values $\{0, 1, 2, \cdots\}$ with the same mass function $f(j)$.

(a) For $a > 0$, find $P(\min(X,Y) < a)$.

$$= 1 - P(\min(X,Y) \geq a) = 1 - P(X \geq a)\,P(Y \geq a)$$
$$= 1 - (P(X \geq a))^2 = 1 - \left(\sum_{j \geq a} f(j)\right)^2$$

(b) For $a > 0$, $P(|X - Y| < a)$.

$$= P(Y - a < X < Y + a) = \sum_{j=0}^{\infty} P(Y - a < X < Y + a \mid Y = j)\,P(Y = j)$$
$$= \sum_{j=0}^{\infty} P(j - a < X < j + a)\, f(j)$$
$$= \sum_{j=0}^{\infty} \left(\sum_{j-a < k < j+a} f(k) \right) f(j)$$

(c) Find $P(X \leq Y \leq 2X)$.

$$= \sum_{j=0}^{\infty} P(X \leq Y \leq 2X \mid X = j)\; P(X = j)$$
$$= \sum_{j=0}^{\infty} P(j \leq Y \leq 2j)\; f(j)$$
$$= \sum_{j=0}^{\infty} \left(\sum_{k=j}^{2j} f(k) \right) f(j)$$

3 (18 points). Two factories A and B manufacture parachutes. Factory A produces 20% of the parachutes and factory B produces 80% of the parachutes. The defective rate for factory A is 1 in 100,000 and the defective rate for factory B is 2 in 100,000.

(a) What is the probability a randomly selected parachute is defective?

$$= P(\text{from } A) \cdot P(\text{defective} \mid \text{from } A) + P(\text{from } B) \cdot P(\text{defective} \mid \text{from } B)$$

$$= 0.2 \times \frac{1}{100000} + 0.8 \times \frac{2}{100000}$$

$$= 1.8 \times 10^{-5}$$

(b) Given that a parachute is defective, what is the probability that it is from factory A?

$$P(\text{from } A \mid \text{defective}) = \frac{P(\text{from } A \text{ and defective})}{P(\text{defective})}$$

$$= \frac{0.2 \times \frac{1}{100000}}{1.8 \times 10^{-5}} = \frac{1}{9}$$

(c) A shipment of 100 parachutes is received from a distribution center (which distribute parachutes from both factories). What is the probability that all of them are not defective? [Leave your answer as it is, do not simplify].

$$(1 - 1.8 \times 10^{-5})^{100}$$

$$\approx 0.9982$$

4 (18 points). Suppose $G(s)$ is the generating function of the random variable X. The random variable X takes non-negative integer values. In your answers, you may use the derivative and integral of G.

(a) Find $E(X^3)$ in terms of the function G.

$$G(s) = E(s^X), \quad G'(s) = E(X s^{X-1}), \quad G'(1) = EX$$

$$G''(s) = E(X(X-1)s^{X-2}), \quad G''(1) = E(X^2 - X)$$

$$G'''(s) = E(X(X-1)(X-2)s^{X-3}), \quad G'''(1) = E(X^3 - 3X^2 + 2X)$$

Thus, $$EX^3 = G'''(1) + 3G''(1) + G'(1)$$

(b) Find $E\Big(\dfrac{1}{X+3}\Big)$ in terms of the function G.

$$G(s) = E(s^X)$$

$$\int_0^1 s^2 G(s)\, ds = \int_0^1 E(s^{X+2})\, ds = E \int_0^1 s^{X+2}\, ds$$

$$= E \left. \frac{s^{X+3}}{X+3} \right|_0^1 = E\left(\frac{1}{X+3}\right)$$

(c) Find $\sum\limits_{n=0}^{\infty} s^n P(X \le n)$ in terms of $G(s)$.

$$\sum_{n=0}^{\infty} s^n P(X \le n) = \sum_{n=0}^{\infty} s^n \sum_{j=0}^{n} P(X=j)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} s^{n-j}\, s^j P(X=j)$$

$$= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} s^{n-j}\, s^j P(X=j)$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s^k\, s^j P(X=j)$$

$$= \left(\sum_{j=0}^{\infty} s^j P(X=j)\right)\left(\sum_{k=0}^{\infty} s^k\right) = \frac{G(s)}{1-s}$$

5 (20 points). Here is a simple random walk. Let $S_n = S_0 + \sum_{i=1}^{n} X_i$, where X_i are independent, and takes the value $-1, 1$. i.e., we allow the walk to pause. Assume

$$P(X_i = 1) = p,\ P(X_i = -1) = q.$$

Let T_k be the time it takes to go from the site 0 to site k for the first time. Let $J_k = E(T_k)$. Obviously $J_0 = 0$.

(a) Find the equation for J_k.

Let T_{ij} be the time it takes to go from i to j sites for the first time. Then, $\forall k > 0$

$$J_k = E(T_k) = E(T_{01} + T_{12} + \cdots + T_{k-1,k}) = k E T_1 = k J_1$$

Also

$$J_1 = E(T_1) = E(T_1 \mid X_1 = 1) P(X_1 = 1) + E(T_1 \mid X_1 = -1) P(X_1 = -1)$$

$$= 1 \cdot p + (J_2 + 1) q$$

$$= 1 + q J_2 = 1 + 2q J_1$$

(*) $\boxed{J_1 = 1 + 2q J_1}$

(b) Find J_k.

① $p > q$, then $2q < 1$, and (*) implies

$$J_1 = \frac{1}{1-2q} = \frac{1}{p-q}, \quad \text{so that}$$

$$J_k = k J_1 = \frac{k}{p-q}, \quad k \geq 0$$

② $p = q$, then (*) becomes $J_1 = 1 + J_1$. The only solution is $J_1 = +\infty$, so that

$$J_k = k J_1 = +\infty, \quad k \geq 1.$$

③ $p < q$, then $2q > 1$, and the only solution to (*) is $J_1 = +\infty$, so that $J_k = +\infty$, $k \geq 1$.

6 (20 points). Suppose that X_i are independent and identically distributed continuous random variable with $EX_i = 0$ and $Var X_i < +\infty$. Which of the following law of large number is valid? Please state whether each law is valid or invalid. For each law that is valid, please quote your theorem or provide a short proof.

(a) $\frac{1}{N}\sum_{j=1}^{N} X_j \to 0$ in distribution.

Valid. Using (b) and Lemma 5 on page 311.

(b) $\frac{1}{N}\sum_{j=1}^{N} X_j \to 0$ in probability.

Valid. Using (c) and Lemma 10 (c) on page 312

(c) $\frac{1}{N}\sum_{j=1}^{N} X_j \to 0$ almost surely.

Valid. Theorem 3 on page 326

(d) $\frac{1}{N}\sum_{j=1}^{N} X_j \to 0$ mean square.

Valid. Theorem 3 on page 326.

7 (18 points). Suppose that S_n is a Martingale with respect to $S_1, S_2, \cdots, S_n$, i.e.,

$$E(S_{n+1}|S_1, \cdots, S_n) = S_n.$$

(a) For $i < j$, prove that $E[S_i^3(S_j - S_i)] = 0$.

$$E[S_i^3(S_j - S_i)] = E[E[S_i^3(S_j - S_i) | S_1, \cdots S_i]]$$

$$= E[S_i^3 E(S_j | S_1, \cdots, S_i)] - ES_i^4$$

$$= ES_i^4 - ES_i^4 = 0$$

(b) Is it true that for $i < j$, prove that $E[S_j^3(S_j - S_i)] = 0$. Prove or give an counter example.

Not True

Simple random walk with $p = q = \frac{1}{2}$, $S_0 = 0$

It is a Martingale with respect to $S_1, \cdots, S_i, \cdots$

$j = 2$, $i = 1$

$$E(S_2^3(S_2 - S_1)) = E((X_1 + X_2)^3 X_2)$$

$$= P(X_1 = 1)P(X_2 = 1)\, 2^3 \cdot 1 + P(X_1 = 1)P(X_2 = -1) \cdot 0^3 \cdot (-1)$$

$$+ P(X_1 = -1)(P(X_2 = 1)\, 0^3 \cdot 1 + P(X_1 = -1)P(X_2 = -1)(-2)^3(-1)$$

$$= \frac{1}{4} \times 8 \cdot 1 + 0 + 0 + \frac{1}{4} \cdot (-8) \cdot (-1)$$

$$= 4 \neq 0$$

8 (18 points). Let $\{W_t\}$ be the standard Wiener process ($W_0 = 0, EW_t^2 = 1$)

(a) Using Itô's formula, find

$$d(W_t)^3$$

(show all your work, please add extra pages if you need more space)

$$d(W_t)^3 = 3(W_t)^2 dW_t + \frac{3\cdot 2}{2!}\cdot W_t\, dt$$

$$= 3(W_t)^2 dW_t + 3W_t\, dt$$

(b) Let $X_t = \int_0^t W_t dt$. Find the integral

$$\int_0^t (W_t)^2 dW_t$$

in terms of X_t.

(show all your work, please add extra pages if you need more space)

By (a)

$$(W_t)^3\Big|_0^t = 3\int_0^t (W_t)^2 dW_t + 3\int_0^t W_t\, dt$$

i,e

$$W_t^3 = 3\int_0^t (W_t)^2 dW_t + 3X_t$$

or

$$\int_0^t (W_t)^2 dW_t = \frac{1}{3}W_t^3 - X_t$$