Math 60850, Final Exam

Dec 6, 2017. Due Monday Dec 11, 2017 at 10:00 am

Instructor: Bei Hu Name: Answers

There is a total of 150 points on this exam. There is a total of 8 problems. *Hand in your exam paper to either the instructor at 174 HURL or to Kathy Phillips at 153 HURL when completed.*

Rules: You may consult books and class notes, go to libraries and search online, use calculators and softwares. *The work should be your own.* You cannot ask for help from anyone other than yourself. If you need clarification on a problem, consult your instructor.

When you quote a theorem or a result, give the page number and identify it. *e.g., (3) Theorem on page 256. you must show your work.* No credit will be given if no work is shown even if the answer is correct.

1 (20 points). A pair of fair dice is tossed. Let X_1 be the value of first dice, X_2 be the value of second dice, and $X = X_1 + X_2$ be the total of two dice. Obviously $P(X = 2) = P(X = 12) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

(a) Find $P(X = k)$ for $2 \leq k \leq 12$.

$P(X=2) = P(X=12) = \frac{1}{36}$

$P(X=3) = P(X=11) = \frac{2}{36}$

$P(X=4) = P(X=10) = \frac{3}{36}$

$P(X=5) = P(X=9) = \frac{4}{36}$

$P(X=6) = P(X=8) = \frac{5}{36}$

$P(X=7) = \frac{6}{36}$

(b) Find EX, $VarX$.

$EX = E(X_1+X_2) = EX_1 + EX_2 = 2EX_1 = 2 \cdot \frac{7}{2} = 7$

Since X_1 and X_2 are independent

$VarX = Var(X_1+X_2) = VarX_1 + VarX_2 = 2VarX_1 = \frac{35}{6}$

(c) Which of the three pairs $a), (X_1, X_2)$; $b), (X, X_1)$; $c), (X, X_2)$ are independent? Circle your answer.

$a)$ independent, not independent;

$b)$ independent, not independent;

$c)$ independent, not independent;

(d) Find conditional expectation $E(X_1|X = 7)$.

Since $E(X_1|X=7) = E(X_2|X=7)$ by symmetry,

$$E(X_1|X=7) = \frac{1}{2}[E(X_1|X=7) + E(X_2|X=7)]$$
$$= \frac{1}{2} E(X_1+X_2|X=7) = \frac{7}{2}$$

2 (18 points). A coin with head probability $P(X=1)=p$ and tail probability $P(X=0)=q=1-p$. The coin in thrown repeatedly until exactly n heads are obtained. Let $Y=$ "total number of tosses". Obviously, $P(Y=n)=p^n$.

(a) Find $P(Y=k)$ for $k=n,n+1,n+2,n+3,\cdots$.

First $k-1$ tosses must produce $n-1$ heads, the last one must be a head

$$P(Y=k)=\binom{k-1}{n-1}p^{n-1}\cdot q^{k-n}\cdot p$$
$$=\binom{k-1}{n-1}p^{n}\cdot q^{k-n},\quad k=n,n+1,\cdots$$

(b) Find the generating function $G_Y(s)$.

Since
$$1=\sum_{k=n}^{\infty}P(Y=k)=\sum_{k=n}^{\infty}\binom{k-1}{n-1}p^n q^{k-n}=\left(\frac{p}{q}\right)^n\sum_{k=n}^{\infty}\binom{k-1}{n-1}q^k$$
we obtain
$$\sum_{k=n}^{\infty}\binom{k-1}{n-1}q^k=\left(\frac{q}{p}\right)^n=\left(\frac{q}{1-q}\right)^n,\quad \forall\, 0<q<1.$$
Thus
$$G_Y(s)=Es^Y=\sum_{k=n}^{\infty}s^k P(Y=k)=\sum_{k=n}^{\infty}s^k\binom{k-1}{n-1}p^n q^{k-n}$$
$$=\left[\sum_{k=n}^{\infty}\binom{k-1}{n-1}(sq)^k\right]\cdot\left(\frac{p}{q}\right)^n=\left(\frac{sq}{1-sq}\right)^n\cdot\frac{p^n}{q^n}=\left(\frac{sp}{1-sq}\right)^n$$

(c) Find EY.

$$EY=G_Y'(1)$$
$$=\frac{n}{p}$$

3 (18 points). Two factories A and B manufacture drones. Factory A produces 25% of the drones and factory B produces 75% of the drones. The defective rate for factory A is 1 in 100 and the defective rate for factory B is 4 in 100.

(a) What is the probability a randomly selected drone is defective?

$$P(\text{def}) = P(\text{def} \mid \text{from A})P(\text{from A}) + P(\text{def} \mid \text{from B})P(\text{from B})$$

$$= \frac{1}{100} \cdot 0.25 + \frac{4}{100} \cdot 0.75$$

$$= \frac{13}{400}$$

(b) Given that a drone is defective, what is the probability that it is from factory A?

$$P(\text{from A} \mid \text{def}) = \frac{P(\text{def and from A})}{P(\text{def})}$$

$$= \frac{P(\text{def} \mid \text{from A})\, P(\text{from A})}{P(\text{def})}$$

$$= \frac{\frac{1}{100} \cdot 0.25}{\frac{13}{400}} = \frac{1}{13}$$

(c) A shipment of 100 drones is received. Let X be the total number of defective drones. Find $EX, VarX$.

$$X_i = \begin{cases} 1, & i\text{th drone is defective} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$X = X_1 + X_2 + X_3 + \cdots + X_{100}$$

and

$X_1, X_2, \cdots, X_{100}$ are independent

Thus

$$EX = 100\, EX_1 = 100 \cdot \frac{13}{400} = \frac{13}{4}$$

$$\text{Var}X = 100\, \text{Var}X_1 = 100 \cdot \left(\frac{13}{400} - \left(\frac{13}{400}\right)^2 \right) = \frac{5031}{1600}$$

4 (18 points). The random variable X takes non-negative integer values. Suppose $G(s)$ is the generating function of the random variable X.

(a) Find $E(X^4)$ in terms of the function G.

$EX = G'(1)$, $E(X(X-1)) = G''(1)$

$E(X(X-1)(X-2)) = G'''(1)$, $E(X(X-1)(X-2)(X-3)) = G^{(4)}(1)$.

Solving these equations

$$EX^4 = G^{(4)}(1) + 6G'''(1) + 7G''(1) + G'(1)$$

(b) Is $\dfrac{G(s/2)+G(s/3)}{G(1/2)+G(1/3)}$ a generating function? Justify your answer.

Yes. This is a power series of s with coefficients satisfying

(i) non-negative

(ii) add-up to 1.

(c) Find $\sum_{n=0}^{\infty} s^n P(X > n)$ in terms of $G(s)$.

$$\sum_{n=0}^{\infty} s^n P(X>n) = \sum_{n=0}^{\infty} s^n \sum_{j=n+1}^{\infty} P(X=j)$$

$$= \sum_{j=1}^{\infty} \sum_{n=0}^{j-1} s^n P(X=j)$$

$$= \sum_{j=1}^{\infty} \frac{1-s^j}{1-s} P(X=j)$$

$$= \frac{1}{1-s} \sum_{j=0}^{\infty} (1-s^j) P(X=j) \qquad \left(\begin{matrix} 1-s^j=0 \\ \text{if } j=0 \end{matrix}\right)$$

$$= \frac{1}{1-s}(1-G(s))$$

5 (20 points). Here is a simple random walk. Let $S_n = S_{n-1} + X_n$ if $1 \le S_{n-1} \le N-1$, where X_i are independent, and takes the value $-1, 1$. Assume

$$P(X_i = 1) = p,\ P(X_i = -1) = q, \quad p + q = 1.$$

We further assume that

(1) N is an absorbing barrier, i.e., the process stops if it reaches N.

(2) 0 is an reflecting boundary, i.e., if $S_{n-1} = 0$, then S_n is assigned to be 1.

Let J_k be the mean duration of the walk.

(a) Show that $J_N = 0$, $J_0 = 1 + J_1$.

$J_N = 0$ since the process stops when it reaches N

If it is at site 0, it will reach site 1 in 1 time unit. Thus $J_0 = 1 + J_1$

(b) Find the equation for J_k.

Conditioned on the first step:

$$J_k = (1 + J_{k+1})p + (1 + J_{k-1})q = pJ_{k+1} + qJ_{k-1} + 1$$

or:

$$pJ_{k+1} - J_k + qJ_{k-1} = -1$$

(b) Suppose $p > q$. Find J_k.

(i) Assuming $J_k = \theta^k$, then the homogenous part:

$\theta^{k-1}(p\theta^2 - \theta + q) = 0$. Thus, $\theta_1 = 1$, $\theta_2 = \frac{q}{p}$

so that $$J_k^h = C_1 + C_2\left(\frac{q}{p}\right)^k$$

(ii) Particular solution $J_k^p = Ak$:

$pA(k+1) - Ak + qA(k-1) = -1$. Thus $A = \frac{1}{q-p}$

(iii) The general solution $J_k = C_1 + C_2\left(\frac{q}{p}\right)^k + \frac{k}{q-p}$

Solving C_1 and C_2 using two boundary conditions

$$J_k = \frac{N-k}{p-q} + \frac{2pq}{(p-q)^2}\left(\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k\right)$$

6 (20 points). Let $\{X(t)\}$ be a simple birth-death process, i.e., the birth intensity $\lambda_n = n\lambda$ and the death intensity $\mu_n = n\mu$. Let $p_j(t) = P(X(t) = j)$.

(a) Find the differential equation system for $p_j(t)$ $(j = 0, 1, 2, 3, \cdots)$.

(show all your work)

From page 271

$$p_j'(t) = \lambda(j-1)p_{j-1}(t) + \mu(j+1)p_{j+1}(t) - (\lambda+\mu)j\,p_j(t), \quad j \geq 1$$

$$p_0'(t) = \mu p_1(t)$$

(b) Find $EX(t)$ for all t, assuming that $X(0) = 1$.

(show all your work)

$y(t) = EX(t)$ satisfies $y(0) = 1$, and

$$y(t) = \sum_{j=0}^{\infty} j\,P(X(t)=j) = \sum_{j=0}^{\infty} j\,p_j(t)$$

$$y'(t) = \sum_{j=0}^{\infty} j\left\{\lambda(j-1)p_{j-1}(t) + \mu(j+1)p_{j+1}(t) - (\lambda+\mu)j\,p_j(t)\right\}$$

$(p_{-1}(t) = 0)$

$$= \underbrace{\sum_{j=0}^{\infty} \lambda(j-1)^2 p_{j-1}(t) + \sum_{j=0}^{\infty} \mu(j+1)^2 p_{j+1}(t) - \sum_{j=0}^{\infty}(\lambda+\mu)j^2 p_j(t)}_{=0}$$

$$+ \sum_{j=0}^{\infty} \lambda(j-1)p_{j-1}(t) - \sum_{j=0}^{\infty} \mu(j+1)p_{j+1}(t)$$

$$= (\lambda-\mu)\,y(t)$$

Thus $y(t) = e^{(\lambda-\mu)t}$

7 (18 points). Suppose that X_j ($j = 1, 2, 3, \cdots$) are independent, identically distributed continuous random variables with density $f(x)$, $f(x) = 0$ for $-\infty < x < 1$. Define $Y_n = (X_1 \cdot X_2 \cdots X_n)^{1/n}$. Please quote the theorems in the book, do not start from the beginning.

(a) Suppose that $E(\ln X_1) = \int_1^{\infty} f(x)\ln(x)dx < \infty$, show that Y_n converges in distribution.

(i) $\ln Y_n = \frac{1}{n}\{\ln X_1 + \ln X_2 + \cdots + \ln X_n\}$.

Since $\ln X_1, \ln X_2, \cdots$, i.i.d and $E(\ln X_1)$ is finite

$$\ln Y_n \xrightarrow{D} \mu = E(\ln X_1)$$

(Theorem 2 on page 193)

(ii) Thus

$$Y_n \xrightarrow{D} e^{\mu}$$

(Theorem 18 on page 316)

(b) Find a condition on $f(x)$ so that Y_n converges almost surely.

If you quote Theorem 3 on page 326, it would require

$$E((\ln X_1)^2) = \int_1^{\infty} f(x)(\ln x)^2 dx < \infty$$

If you quote Theorem 1 on page 329, no additional assumptions are necessary.

8 (18 points). Let $\{W_t\}$ be the standard Wiener process ($W_0 = 0, EW_t = 0, EW_t^2 = t$)

(a) Let $r > 0, \sigma > 0$ be positive constants. Let X_t satisfy the geometric Brownian motion, i.e.,

$$\frac{dX_t}{X_t} = rdt + \sigma^2 dW_t.$$

Using Itô's formula, find

$$d(\ln |X_t|)$$

(show all your work)

Itô's formula Theorem 2 on page 545

$$d(\ln|X_t|) = \left\{ \frac{1}{X_t} \cdot rX_t - \frac{1}{2}\frac{1}{X_t^2}(\sigma^2 X_t)^2 \right\} dt + \frac{1}{X_t}(\sigma^2 X_t)\, dW_t$$

$$= \left(r - \frac{\sigma^4}{2}\right) dt + \sigma^2 dW_t$$

(b) Let $Y_t = (1 + t^2) \cos W_t$. Find dY_t.

(show all your work)

Itô's formula Theorem 4 on page 545

$$dY_t = \left\{ 2t \cos W_t - \frac{1}{2}(1+t^2)\cos W_t \right\} dt - (1+t^2)\sin W_t \, dW_t$$