ACMS 60850: Applied Probability, Midterm Test
Oct 7, 2015.

Instructor: Bei Hu **Name:**____________________________________

There is a total of 100 points on this exam plus a possible 5 bonus points. This is a 75 minutes test.

1. (10 points) State precisely

(a) The definition of mass function $f(x)$ of a discrete random variable.

Ans: $f(x) = P(X = x)$. There is a countable set $\{x_j\}$, $P(X = x) = 0$ for $x \neq x_j$.

(b) The definition of covariance of random variables X and Y.

Ans: (i), $Cov(X, Y) = E(XY) - EX \cdot EY$

or (ii), $Cov(X, Y) = E[(X - EX)(Y - EY)]$

2. (20 points) Determine whether each of the following statements is true or false. No justification is needed.

(a) If the events A and B are disjoint, then the events A and B must also be independent.

True ___ False ___

False

(b) For any random variables X and Y, we always have $E(XY) = E(X) \cdot E(Y)$.

True ___ False ___

False, Need to be uncorrelated.

(c) A random variable X must either be continuous or discrete.

True ___ False ___

False

(d) Suppose that $F(x) = P(X \leq x)$ is the distribution of the random variable X. Then it is always true that $F(x)$ is right-continuous, i.e., $F(x+h) \to F(x)$ as $h \searrow 0$ $(h > 0)$.

True ___ False ___

True, This is one of the properties of probability space.

3. (15 points) Factories A and B produce toy drones. Factory A produces 3 times as many drones as factory B (i.e, for every 4 drones, 3 from A and 1 from B). A drone is considered defective if it cannot stay in the air for specified amount of time. The probability that a drone produced by factory A is defective is 0.1 and the probability that a drone produced by factory B is defective is 0.05. Show all your work.

(a) A drone is selected at random. What is the probability that it is defective?

Ans: $P(Def) = P(Def|A)P(A) + P(Def|B)P(B) = 0.1 \cdot \frac{3}{4} + 0.05 \cdot \frac{1}{4} = 0.0875$

(b) A drone is selected at random and is found to be defective. What is the probability it came from factory A?

Ans: $P(A|Def) = \frac{P(A\&Def)}{P(Def)} = \frac{P(Def|A)P(A)}{P(Def)} = \frac{0.1 \cdot \frac{3}{4}}{0.0875} = 0.8571428$

4. (15 points) A daycare center with 100 children.

(a) Let's make the assumptions mathematically rigorous. Assume $P(B) = P(G) = \frac{1}{2}$, assume also that conceiving a baby of any gender is independent of any other conceptions, assume further each conception resulted exactly one child either a boy or a girl. What is the expected number of boys in this daycare center? Show all your work.

Ans: Let $X_i = 1$ if ith child is a boy, and $X_i = 0$ otherwise. Then

$$EX_i = \frac{1}{2},$$

and

$$E\sum_{i=1}^{100} X_i = 100 \cdot \frac{1}{2} = 50.$$

(b) Continued from (a). What is the probability that this daycare center has exactly 50 girls? (Please give your answers. Do not simplify). Show all your work.

Ans:

$$\binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50} = \binom{100}{50}\left(\frac{1}{2}\right)^{100}$$

5. (15 points)

Let X and Y be independent discrete random variables with mass function $f_X(x)$ and $f_Y(x)$. Find the following in terms of f_X and f_Y.

(a) $P(\min\{X,Y\} \le x)$.

Ans:

$$\begin{aligned} P(\min\{X,Y\} \le x) &= 1 - P(\min\{X,Y\} > x) \\ &= 1 - P(X > x, Y > x) \\ &= 1 - P(X > x)P(Y > x) \\ &= 1 - \sum_{u>x} f_X(u) f_Y(u) \end{aligned}$$

(b) $P(X = Y)$.

Ans:

$$\begin{aligned} P(X = Y) &= \sum_u P(X = u, Y = u) \\ &= \sum_u f_X(u) f_Y(u) \end{aligned}$$

(c) $P(X + Y = x)$.

Ans:

$$\begin{aligned} P(X + Y = x) &= \sum_u P(X = u, Y = x - u) \\ &= \sum_u f_X(u) f_Y(x - u) \end{aligned}$$

6. (15 points) (Single sided absorbing barrier). Consider the simple random walk: $S_n = S_0 + \sum_{i=1}^{n} X_i$, where X_i are independent, and takes the value -1 and 1. Assume that $P(X_i = 1) = p, \quad P(X_i = -1) = 1 - p = q$, $p > 0, 1 - p = q > 0$. The stopping rule is the site 0 is hit.

(a) Let J_k be the probability of hitting 0 at the stopping time while starting at site k ($S_0 = k$). Find the equation for J_k for $k \geq 1$. Obviously $J_0 = 1$.

Ans: $J_k = P(X_1 = 1)J_{k+1} + P(X_1 = -1)J_{k-1} = pJ_{k+1} + qJ_{k-1}, \quad k \geq 1$.

(b) Assuming $p \neq q$. Find the general solution for J_k containing two constants.

Ans: $J_k = \theta^k$, then $\theta^k = p\theta^{k+1} + q\theta^{k-1}$, or $p\theta^2 - \theta + (1 - p) = 0$. We can factorize this into $\big(p\theta - (1 - p)\big)(\theta - 1) = 0$, so that $\theta = 1, \dfrac{1 - p}{p}$. Thus the answer is

$$J_k = C_1 \cdot 1^k + C_2\Big(\frac{1 - p}{p}\Big)^k = C_1 + C_2\Big(\frac{q}{p}\Big)^k.$$

(c) We only have one boundary condition $J_0 = 1$. In the case $p > q$, determine two constants found in the general solution in (b).

Ans: $J_0 = 1$ implies $C_1 + C_2 = 1$. When k is very large, the probability of hitting 0 (since $p > q$) should be very small, or, mathematically $\lim_{k\to\infty} J_k = 0$. This gives $C_1 = 0$. Thus

$$J_k = \Big(\frac{q}{p}\Big)^k.$$

(d) (Extra 5 bonus points - complete only if you have time) In the case $p < q$, determine two constants found in the general solution in (b).

Ans: In this case $J_k \leq 1$ implies $C_2 = 0$. Thus

$$J_k \equiv 1, \qquad k \geq 1.$$

7. (10 points) Find the constant C so that $f(x) = \dfrac{C}{1+x^2}$ is a density function.

Ans:

$$\int_{-\infty}^{\infty} \frac{C}{1+x^2} dx = 1,$$

so that

$$C = \Big(\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \Big)^{-1} = \frac{1}{\pi}.$$

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