Ph. D. Qualifying Exam: Applied Probability. June, 2018

ACMS Applied Probability Qualifying exam committee.

Candidate:____

There are 8 problems. Show all your work.

1. A die is rolled until two different numbers appear. Let T be the total number of times the die is rolled. Obviously $P(T=0) = P(T=1) = 0$. Find ET and $VarT$. For example

$$\begin{aligned}&\{1,1,2\}, T = 3;\\ &\{1,0\}, T = 2;\\ &\{0,0,0,0,5\}, T = 5;\\ &\{4,4,4,4,4,6\}, T = 6;\end{aligned}$$

Sol. For $k \geq 2$,

$$P(T = k) = \left(\frac{1}{6}\right)^{k-2} \cdot \frac{5}{6},$$

so that

$$\begin{aligned} ET &= \sum_{k\geq 2} k \cdot \left(\frac{1}{6}\right)^{k-2} \cdot \frac{5}{6} = \frac{11}{5},\\ ET^2 &= \sum_{k\geq 2} k^2 \cdot \left(\frac{1}{6}\right)^{k-2} \cdot \frac{5}{6} = \frac{127}{25},\\ VarT &= ET^2 - (ET)^2 = \frac{6}{25}.\end{aligned}$$

2. Suppose X and Y are independent continuous random variables with uniform distributions on $[0, 1]$.

(a) Find the density function for $X + 2Y$;

Sol.

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

$$f_{2Y}(y) = \begin{cases} 1/2, & 0 < y < 2 \\ 0, & otherwise \end{cases}$$

$$\begin{aligned} f_{X+2Y}(z) &= f_X * f_{2Y} = \int f_X(x) f_{2Y}(z-x)dx = \frac{1}{2}\Big(\min(1,z) - \max(0, z-2)\Big)^+ \\ &= \begin{cases} z/2, & 0 < z < 1, \\ 1/2, & 1 \le z < 2, \\ (3-z)/2, & 2 \le z < 3, \\ 0, & otherwise. \end{cases} \end{aligned}$$

This is a trapezoid with vertices at $(0, 0), (3, 0), (2, 1/2), (1, 1/2)$.

(b) Find the joint density function for $X - Y, X + Y$.

Sol. The joint density for X, Y is $f_{X,Y}(x, y) = f_X(x)f_Y(y) = 1$ for $0 < x < 1, 0 < y < 1$ and 0 otherwise. Under the change of variables $Z_1 = X - Y, Z_2 = X + Y$ the Jacobian is

$$\frac{\partial(z_1, z_2)}{\partial(x, y)} = \begin{vmatrix} 1, & -1 \\ 1, & 1 \end{vmatrix} = 2,$$

so that

$$\frac{\partial(x, y)}{\partial(z_1, z_2)} = \frac{1}{2}.$$

It follows that

$$f_{Z_1,Z_2}(z_1, z_2) = \begin{cases} \frac{1}{2}, & \text{for } 0 < z_1 + z_2 < 2,\ 0 < z_2 - z_1 < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The value of 1/2 is on a square with vertices's at $(0, 0), (1, 1), (0, 2), (-1, 1)$.

3. Suppose that $\{X_n\}$ is a sequence of random variables, and X is a random variable.

(a) If $\{X_n\}$ converges to X mean square, is it true that $\{X_n\}$ also converges to X in probability? Prove or given an counter example.

True.

$$P(|X_n - X| \geq \epsilon) \leq \frac{1}{\epsilon^2} E|X_n - X|^2 \to 0.$$

(b) If $\{X_n\}$ converges to X in probability, is it true that $\{X_n\}$ also converges to X mean square? Prove or given an counter example.

Not necessarily. Example.

$$X \equiv 0, \qquad P(X_n = 0) = 1 - \frac{1}{n}, \quad P(X_n = n) = \frac{1}{n}.$$

Clearly,

$$P(|X_n - X| \geq \epsilon) = \frac{1}{n} \to 0,$$

but

$$E|X_n - X|^2 = E|X_n|^2 = n^2 \cdot \frac{1}{n} \to \infty.$$

(c) Let X_n, X, Y_n, Y be random variables on the same probility space. If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$ both converges in distribution? Is it true $X_n + Y_n \xrightarrow{D} X + Y$ also converges in distribution? Prove or given an counter example.

Ans: No. Example: $P(X = 0) = P(X = 1) = 1/2$, $X_n = X$, $Y_n = 1 - X$. $X_n \to X$ in distribution, $Y_n \to X$ in distribution. Then $X_n + Y_n \equiv 1 \neq 2X$.

4. Factory A produces 70 % of a special brand of umbrellas with defective rate 2 %. Factory B produces the remaining 30 % of the same umbrellas with defective rate 1 %.

(a) What is the defective rate of a randomly purchased umbrella of this brand of this product?

Sol.

$$P(def) = P(def|A)P(A) + P(def|B)P(B) = \frac{2}{100}\frac{7}{10} + \frac{1}{100}\frac{3}{10} = \frac{17}{1000}.$$

(b) Given that an umbrella of this brand is defective, what is the probability that it is from Factory B?

Sol.

$$P(B|def) = \frac{P(def \text{ and } B)}{P(def)} = \frac{P(def|B)P(B)}{P(def)} = \frac{\frac{1}{100}\frac{3}{10}}{\frac{17}{1000}} = \frac{3}{17}.$$

(c) 25% of umbrellas from Factory A are blue and the rest are other colors. Defective rate does not vary for different colors. All umbrellas from Factory B are blue. Given that a **blue** umbrella of this brand is defective, what is the probability that it is from Factory B?

Sol.

$$P(A \text{ and } Blue) = P(Blue|A)P(A) = \frac{25}{100}\frac{7}{10} = \frac{7}{40}$$

$$P(B \text{ and } Blue) = P(Blue|B)P(B) = \frac{100}{100}\frac{3}{10} = \frac{3}{10}$$

$$P(B|def \text{ and } Blue) = \frac{P(def \text{ and } Blue \text{ and } B)}{P(def \text{ and } Blue)}$$

$$= \frac{P(def|B \text{ and } Blue)P(B \text{ and } Blue)}{P(def|B \text{ and } Blue)P(B \text{ and } Blue) + P(def|A \text{ and } Blue)P(A \text{ and } Blue)}$$

$$= \frac{\frac{1}{100}\frac{3}{10}}{\frac{1}{100}\frac{3}{10} + \frac{2}{100}\frac{7}{40}} = \frac{6}{13}.$$

5. Here is a simple random walk. Let $S_n = S_0 + \sum_{i=1}^{n} X_i$, where X_i are independent, and takes the value $-1, 1$. Assume

$$P(X_i = 1) = p, \; P(X_i = -1) = q, \quad p + q = 1, \quad p > q.$$

(a) Give a definition of a martingale.

Def. $\{S_n : n \geq 1\}$ is a martingale with respect to $\{X_n : n \geq 1\}$ if

(i) $E|S_n| < \infty$ for all n, and

(ii) $E(S_{n+1}|X_1, X_2, \cdots, X_n) = S_n$.

(b) Show that $M_n = |S_n|^2 - n$ is a martingale with respect to $\{X_n\}$. (Show all your work)

Proof. Obviously $E|S_n|^2 < \infty$.

Since $|S_{n+1}|^2 = |X_{n+1} + S_n|^2 = |S_n|^2 + 2X_{n+1}S_n + |X_{n+1}|^2 = |S_n|^2 + 2X_{n+1}S_n + 1$,

$$\begin{aligned} E(|S_{n+1}|^2 - (n+1)\Big|X_1, \cdots, X_n) &= (|S_n|^2 + 2S_n E X_{n+1} + 1) - (n+1) \\ &= |S_n|^2 - n. \end{aligned}$$

6. Suppose also that the random variables X_i are all independent, and is of Poisson distribution with parameter $\lambda_i > 0$, i.e.,

$$P(X_i = k) = \frac{e^{-\lambda_i}\lambda_i^k}{k!}, \quad k = 0, 1, 2, 3, \cdots .$$

(a) Find the generating function for each X_i.

Sol.

$$G_i(s) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda_i}\lambda_i^k}{k!} = e^{-\lambda_i} \sum_{k=0}^{\infty} \frac{(s\lambda_i)^k}{k!} = e^{(s-1)\lambda_i}.$$

(b) Let $Y = X_1 + X_2 + X_3 + \cdots X_n$. Find the generating function for Y.

Sol.

$$G_Y(s) = G_1(s)G_2(s)\cdots G_n(s) = e^{(s-1)(\lambda_1+\cdots+\lambda_n)}.$$

(c) Find $P(Y = k)$ for $k = 0, 1, 2, 3, \cdots$.

Sol. Expanding answers from (b)

$$P(Y = k) = \frac{e^{-(\lambda_1+\cdots+\lambda_n)}(\lambda_1 + \cdots + \lambda_n)^k}{k!}$$

7. A die is rolled repeatedly. Which of the following are Markov Chains? For those that are, **supply the transition matrix.**

(a) $S_n =$"the sum of all rolls up to nth roll."

Sol. Yes. Since $S_n = S_{n-1} + X_n$, the answer is "Yes"

$$p_{ij} = P(S_n = j|S_{n-1} = i) = P(X_n = j - i) = \begin{cases} 1/6, & \text{if } 1 \le j - i \le 6, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $Y_n =$"the sum of $(n-1)$th roll and nth roll."$= X_n + X_{n-1}$ (Assume $Y_0 = 0, Y_1 = X_1$).

Sol. No. $Y_n = X_n + X_{n-1}$.

$$P(Y_3 = 3|Y_1 = 1, Y_2 = 3) = P(Y_3 = 3|X_1 = 1, X_2 = 2) = P(X_3 = 1) = 1/6,$$

but

$$\begin{aligned} P(Y_3 = 3|Y_2 = 3) &= P(Y_3 = 3|Y_1 = 1, Y_2 = 3)P(Y_1 = 1) + P(Y_3 = 3|Y_1 = 2, Y_2 = 3)P(Y_1 = 2) \\ &= \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} = \frac{1}{18} \end{aligned}$$

(c) $Z_n =$"total numbers of 6's up to nth roll."

Sol. "Yes."

$$p_{ij} = P(Z_n = j|Z_{n-1} = i) = \begin{cases} 1/6, & \text{if } j = i + 1, \\ 5/6, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

8. Let W_t be the standard Brownian motion with $W_0 = 0$, $EW_t = 0$ and $VarW_t = t$. Suppose that $0 < s < t$.

(a) Find

$$E(W_t^3|W_s)$$

Sol.

$$W_t^3 = (W_t - W_s + W_s)^3 = (W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3,$$

so that

$$\begin{aligned} E(W_t^3|W_s) &= E\Big((W_t - W_s)^3|W_s\Big) + 3W_s E\Big((W_t - W_s)^2|W_s\Big) + 3W_s^2 E\Big((W_t - W_s)|W_s\Big) + W_s^3 \\ &= EW_{t-s}^3 + 3W_s EW_{t-s}^2 + 3W_s^2 EW_{t-s} + W_s^3 \\ &= 3(t-s)W_s + W_s^3. \end{aligned}$$

(b) Using Itô's formula, find $d(W_t)^3$ and $d(tW_t)$.

Sol.

$$d(W_t)^3 = 3(W_t)^2 dW_t + \frac{1}{2} \cdot 3 \cdot 2W_t dt = 3(W_t)^2 dW_t + 3W_t dt.$$

$$d(tW_t) = tdW_t + W_t dt$$

(c) Find

$$\int_0^T \Big(W_t^2 - t\Big) dW_t$$

Sol. From (b)

$$(W_t^2 - t)dW_t = \frac{1}{3} d(W_t)^3 - d(tW_t)$$

so that

$$\int_0^T \Big(W_t^2 - t\Big) dW_t = \frac{1}{3}(W_t)^3 - tW_t.$$