Ph. F. Qualifying Exam: Applied Probability. June, 2018

ACMS Applied Probability Qualifying exam committee.

Candidate:____

There are 8 problems. Show all your work.

1. A die is rolled until two different numbers appear. Let T be the total number of times the die is rolled. Obviously $P(T=0)=P(T=1)=0$. Find ET and $VarT$. For example

$$\{1,1,2\}, T=3;$$
$$\{1,0\}, T=2;$$
$$\{0,0,0,0,5\}, T=5;$$
$$\{4,4,4,4,4,6\}, T=6;$$

2. Suppose X and Y are independent continuous random variables with uniform distributions on $[0, 1]$.

(a) Find the density function for $X + 2Y$;

(b) Find the joint density function for $X - Y, X + Y$.

3. Suppose that $\{X_n\}$ is a sequence of random variables, and X is a random variable.

(a) If $\{X_n\}$ converges to X mean square, is it true that $\{X_n\}$ also converges to X in probability? Prove or given an counter example.

(b) If $\{X_n\}$ converges to X in probability, is it true that $\{X_n\}$ also converges to X mean square? Prove or given an counter example.

(c) Let X_n, X, Y_n, Y be random variables on the same probility space. Suppose $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$ both converges in distribution. Is it true $X_n + Y_n \xrightarrow{D} X + Y$ also converges in distribution? Prove or given an counter example.

4. Factory A produces 70 % of a special brand of umbrellas with defective rate 2 %. Factory B produces the remaining 30 % of the same umbrellas with defective rate 1 %.

(a) What is the defective rate of a randomly purchased umbrella of this brand of this product?

(b) Given that an umbrella of this brand is defective, what is the probability that it is from Factory B?

(c) 25% of umbrellas from Factory A are blue and the rest are other colors. Defective rate does not vary for different colors. All umbrellas from Factory B are blue. Given that a **blue** umbrella of this brand is defective, what is the probability that it is from Factory B?

5. Here is a simple random walk. Let $S_n = S_0 + \sum_{i=1}^{n} X_i$, where X_i are independent, and takes the value $-1, 1$. Assume

$$P(X_i = 1) = p, \; P(X_i = -1) = q, \quad p + q = 1, \quad p > q.$$

(a) Give a definition of a martingale.

(b) Show that $M_n = |S_n|^2 - n$ is a martingale with respect to $\{X_n\}$. (Show all your work)

6. Suppose also that the random variables X_i are all independent, and is of Poisson distribution with parameter $\lambda_i > 0$, i.e.,

$$P(X_i = k) = \frac{e^{-\lambda_i}\lambda_i^k}{k!}, \quad k = 0, 1, 2, 3, \cdots .$$

(a) Find the generating function for each X_i.

(b) Let $Y = X_1 + X_2 + X_3 + \cdots + X_n$. Find the generating function for Y.

(c) Find $P(Y = k)$ for $k = 0, 1, 2, 3, \cdots$.

7. A die is rolled repeatedly. Which of the following are Markov Chains? For those that are, **supply the transition matrix.**

(a) $S_n =$"the sum of all rolls up to nth roll."

(b) $Y_n =$"the sum of $(n-1)$th roll and nth roll." $= X_n + X_{n-1}$ (Assume $Y_0 = 0, Y_1 = X_1$).

(c) $Z_n =$"total numbers of 6's up to nth roll."

8. Let W_t be the standard Brownian motion with $W_0 = 0$, $EW_t = 0$ and $VarW_t = t$. Suppose that $0 < s < t$.

(a) Find

$$E(W_t^3|W_s)$$

(b) Using Itô's formula, find $d(W_t)^3$ and $d(tW_t)$.

(c) Find

$$\int_0^T \left(W_t^2 - t\right) dW_t$$