Ph. D. Qualifying Exam: Applied Probability. June, 2020

ACMS Applied Probability Qualifying exam committee.

Candidate:____

There are 8 problems. Each problem from problems 1 to 4 is worth 10 points. Each problem from problems 5 to 8 is worth 15 points. Show all your work.

1. (10 points). Toss a coin until you get two consecutive heads. This is a fair coin with $P(H) = P(T) = 1/2$.

(a) Let X be the total number of tosses. Then

$$\{X = 0\} = \{X = 1\} = \emptyset,$$
$$\{X = 2\} = \{HH\},\ \{X = 3\} = \{THH\},\ \{X = 4\} = \{TTHH, HTHH\}, \cdots,$$

For $n \geq 3$, find $P(X = n)$ in terms of $P(X = n - 1)$ and $P(X = n - 2)$.

(b) What is the expected total number of tosses?

2. (10 points). Let D be the region of a unit disk. Assume the joint density $f_{X,Y}(x,y)$ is given by

$$f_{X,Y}(x,y) = \frac{1}{\pi} I_D(x,y),$$

where I_D is the indicator function of D.

(a) Find the marginal density $f_X(x)$ and $f_Y(y)$.

(b) Are X, Y independent? Justify your answer.

(c) Find the conditional density $f_{X|Y}(x|y)$.

3. (10 points). Suppose X and Y are independent random variables with exponential distribution with density function $f(x) = \begin{cases} x^{-2}, & 1 < x < \infty \\ 0, & x \leq 1 \end{cases}$. Find the probability density function for

(a) $\sqrt{X}$;

(b) XY;

4. (10 points). Factory A produces 65 % of a special brand of chips with defective rate 6 %. Factory B produces the remaining 35 % of the same umbrellas with defective rate 5 %.

(a) What is the defective rate of a randomly selected chip of this brand of this product?

(b) Given that an chip of this brand is defective, what is the probability that it is from Factory A?

5. (15 points). Suppose $G(s) = se^{s-1}$ is the generating function of the random variable X. The random variable X takes non-negative integer values

(a) Find $P(X = 0)$, $P(X = 1)$, $E(X)$ and $E(X^2)$.

(b) Let X_1, X_2 be independent, identically distributed random variables with the same mass function as X. Find

$$E\frac{1}{X_1 + X_2}$$

6. (15 points). Suppose that $\{S_n\}$ is a sequence of random walk with $S_n = X_0+X_1+\cdots+X_n$, where $X_0 = 0$, $P(X_i = 1) = p$ and $P(X_i = -1) = q$ where $p+q = 1$, and $X_1, X_2, \cdots, X_n, \cdots$ are i.i.d,

(a) Prove that S_n is a Martingale if and only if $p = q = \frac{1}{2}$.

(b) Suppose that $p \neq q$. Find the constants c_n $(n = 1, 2, 3, \cdots)$ so that $M_n = S_n + c_n$ is a Martingale.

7. (15 points). Let $\{S_n\}$ be a homogenous Markov chain taking integer values with the transition probability

$$p_{ij} = \begin{cases} 9/10 & \text{if } j = i+1, \\ 1/10 & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find all persistent states.

(b) Find all states i and j such that $i \to j$, i.e., i communicates with j.

(c) Find all states i and j such that $i \leftrightarrow j$, i.e., i and j intercommunicate.

8. (15 points). Let B_t denote the Standard Brownian Motion $EB_t = 0$ amd $VarB_t = t$.

(a) Let $Y_t = (B_t + 5t)^2$. Find dY_t.

(b) Suppose X_t is a geometric Brownian Motion representing an assert, i.e.,

$$\frac{dX_t}{X_t} = rdt + \sigma dB_t,$$

where the constant r represents an interest rate and the constant σ represents the volatility. Find

$$d \ln |X_t|.$$

(c) Following (b), find X_t

(d) Two investments (both satisfy geometric Brownian Motions) with investment B twice the interest rate and volatility of investment A, and the interest rate and volatility for investment A are $r = 0.05$ and $\sigma = 0.2$. Which one is better in average? Justify your answer.