Sample Final 2 Solutions - ACMS 20550

Multiple Choice Answers: 1(e), 2(c), 3(b), 4(a), 5(e), 6(d), 7(d), 8(b), 9(e), 10(d), 11(c), 12(c), 13(d), 14(c), 15(e).

Partial Credit Answers:

16a. Begin with calculating for $\phi(x, y, z) = x^2 + y^2 - 2z$ that

$$\sec\gamma = \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{|\phi_z|} = \frac{\sqrt{4x^2 + 4z^2 + 4}}{2} = \sqrt{1 + x^2 + y^2}$$

Now integrate over the circle $x^2 + y^2 \leq 9$ using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ with $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$ to get,

$$\iint_A \sec\gamma \, \mathrm{d}A = 2\pi \int_{r=0}^{3} r\sqrt{1 + r^2} \, \mathrm{d}r$$

$$= 2\pi \left[\frac{1}{3}(1 + r^2)^{3/2}\right]_{r=0}^{3} = \boxed{\frac{2\pi}{3}\left[10\sqrt{10} - 1\right]}$$

16b. Applying the Divergence Theorem with $\mathbf{F} = 2z^2x\mathbf{i} + z\mathbf{j} - y\mathbf{k}$ gives

$$\oiint_S \mathbf{F} \cdot \mathbf{n} \mathrm{d}s = \iiint_V \nabla \cdot \mathbf{F} \, \mathrm{d}V = \iiint_V 2z^2 \mathrm{d}V$$

In cylindrical polar coordinates with $0 \leq r \leq 2$, $0 \leq z \leq 1$ and $0 \leq \theta \leq 2\pi$, we have that

$$\iiint_V 2z^2 \mathrm{d}V = 2\int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{2} z^2 \, r \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z = 2\pi(2^2)\int_{z=0}^{1} z^2 \, \mathrm{d}z = \boxed{\frac{8\pi}{3}}$$

17. The Lagrangian is

$$F = xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

where λ is a Lagrange multiplier. Taking the partial derivatives

$$\frac{\partial F}{\partial x} = 0: \qquad yz + \lambda\frac{2x}{a^2} = 0 \qquad (1)$$

$$\frac{\partial F}{\partial y} = 0: \qquad xz + \lambda\frac{2y}{b^2} = 0 \qquad (2)$$

$$\frac{\partial F}{\partial z} = 0: \qquad xy + \lambda\frac{2z}{c^2} = 0 \qquad (3)$$

$$\frac{\partial F}{\partial \lambda} = 0: \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

To incorporate all the equations take $x \times (1) + y \times (2) + z \times (3)$ which gives

$$3xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0, \quad \Longrightarrow \quad \lambda = -\frac{3xyz}{2}.$$

Substituting this expression for λ back into (1),(2), (3) gives

$$yz(1-\frac{3x^2}{a^2})=0, \qquad xz(1-\frac{3y^2}{b^2})=0, \qquad yx(1-\frac{3z^2}{c^2})=0$$

which implies $\boxed{(x,y,z)=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)}$.

18. The formula for a_n is $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x$. For the given even function

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}(\pi-x)\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\frac{(\pi-x)}{n}\sin nx\Big|_0^{\pi} - \int_0^{\pi}\frac{-1}{n}\sin nx\,\mathrm{d}x\right) = \frac{2}{n\pi}\int_0^{\pi}\sin nx\,\mathrm{d}x \\
&= \frac{-2}{\pi n}\left[\frac{\cos nx}{n}\Big|_0^{\pi}\right] = \frac{2}{\pi n^2}\left[1-\cos n\pi\right] = \frac{2}{\pi n^2}\left[1-(-1)^n\right] \\
&= \begin{cases} 0 & n \text{ even}, \\ \frac{4}{\pi n^2} & n \text{ odd} \end{cases}
\end{aligned}$$

The first mode a_0 is

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}(\pi-x) = \pi$$

The final Fourier series of $f(x)$ is

$$\boxed{f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}}$$