Test 2, Version Gold,
1(c), 2(d), 3(a), 4(d), 5(b), 6(e), 7(c), 8(b), 9(a), 10(c)

Problem 11

(a) The Taylor series for $\sin x$ is given by $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$. Estimate the error for $\sin 1$ using only 2 terms, namely, estimate $\left|\sin 1 - \left(1 - \frac{1}{3!}\right)\right|$. No need to simply your answer – You may leave your answer as a fraction containing factorials.
Hint: Is this series alternating and absolutely decreasing?

Sol. We first substiitute $x = 1$ into the series expression. The series is alternating with the terms $\frac{1}{(2n+1)!}$ decreasing to 0.
The error estimate is the "next term", which is $\frac{1}{5!}$.

(b) Compute $\int_0^{2\pi} \sin 3x \cdot \cos 5x \, dx$. Show all your work.
Hint: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Sol.

$$\begin{aligned}
\int_0^{2\pi} \sin 3x \cdot \cos 5x \, dx &= \frac{1}{4i}\int_0^{2\pi}(e^{i3x} - e^{-i3x})(e^{i5x} + e^{-i5x})dx \\
&= \frac{1}{4i}\int_0^{2\pi}(e^{i8x} + e^{-i2x} - e^{i2x} - e^{-i8x})dx \\
&= \frac{1}{4i}\left(\frac{1}{8i}e^{i8x} + \frac{1}{-2i}e^{-i2x} - \frac{1}{2i}e^{i2x} - \frac{1}{-8i}e^{-i8x}\right)_{x=0}^{2\pi} = 0
\end{aligned}$$

Problem 12

(a) Compute $\frac{d}{dx}\int_0^{x^3} \frac{\sin(xt)}{t}dt$ using Leibniz rules, *no credit will be given for using a series.* Show all your work.

Sol.

$$\begin{aligned}\frac{d}{dx}\int_0^{x^3} \frac{\sin(xt)}{t}dt &= \frac{\sin(x\cdot x^3)}{x^3}\frac{\partial}{\partial x}x^3 + \int_0^{x^3}\frac{\partial}{\partial x}\frac{\sin(xt)}{t}dt \\ &= \frac{\sin(x^4)}{x^3}\cdot 3x^2 + \int_0^{x^3} \cos(xt)dt \\ &= \frac{3\sin(x^4)}{x} + \frac{\sin(xt)}{x}\Big|_{t=0}^{x^3} = \frac{4\sin(x^4)}{x}.\end{aligned}$$

(b) Find the point on the plane $x + 2y - 3z = 6$ for which $f(x, y, z) = \frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2$ is minimum. Show all your work.

Sol. Lagrange Method.

$$F(x, y, z) = \frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2 + \lambda(x + 2y - 3z - 6).$$

$$\begin{aligned}0 &= \frac{\partial F}{\partial x} = x + \lambda \quad \Rightarrow x = -\lambda \\ 0 &= \frac{\partial F}{\partial y} = 2y + 2\lambda \quad \Rightarrow y = -\lambda \\ 0 &= \frac{\partial F}{\partial z} = 3z - 3\lambda \quad \Rightarrow z = \lambda.\end{aligned}$$

Substituting into $x + 2y - 3z = 6$, we find $(-\lambda) + 2(-\lambda) - 3\lambda = 6$, so that $\lambda = -1$.
Thus $x = 1, y = 1, z = -1$.