Multiple Choice: 1e, 2b, 3c, 4d, 5b, 6e, 7a, 8e, 9d, 10c

Test 1, Problem 11

Find the Taylor series of the following functions. Write your answer in the format of a summation. For example, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$.

(a) $\dfrac{1}{1+x^2}$.

Sol. Substituting x with $(-x^2)$ in the above formula, we get

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

(b) $\arctan x$ (use the result from (a))

Sol. $\arctan x = \displaystyle\int_0^x \frac{1}{1+u^2}du = \sum_{n=0}^{\infty}\int_0^x (-1)^n u^{2n} du = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

Problem 12,

The formula $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is valid for all x.

(a) For $|x| \leq 1$, write the formula **for the error estimates** covered in lecture using n-terms

$$\sum_{k=0}^{n} \frac{x^k}{k!}$$

as the approximation for e^x.

Sol.

$$\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c}{(n+1)!}x^{n+1}, \quad |x| \leq 1, \quad -1 \leq c \leq 1.$$

(b) Using (a), for $|x| \leq 1$, find out the minimum number n needed in order to have an error less than 0.001. Justify your answer. For those without a calculator, we list the first few factorial numbers and the approximation of the number e here
$(2! = 2,\ 3! = 6,\ 4! = 24,\ 5! = 120,\ 6! = 720,\ 7! = 5040,\ 8! = 40320$ and $e \approx 2.718$)

Sol. Want
$\left|\dfrac{e^c}{(n+1)!}x^{n+1}\right| \leq \dfrac{e^1}{(n+1)!} \leq \dfrac{2.718}{(n+1)!} < 0.001$. Thus $(n+1)! > 2718$, so that $n+1 \geq 7$,
Answer: $n = 6$

Probelm 13,
Use Taylor series to find

$$\frac{d^8}{dx^8}\left(x^4[1-\cos(x^2)]\right) \text{ at } x = 0.$$

Show all your work.

Sol. The only term that matters is the term involving x^8. The terms with order less than x^8 are all gone after differentiation. The terms with order greater than x^8 are all 0 after plug in $x = 0$. Using the Taylor expansion for cos we find that the x^8 order term is $\frac{x^8}{2}$. Thus the answer is

$$\frac{8!}{2}$$