**Math 10250, Exam 3 Solutions, Fall 2008**

1. If $f(x) = 3x^5 - 10x^4$ , then $f'(x) = 15x^4 - 40x^3$ and $f''(x) = 60x^3 - 120x^2 = 60x^2(x-2)$. The second derivative changes sign ONLY at $x = 2$. So $f(x)$ has $\boxed{\text{ONE}}$ inflection point.

2. If $f(x) = 2x^3 + 3x^2 - 72x + 5$, then $f'(x) = 6x^2 + 6x - 72 = 6(x^2 + x - 12) = 6(x-3)(x+4)$. The critical points of $f(x)$ are $\boxed{x = 3 \text{ and } x = -4}$.

3. Take the derivative, with respect to $x$, of both sides of $x^2 + y^2 = 25$ to get $2x + 2y\frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = \frac{-x}{y}$.

The slope of the line tangent to $x^2 + y^2 = 25$ at the point $(3, -4)$ is $\frac{dy}{dx}$ evaluated at $(3, -4)$, and this is equal to $\frac{-3}{-4} == \boxed{3/4}$.

4. Integrate both sides of $\dfrac{dy}{dx} = 1 - \dfrac{1}{x}$ to see that $y = x - \ln|x| + C$. Plug in the point $(1, 3)$ to learn $3 = 1 - 0 + C$; so, $C = 2$. The positive number $x = 1$ is in the domain of our solution and 0 is not in the domain of our solution. So, the domain of our solution is contained in the set of positive numbers and the absolute value is not needed. The solution of the Initial Value Problem $\dfrac{dy}{dx} = 1 - \dfrac{1}{x}$, $y(1) = 3$ is $\boxed{y = x - \ln x + 2}$.

5. We must find the global maximum of $h(t)$. The graph of $h(t)$ is a parabola which opens down. The global maximum occurs at the point where $h'(t) = 0$. We see $h'(t) = -32t + 64$. Thus $h'(t) = 0$ when $t = 2$. The maximum height of this projectile is $h(2) = -16(4) + 64(2) + 300 = \boxed{364 \text{ feet}}$.

6. We find the global maximum point of $G(w) - E(w) = .4w^3 - .1w^4$, for $w \geq 0$. Let $F(w) = .4w^3 - .1w^4$. We see that $F'(w) = 1.2w^2 - .4w^3 = .4w^2(3 - w)$. Thus $F(w)$ is increasing for all $w < 3$ and $F(w)$ is decreasing for all $w > 3$. The absolute maximum of $F(w)$ occurs at $\boxed{w = 3 \text{ pounds}.}$

7. We find the value of $\frac{dq}{dt}$ on the day of interest. The chain rule tells us that $\frac{dq}{dt} = \frac{dq}{dp}\frac{dp}{dt} = -300e^{-.05p}\frac{dp}{dt}$. The problem tells us that on the day of interest $p = 10$ and $\frac{dq}{dt} = -2$. Thus, on the day of interest, $\frac{dq}{dt} = -300e^{-.05(10)}(-2) = 600e^{-.5} = \boxed{\dfrac{600}{\sqrt{e}} \text{barrels per day}}$

8. We maximize $P(x) = R(x) - C(x) = 500x - x^2 - (100x + 600) = -600 + 400x - x^2$. We see that the graph of $P(x)$ is a parabola which opens down. The maximum profit occurs where $P'(x) = 0$. We compute $P'(x) = 400 - 2x$. Thus, $P'(x) = 0$ when $x = 200$. To maximize profit, the company should make $\boxed{\text{200 CD players each week.}}$

9. We see that $f'(5) > 0$, so $f(x)$ is increasing at $x = 5$. We also see that $f'(3) > 0$ so $\boxed{f(x) \text{ is increasing at } x = 3}$. The only local extreme point of $f(x)$ is the local minimum point $(2, f(2))$. We see that $f'(0) < 0$ so $f(x)$ is decreasing at $x = 0$.

10. We see that $f''(x) = \frac{1}{x^2}$, which is positive for all non-zero $x$. Thus, $f(x)$ is concave up for all $x$ in the domain of $f$. We see that $f'(1) = 0$ and $f''(1) > 0$, so $(1, f(1))$ is a local minimum point of $f(x)$. We see that $f'(-1) = 2$, so $f(x)$ is increasing at $x = -1$. The function $f(x)$ never changes concavity, so there are no points of inflection. The function $f(x)$ is decreasing for positive $x$'s near zero (because $f'$ for such an $x$ is negative); hence, it is $\boxed{\text{impossible for } \lim\limits_{x \to 0^+} f(x) = -\infty.}$

11a. We see that $\displaystyle\int \frac{4}{x^5}\,dx = \boxed{\frac{-1}{x^4} + C}$ because the derivative of $-x^{-4}$ is $4x^{-5}$.

11b. To compute $\displaystyle\int t^2\sqrt{t^3+1}\,dt$, let $u = t^3 + 1$. It follows that $du = 3t^2 dt$. The integral is equal to $\frac{1}{3}\displaystyle\int u^{1/2}\,du = \frac{1}{3}\frac{2}{3}u^{3/2} + C = \boxed{\frac{2}{9}(t^3+1)^{3/2} + C}$. **Check.** The derivative of the proposed answer is

$\frac{3}{2}\frac{2}{9}(t^3+1)^{1/2}3t^2 = t^2\sqrt{t^3+1}$. ✓

12. The base of the box is $x \times x$. The height of the box is $h$. We are told that the volume of the box is 20 cubic feet; so, $x^2 h = 20$ and $h = 20/x^2$. The cost of the box is 3 times the area of the bottom plus 2 times the area of the top plus 1 times the area of the four sides; so,

$$C = 3x^2 + 2x^2 + 4xh = 5x^2 + 4x(20/x^2) = 5x^2 + 80/x.$$

We minimize $C(x)$ for $x > 0$. We notice that $\lim_{x\to 0^+} C(x) = \infty$ and $\lim_{x\to\infty} C(x) = \infty$, so the local minimum of $C(x)$ is also the global minimum. At any rate, $C'(x) = 10x - 80/x^2 = \frac{10x^3-80}{x^2} = \frac{10(x^3-8)}{x^2}$. Thus, $x = 2$ is the only critical point of $C(x)$ and $(2, C(2))$ is the local and global minimum of $C(x)$. The minimum cost of the box is $\boxed{C(2) = \$60}$.

13. We compute $N'(t) = 500e^{-t}(4t^3 - t^4) = 500t^3e^{-t}(4-t)$. We see that $N'(t) = 0$ for $t = 0$ or $t = 4$. The maximum of $N$ occurs either at an end point or a critical point. We compute $N(0) = 0$, $N(4) = \frac{(500)4^4}{e^4} \cong 2344$, and $N(6) = \frac{(500)6^4}{e^6} \cong 1606$. We conclude that the maximum harvest occurs in week $\boxed{4}$.

14. Let $x$ be the length of each side of the square. The area of the square is $A = x^2$. We must find $\frac{dA}{dt}$ at the appropriate moment. The chain rule tells us that $\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = 2x\frac{dx}{dt}$. The moment of interest to us has $x = 100\text{cm}$ and $\frac{dx}{dt} = 2\text{cm/sec}$. The answer is $2(100)2\,\text{cm}^2/\text{sec} = \boxed{400\,\text{cm}^2/\text{sec}}$.

15. (a) We see that $f'(x)$ is positive for $x > 1$ and $f'(x)$ is negative for $x < 1$. Thus $\boxed{f(x) \text{ is increasing for } x > 1 \text{ and } f(x) \text{ is decreasing for } x < 1.}$

(b) Part (a) shows that $\boxed{(1, f(1)) \text{ is a local minimum point and this is the only local extreme point.}}$

(c) We have $f'(x) = e^{-x}(x-1)$. It follows that $f''(x) = e^{-x} - e^{-x}(x-1) = e^{-x}(1+1-x) = \frac{2-x}{e^x}$. Thus, $f''(x)$ is positive for $x < 2$ and $f''(x)$ is negative for $x > 2$. We conclude $\boxed{f(x) \text{ is concave up for } x < 2 \text{ and } f(x) \text{ is concave down for } x > 2.}$

(d) We see from part (c) that $\boxed{(2, f(2)) \text{ is the only inflection point of } f(x).}$

(e) $\boxed{\text{The function } f(x) \text{ does not have any vertical asymptotes. The } x\text{-axis is a horizontal asymptote.}}$