1. (35 points). True or False questions. Proofs or counter examples are not needed.

(a). Consider the Fourier Sine series for the function \( f(x) = x^2 \) on the interval \( 0 \leq x \leq 1 \).
This series converges pointwise for all \( 0 \leq x \leq 1 \).
False. The extended function on \((-\infty, \infty)\) will have a jump at \( x = 1 \). Would have be true if the question is for Cosine Series.

(b). If \( u \) satisfies the equation \( u_{tt} = 2u_{xx} \) in the domain \( \{(x,t); \ -\infty < x < \infty, \ 0 < t < \infty\} \) with the initial data \( u(x,0) = u_0(x) \), then there is a finite speed of propagation.
True.

(c). The heat equation \( u_t = 3u_{xx}, \ -\infty < x < \infty \) is well posed for \( t < 0 \).
False. Heat equations are not well posed in the negative time direction.

(d). Suppose that \( X_1 \) and \( X_2 \) satisfy (i), \( -X_1''(x) = \lambda_1 X_1(x) \) and \( -X_2''(x) = \lambda_2 X_2(x) \) on the interval \( a \leq x \leq b \), (ii), both \( X_1 \) and \( X_2 \) satisfies the homogenous Robin boundary conditions at both \( x = a \) and \( x = b \), (iii), \( \lambda_1 \neq \lambda_2 \). Then \( X_1 \) and \( X_2 \) are orthogonal to each other.
True.

(e). If a series converges uniformly on a finite interval, then it also converges in the Mean-square sense.
True.

(f). If \( f \) is continuously differentiable on \((0,3)\) such that \( \lim_{x \to 0^+} f'(x) \) exists, then its even extension is always differentiable on \((-3,3)\).
False. Need also \( \lim_{x \to 0^+} f'(x) = 0 \).

(g). There is a maximum principle for the wave equation \( u_{tt} = c^2 u_{xx} \), where \( c \) is a positive constant.
False. There is no maximum principle for wave equatio.
2. (10 points). Classify each of the following equations as (i) elliptic, parabolic, hyperbolic, (ii) linear, nonlinear.

(a). \( u_{xx} - 4u_{xy} + u_{yy} = (\sin(x^2 + y^2)) \ u. \)
hyperbolic, linear

(b). \( u_{xx} - 4u_{xy} + 4u_{yy} + 3u_{x} = \sin(u^2). \)
parabolic, nonlinear

(c). \( u_{xx} - 4u_{xy} + 10u_{yy} + 4u_{x} = 5u. \)
elliptic, linear

(d). \( u_{t} - 4u_{xx} = e^{u}. \)
parabolic, nonlinear

(e). \( u_{xx} + u_{yy} = \frac{u}{1 + x^2 + y^2}. \)
elliptic, linear
3a. (10 points). State precisely the Mean-value property for the harmonic functions.
See book.

3b. (10 points). State precisely the uniform convergence theorem for a Classical Fourier Series.
See book.
4. (10 points). Find the general solution to the problem \( u_{xx} + 3u_{xy} - 10u_{yy} = 0 \).

\textbf{Sol.}

The equation decomposes into \((\partial_x + 5\partial_y)(\partial_x - 2\partial_y)u = 0\), so that the final answer is

\[ u(x, y) = f(5x - y) + g(2x + y). \]
5. (10 points). Find the general solution to the problem \( u_x + x^2 u_y = 3 \cos(3x) \).

**Sol.** The characteristic ODE is

\[
\frac{dy}{dx} = x^2
\]

which integrates to \( y = \frac{1}{3} x^3 + C \). Thus the equation reduces to

\[
\frac{d}{dx} u(x, \frac{1}{3} x^3 + C) = 3 \cos(3x),
\]

which integrates to

\[
u(x, \frac{1}{3} x^3 + C) = \sin(3x) + C_1.
\]

We rewrite this as \( u(x, \frac{1}{3} x^3 + C) - \sin(3x) = C_1 \), which implies

\[
u(x, \frac{1}{3} x^3 + C) - \sin(3x) = u(0, 0 + C) - \sin 0.
\]

Letting \( \frac{1}{3} x^3 + C = y \), we find that

\[
u(x, y) = \sin(3x) + g(y - \frac{1}{3} x^3).
\]
6. (15 points). Solve

\[ u_t = 2u_{xx}, \quad 0 < x < \infty, \quad t > 0, \]
\[ u_x(0, t) = t^2, \quad t > 0, \]
\[ u(x, 0) = 0, \quad 0 \leq x < \infty. \]

**Sol.**

Step 1. Reduce to homogenous boundary condition:

Let \( v(x, t) = u(x, t) - t^2x \), then \( v \) satisfies

\[ v_t = 2v_{xx} - 2tx, \quad 0 < x < \infty, \quad t > 0, \]
\[ v_x(0, t) = 0, \quad t > 0, \]
\[ v(x, 0) = 0, \quad 0 \leq x < \infty. \]

Step 2. Since this is a **homogenous Neumann** boundary condition, we use even extension to a function \( w(x, t) \), and \( w \) satisfies (note here the even extension of \( 2tx \) is \( 2t|x| \)).

\[ w_t = 2w_{xx} - 2t|x|, \quad -\infty < x < \infty, \quad t > 0, \]
\[ w(x, 0) = 0, \quad -\infty < x < \infty. \]

Thus, for \( k = 2 \),

\[
    w(x, t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-(x-\xi)^2/(4k(t-\tau))} (-2\tau|x|) \, d\xi \, d\tau
\]

and

\[
    u(x, t) = t^2x + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-\tau)}} e^{-(x-\xi)^2/(4k(t-\tau))} (-2\tau|x|) \, d\xi \, d\tau, \quad x \geq 0, \quad t \geq 0.
\]
7. (15 points). Find a harmonic function \( u(x, y) \) on a rectangle \( \{0 < x < 1, 0 < y < 1\} \) with the following boundary conditions:

\[
\begin{align*}
\text{x = 0:} & \quad u(0, y) = 0, & \text{x = 1:} & \quad u_x(1, y) = 0, \\
\text{y = 0:} & \quad u(x, 0) = 0, & \text{y = 1:} & \quad u(x, 1) = 2x - x^2.
\end{align*}
\]

**Sol.** Use separation of variables:

\[
u(x, y) = X(x)Y(y)
\]

which leads to ODEs

\[
X'' + \lambda X = 0, \quad X(0) = 0, \quad X'(1) = 0 \tag{1}
\]

and

\[
Y'' - \lambda Y = 0, \quad Y(0) = 0. \tag{2}
\]

We need to discuss all three cases \( \lambda < 0 \), \( \lambda = 0 \), and \( \lambda > 0 \). After all these discussions, we found that only when \( \lambda > 0 \), equation has non-trivial solution:

\[
X(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x).
\]

The condition \( X(0) = 0 \) gives \( B = 0 \), and then the condition \( X'(1) = 0 \) gives \( \sqrt{\lambda} = (n + 1/2)\pi \).

Plug in \( \lambda = (n + 1/2)^2\pi^2 \) into (2) and solve the ODE, we obtain

\[
Y(y) = \sinh(n + 1/2)\pi y.
\]

Thus

\[
u(x, y) = \sum_{n=1}^{\infty} A_n \sin([n + 1/2]\pi x) \cdot \sinh([n + 1/2]\pi y).
\]

Finally, solving

\[
2x - x^2 = \sum_{n=1}^{\infty} A_n \sin([n + 1/2]\pi x) \cdot \sinh([n + 1/2]\pi y)
\]

gives

\[
A_n = \frac{2}{\sinh((n + 1/2)\pi)} \int_0^1 (2x - x^2) \sin([n + 1/2]\pi x) dx.
\]
8. (10 points). Find the harmonic function $u$ in the semidisk $\{ r < 1, \ 0 < \theta < \pi \}$ with $u$ vanishing on the diameter $(\theta = 0, \pi)$ and

$$u = \sin \theta + 6 \sin(6\theta) - 7 \sin(7\theta) \text{ on } r = 1.$$ 

**Sol.** The separation of variables $u = R(r)\Theta(\theta)$ applied to the Laplace equation in polar coordinates

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

gives

$$R(r) = (Cr^\alpha + Dr^{-\alpha}), \quad \Theta(\theta) = A \sin \alpha \theta + B \cos \alpha \theta;$$

or

$$R(r) = C_0 + D_0 \ln r, \quad \Theta(\theta) = A_0.$$ 

In case 1, the boundary condition $\Theta(0) = 0$ implies that $B = 0$, and the boundary condition $\Theta(\pi) = 0$ implies that $\alpha = n$. In case 2, the boundary condition implies $C_0 = D_0 = 0$.

Thus

$$u(r, \theta) = \sum_{n=1}^{\infty} (A_n r^n \sin n\theta + B_n r^{-n} \sin n\theta).$$ 

Since the domain contains 0, we must have $B_n = 0$. Thus we can match the coefficients to obtain

$$u = r \sin \theta + 6r^6 \sin(6\theta) - 7r^7 \sin(7\theta).$$