Corrections to "Quantum Theory for Mathematicians"

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1. Additional material

Please see the links for additions to Sections 9.2, 17.5, and 18.5.

2. Notes

Section 1.1.3 Blackbody radiation. The concept of a "mode" of the electromagnetic field is not defined. The idea is as follows. Maxwell's equations are linear wave equations. The general solution for the electromagnetic field in the box (with appropriate boundary conditions) is a linear combination of special solutions called *standing waves* or *modes.* Rather than go into the details of these solutions—which would require a lengthy discussion of Maxwell's equations—we illustrate the idea with the closely related equation of a vibrating string,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

on an interval $[0, L]$. Here $u(x,t)$ is the displacement of the string and c is the speed of wave propagation. We impose the boundary conditions $u(0,t) = u(L,t) = 0$, which mean that the string is held fixed at the ends of the interval.

The general solution of the equation with the given boundary conditions is a linear combination of the following special solutions:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

and

$$v_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right),$$

for $n = 1, 2, 3, \ldots$. These are "modes" in which the solution maintains a fixed sinusoidal shape as a function of x, with an amplitude that varies sinusoidally in time. The general solution of Maxwell's equations in a box has a similar decomposition in terms of modes.

Proposition 3.1. The proof does not require results on uniqueness of the moment problem. Rather, we may argue as follows. Assume that $\int_{\mathbb{R}} x^m \; d\mu = \lambda^m$ for all m (or even just for $m = 0$, 1, and 2). Then

$$\begin{aligned}
\int_{\mathbb{R}} (x-\lambda)^2 \; d\mu &= \int_{\mathbb{R}} (x^2 - 2x\lambda + \lambda^2) \; d\mu \\
&= \int_{\mathbb{R}} x^2 \; d\mu - 2\lambda \int_{\mathbb{R}} x \; d\mu + \lambda^2 \int_{\mathbb{R}} 1 \; d\mu \\
&= \lambda^2 - 2\lambda^2 + \lambda^2 \\
&= 0.
\end{aligned}$$

But since the function $(x-\lambda)^2$ is non-negative and equal to zero only at $x=\lambda$, the only way that $\int_{\mathbb{R}}(x-\lambda)^2\ d\mu$ can be zero is if μ is entirely concentrated at the point λ. After all, if μ assigned positive measure to the set $\mathbb{R}\setminus\{0\}$, elementary results in measure theory would tell us that the integral of the strictly positive function $(x-\lambda)^2$ over $\mathbb{R}\setminus\{\lambda\}$ would be positive.

Proof of Propositions 12.10 and 12.11. The argument can be simplified as follows. If $f=(X-\alpha I)\psi$ and $g=(P-\beta I)\psi$, then equality will hold in the last line of (12.21) if and only if $\operatorname{Im}\langle f,g\rangle=\|f\|\,\|g\|$. If this condition holds, then $|\langle f,g\rangle|\geq\|f\|\,\|g\|$, which means, by the Cauchy–Schwarz inequality that $|\langle f,g\rangle|=\|f\|\,\|g\|$. By the condition for equality in the Cauchy–Schwarz inequality, we must have $f=0$, $g=0$, or $f=cg$ for some nonzero constant c. But the only way $f=cg$ will lead to the condition $\operatorname{Im}\langle f,g\rangle=\|f\|\,\|g\|$ is if $c=i\gamma$ for some $\gamma<0$. (Recall that we take our inner products to be conjugate-linear in the *first* factor.) Thus, the real number $\delta:=-\gamma$ in (12.24) is necessarily positive, and we do not need to consider the case $\delta<0$ in Proposition 12.21.

Proposition 17.10. We will also construct the irreducible representations of $\mathsf{SO}(3)$ at the group level directly using spherical harmonics, in Section 17.6.

Chapters 16, 17, and 18. Although much of motivation for the material in Chapters 16 and 17 is the analysis of the hydrogen atom in Chapter 18, it is possible read much of Chapter 18 without going through Chapters 16 and 17 first. In particular, the analysis from Proposition 18.1 through Theorem 18.3 is self contained. On the other hand, the *motivation* for considering the particular form of the functions in Proposition 18.1 comes from Proposition 17.19, which in turn relies on the material in Chapter 16. Furthermore, Theorem 18.4 (the claim that we have *all* the eigenvectors of the hydrogen-atom Hamiltonian with negative eigenvalues) is also strongly dependent on Proposition 17.19.

3. Corrections

Chapter 1

p. 5, the units (in the "cgs" system) in the numerical value of $\hbar$ should be included, as follows:

$$\hbar=1.054\times10^{-27}\ \mathsf{g}\cdot\mathsf{cm}^2\cdot\mathsf{s}^{-1}.$$

Chapter 2

p. 27, second-to-last line, change $-F^{k,j}(\mathbf{x}^k,\mathbf{x}^j)$ to $-F^{k,j}(\mathbf{x}^j,\mathbf{x}^k)$ (i.e., the variables should always be written in the same order).

p. 30, the formulas for $\mathbf{x}^1$ and $\mathbf{x}^2$ should read

$$\begin{aligned}\mathbf{x}^1 &= \mathbf{c}+\frac{m_2}{m_1+m_2}\mathbf{y}\\ \mathbf{x}^2 &= \mathbf{c}-\frac{m_1}{m_1+m_2}\mathbf{y}.\end{aligned}$$

(That is, the $\mathbf{c}$ terms should be outside the fractions.)

p. 35, proof of Prop. 2.23, the initial expression for $\{f,\{g,h\}\}$ should be

$$\sum_j\sum_k \frac{\partial f}{\partial x_j}\frac{\partial}{\partial p_j}\left(\frac{\partial g}{\partial x_k}\frac{\partial h}{\partial p_k}-\frac{\partial g}{\partial p_k}\frac{\partial h}{\partial x_k}\right)$$
$$-\sum_j\sum_k \frac{\partial f}{\partial p_j}\frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial x_k}\frac{\partial h}{\partial p_k}-\frac{\partial g}{\partial p_k}\frac{\partial h}{\partial x_k}\right).$$

(That is, some of the j's should be k's and there should then be a sum over k.) Eq. (2.27) should then read

$$\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j\partial x_k}\frac{\partial h}{\partial p_k}+\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k}\frac{\partial^2 h}{\partial p_j\partial p_k}$$
$$-\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j\partial p_k}\frac{\partial h}{\partial x_k}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_k}\frac{\partial^2 h}{\partial p_j\partial x_k},$$

where all terms are summed over j and k. The representative term at the end of the proof should then be

$$\frac{\partial h}{\partial x_j}\frac{\partial f}{\partial x_k}\frac{\partial^2 g}{\partial p_j\partial p_k}.$$

p. 36 change $f(x,p)=p_j$ to $g(x,p)=p_j$.

p. 38, line 4, the notation X_H is not explained; it is the vector field on the right-hand side of (2.28).

p 40, first two lines, should read "which holds if and only if $\{H,f\}=0$" (i.e., the second $\{f,H\}$ should be $\{H,f\}$).

p. 49, Exercise 11, the displayed equation should read

$$H(\mathbf{x}^1+\mathbf{a},\ldots,\mathbf{x}^N+\mathbf{a},\mathbf{p}_1,\ldots,\mathbf{p}^N),\quad \mathbf{a}\in\mathbb{R}^N,$$

That is to say, the Hamiltonian is assumed to be invariant under translations in the *position* variables only.

p. 52, the expression for $\tilde{E}$ should read

$$\tilde{E}=\frac{1}{2}\left|\dot{\mathbf{x}}\right|^2-\frac{GM}{|\mathbf{x}|}.$$

(The square on the $|\dot{\mathbf{x}}|$ term is missing.)

Chapter 3

p. 57, first full paragraph, third-to-last line, change "may" to "many": "for many purposes."

p. 57, Proposition 3.4(2), change "eigenvector" to "eigenvalue": "Suppose λ is an eigenvalue for A ..."

p. 63, Prop 3.8, ends with a comma, which should be a period.

Chapter 4

p. 103, just after Eq. (4.26), there are four occurrences of the symbol ψ that should be ψ_0 (including cases where $\hat{\psi}$ should be $\hat{\psi}_0$).

p. 105, Eq. (4.29), there is a missing factor of $2m$ in the denominator in the second expression—because the Hamiltonian is $P^2/(2m)$ rather than P^2. The first equality should therefore read:

$$\frac{d}{dt}\left\langle X\right\rangle_{\psi(t)}=\left\langle\frac{i}{\hbar}\frac{1}{2m}(-2i\hbar P)\right\rangle_{\psi(t)}.$$

Chapter 8

p. 161, last line, should read "so that $\mathcal{C}(\sigma(A);\mathbb{R}) \subset \mathcal{F}$."

p. 167, Exercise 4, the hint should refer to Exercise 3, not to Exercise 17 (which does not exist).

Chapter 9

p. 174, second-to-last line of the first full paragraph, change $A^{cl}\psi = \phi$ to $(A^{cl})^*\psi = \phi$.

p. 175, last line of Proposition 9.14, change "for all A in $\mathrm{Dom}(A)$" to "for all ψ in $\mathrm{Dom}(A)$".

p. 182, second line of Prop. 9.27, change f to ψ: "the space of continuously differentiable functions ψ..."

p. 183, line 4, there is a missing $\hbar$ in the exponent: The exponential should be $e^{i\lambda x/\hbar}$.

p. 183, last line of the proof of Lemma 9.28, there is a missing $\hbar$; it should read "$A^*\phi = -i\hbar\ d\phi/dx$".

p. 185, after Eq. (9.14), the reference to Proposition A.5 should actually be to *Equation* (A.5).

p. 186, between (9.15) and (9.16), there is a repeated "thus."

p. 186, starting from the third line, in the proof of Prop. 9.30, there are several places where the roles of ϕ and ψ are reversed. It should read as follows.

Meanwhile, suppose $\phi \in \mathrm{Dom}(V(\mathbf{X})^*)$, meaning that

$$\psi \mapsto \int_{\mathbb{R}^n} \overline{\phi(\mathbf{x})} V(\mathbf{x})\psi(\mathbf{x})\ d\mathbf{x}, \quad \psi \in \mathrm{Dom}(V(\mathbf{X}))$$

is a bounded linear functional. This linear functional has a unique bounded extension to $L^2(\mathbb{R}^n)$ and, thus, there exists a unique $\chi \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \overline{\phi(\mathbf{x})} V(\mathbf{x})\psi(\mathbf{x})\ d\mathbf{x} = \int_{\mathbb{R}^n} \overline{\chi(\mathbf{x})}\psi(\mathbf{x})\ d\mathbf{x},$$

or

$$\int_{\mathbb{R}^n} [\overline{\phi(\mathbf{x})} V(\mathbf{x}) - \overline{\chi(\mathbf{x})}]\psi(\mathbf{x})\ d\mathbf{x} = 0$$

for all $\psi \in \mathrm{Dom}(V(\mathbf{X}))$.

Taking $\psi = (\phi V - \chi)\mathbf{1}_{E_m}$, we see that $\phi V - \chi$ is zero almost everywhere on E_m, for all m, hence zero almost everywhere on $\mathbb{R}^n$. Thus, ϕV is equal to the square-integrable function χ as an element of $L^2(\mathbb{R}^n)$. This shows that $\phi \in \mathrm{Dom}(V(\mathbf{X}))$.

p. 193, third line from the bottom, the reference to Sect. 12.2 should be to Sect. 9.6.

pp. 194-195, Lemma 9.42 and its proof. The factor of 5/4 in the first term in the expression for m_α should be 5, and similarly in the last displayed equation in the proof.

p. 198, Exercise 14 of Chapter 9, the suggested strategy of proof is unclear. Here is a revised version of the exercise.

14. Let $\mathrm{Dom}(\Delta) \subset L^2(\mathbb{R}^n)$ denote the domain of the Laplacian, as given in Proposition 9.34, and assume $n \leq 3$.

    (a) Show that each $\psi \in \mathrm{Dom}(\Delta)$ is continuous.

Hint: Express $\hat{\psi}$ as the product of two L^2 functions and use Proposition A.14.

(b) Show that for all $\varepsilon > 0$, there exists a constant c_ε such that

$$|\psi(\mathbf{x})| \leq c_\varepsilon \|\psi\| + \varepsilon \|\Delta\psi\| .$$

Chapter 11

p. 228, just before (11.5), should be: "the *raising operator*, given by..."

p. 232, Eq. (11.12), and the lines just before it, the normalization constant is wrong. The constant should be $\pi^{-1/4}D^{-1/2}$, which gives

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\frac{m\omega}{2\hbar}x^2\right\}.$$

p. 235, Theorem 11.4, same issue with the normalization constant.

Chapter 12

p. 245, third line from bottom, change $\psi(0)$ to $\psi(-1)$: "functions ψ such that $\psi(-1) = \psi(0) = 0$."

Chapter 13

p. 261, after Eq. (13.11), X should be x: "A is the Weyl quantization of $(a_1x + b_1p)$".

p. 262, Eq. (13.15), $\mathbf{a} \cdot \mathbf{p}$ should be $\mathbf{a} \cdot \mathbf{x}$. It should be

$$Q_{\mathrm{Weyl}}((\mathbf{a} \cdot \mathbf{x} + \mathbf{b} \cdot \mathbf{p})^j) = \cdots$$

p. 263, first line of Eq. (13.23), β should be $\mathbf{b}$. It should read

$$(2\pi\hbar)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\mathbf{a}\cdot\mathbf{b})/2} \cdots$$

Chapter 14

p. 285, first paragraph of the proof of Example 14.15, there is an m_x that should be $m_{x,a}$: "where $m_{x,a}$ is the unique integer..."

p. 285, just above Eq. (14.11), $L^2([0,1])$ should be $L^2([-1,1])$.

p. 291, first paragraph, change "A's and B's" to "A_j's and B_j's".

p. 292, sixth line of first full paragraph, change the second occurence of $e^{i(\mathbf{a}\cdot\mathbf{A}+\mathbf{b}\cdot\mathbf{B})}$ to $e^{i(\mathbf{a}\cdot\mathbf{X}+\mathbf{b}\cdot\mathbf{P})}$.

p. 296, displayed equation in the third line of the proof, change $P_{\mathbf{z}_0}$ to $P_{\mathbf{z}_0,R}$.

Chapter 16

p. 343, proof of Example 16.21, the relation $e^{t\mathrm{trace}(X)} = 0$ should be $e^{t\mathrm{trace}(X)} = 1$.

p. 344, first full paragraph, the reference to Proposition 16.22 should be to Example 16.22.

p. 366, Exercise 16(a), change $\mathsf{u}(n)/\{ia\mathbb{R}\}$ to $\mathsf{u}(n)/\{iaI\}$. (That is, we quotient out by the pure-imaginary multiples of the identity).

Chapter 17

p. 369, Eq. (17.2) is missing an equals sign.

p. 372, Eq. (17.12), the v_j on the right-hand of the equality should be v_{j-1}:

$$L^+ v_j = j(2\mu + 1 - j)v_{j-1}.$$

p. 372, fourth line, change $j = N$ to $j = N + 1$: "Applying (17.12) with $j = N + 1$".

p. 378, the expression for the Gaussian measure does not agree with the $\hbar = 1$ case of the definition (14.25) of $\mu_\hbar$. (There should not be a factor of 2 in the exponent and $\pi^{3/2}$ should be π^3.) The inner product should therefore be

$$\langle p, q\rangle_{\mathcal{P}} = \int_{\mathbb{C}^3} \overline{p_{\mathbb{C}}(\mathbf{z})}q_{\mathbb{C}}(\mathbf{z})\frac{e^{-|\mathbf{z}|^2}}{\pi^3}\ d^6 z.$$

p. 383, fourth paragraph, the assertion that $\rho_{l_0,k}(\rho_{l_0,k_0})^{-1}$ is a multiple of the identity should be replaced with "a multiple of the obvious identification of $V_{l_0} \otimes g_{k_0}$ with $V_{l_0} \otimes g_k$."

p. 386, just above Eq. (17.19), remove the spurious reference to "17.22". (The reference to Eq. (17.18) is correct.)

Chapter 18

p. 399, Eq. (18.15), the roles of k and $k+1$ in the denominator are reversed. It should read

$$a_{k+1} = a_k \frac{k + l + 1 - \lambda}{(k + 1)(k + 2(l + 1))}.$$

pp. 400-401, the discussion beginning at the bottom of p. 400 is incorrect: *One* of the solutions, say $g_1(\rho)$ is indeed given by a power series starting from ρ^0. The second solution has the form

$$g_2(\rho) = Cg_1(\rho)\log(\rho) + \sum_{k=0}^{\infty} b_k \rho^{r+k},$$

where $r = -(2l + 1)$ and $b_0 \neq 0$. (The log term is missing in the current statement.) This result is an application of the Frobenius method in the case that the roots of the indicial polynomial differ by an integer. See, for example, Section 5.3 of [E. Kreyszig, Advanced Engineering Mathematics, 10th edition, Wiley, 2011]. Since the dominant behavior of $g_2(\rho)$ near $\rho = 0$ is still of order $\rho^{-(2l+1)}$, the subsequent analysis remains unchanged.

p. 417, Exercise 3, the last matrix in the list should be labeled as $\mathbf{k}$ rather than $\mathbf{j}$. That is,

$$\mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

p. 416, Exercise 1, start of second paragraph, change $\hat{H}_2$ to $\hat{H}$: "Show that $\hat{H}$ can be expressed..."

p. 418, Exercise 9. The claim in the exercise is incorrect; see the correction for pp. 400-401.

Chapter 19

p. 425, Proposition 19.10, the reference should be to Notation 3.28 rather than Notation 3.29.

Chapter 20

P. 442, Eqns. (20.1) and (20.2), the limits should be set equal to 0.

p. 450, Definition 20.4, change $\mu(E)$ to $\mu^\sigma(E)$ in the displayed equation.

Chapter 21

p. 462, just after Proposition 21.10, change "function" to "functions": "space of smooth functions on N..."

Chapter 22

p. 472, second displayed equation, change X_f to X in the first two lines.

p. 479, fourth line from bottom, change $Q_{\text{pre}}(f)$ to $Q_{\text{pre}}(p_j^2)$.

Chapter 23

p. 485, last line, change $\pi(x,z) = z$ to $\pi(x,z) = x$.

p. 486, just after Definition 23.2, change "fs is a section of s" to "fs is a section of L".

p. 488, Eq. (23.5), on the right-hand side, change $\theta(\gamma(t))$ to $\theta(\dot{\gamma}(t))$.

p. 492, Definition 23.16, $T_z^{\mathbb{C}}(X)$ should be $T_z^{\mathbb{C}}(N)$.

p. 501, Example 23.30, there is a sign error in both expressions for ω, and a factor of 4 missing in the second term. It should read

$$\omega = 4(1 - |z|^2)^{-2}\ dy \wedge dx = 4(1 - r^2)^{-2}\ r\ d\phi \wedge dr.$$

The sign in the formula for θ then needs to be changed as well:

$$\theta = -2\frac{r^2}{1 - r^2}\ d\phi.$$

p. 503, second displayed equation, the complex conjugate is missing on both occurrences of the variable z_{k_q}.

p. 515, in the second line of the displayed equation above Eq. (23.36), the last term has the roles of f and g reversed. The last term should be

$$i\hbar[\gamma(X_f), Q_{\text{pre}}(g)].$$

p. 520, Definition 23.52, there is a sign error in the displayed equation. It should read

$$Q(f)s = (Q_{\text{pre}}(f)\mu) \otimes \nu + i\hbar\mu \otimes \mathcal{L}_{X_f}\nu,$$

as in Definition 23.44 in the real case.

p. 520, Example 23.53, there is a sign error in the computation of $\mathcal{L}_{X_H}\sqrt{dz}$. When this correction is combined with the previous one, the result of the example is unchanged.

Appendix A

p. 531, Definition A.15, in the first displayed equation, change $\lim_{x\to\pm\infty}$ to $\lim_{|\mathbf{x}|\to\infty}$.

p. 536, Definition A.33, in the displayed formula, change S_n to S_N.