1 Revision to Section 17.5: Spin

We classified irreducible finite-dimensional representations of the Lie algebra $\mathsf{so}(3)$ by their "spin" l, where l is the largest eigenvalue for the operator $L_3 = i\pi(F_3)$. The possible values for l are non-negative integers $(0, 1, 2, \ldots)$ and the positive half-integers $(1/2, 3/2, \ldots)$. Inside $L^2(S^2)$ and $L^2(\mathbb{R}^3)$, however, we found only irreducible representations of $\mathsf{so}(3)$ with *integer* spin. It is easy to understand why the half-integer spin representations do not occur: They do not correspond to any representation of the *group* $\mathsf{SO}(3)$ (Proposition 17.10). Since $L^2(S^2)$ and $L^2(\mathbb{R}^3)$ both carry a natural unitary action Π of the group $\mathsf{SO}(3)$, any finite-dimensional subspace that is invariant under the associated Lie algebra representation π will also be invariant under Π and thus constitute a representation of $\mathsf{SO}(3)$.

Although the half-integer representations π_l of the Lie algebra $\mathsf{so}(3)$ cannot be exponentiated to representations of $\mathsf{SO}(3)$, they can be exponentiated to representations of the universal cover $\mathsf{SU}(2)$ of $\mathsf{SO}(3)$, as in the proof of Proposition 17.10. For a half-integer l, the associated representation Π_l' of $\mathsf{SU}(2)$ satisfies $\Pi_l'(-I) = -I$, which means that Π_l' does not factor through $\mathsf{SO}(3) \cong \mathsf{SU}(2)/\{I, -I\}$. If, however, we think about *projective* representations, we see that $[-I]$ *is* the identity element in $\mathsf{PU}(V)$. Thus, even when l is a half-integer, we get a well-defined projective representation Π_l of $\mathsf{SO}(3)$ that satisfies

$$\Pi_l(e^{tX}) = [e^{t\pi_l(X)}]$$

for all $X \in \mathsf{so}(3)$, where $[U]$ denotes the image of $U \in \mathsf{U}(V)$ in $\mathsf{PU}(V)$. (Compare the general result in Theorem 16.47.)

In this section, we will argue that the half-integer projective representations of $\mathsf{SO}(3)$ do, in fact, occur in nature. Notably, we will explain the key role played by the two-dimensional projective representation $\Pi_{1/2}$ in the quantum theory of electrons.

Up to this point, we have taken the Hilbert space for an electron moving in $\mathbb{R}^3$ to be $L^2(\mathbb{R}^3)$. Various experiments in the 1920's, however, indicated that this proposal is not quite correct: The Hilbert space needs to be modified by the inclusion of "internal degrees of freedom" known as "spin." The experiment that most clearly demonstrates the concept of the spin of an electron is the Stern–Gerlach experiment. Otto Stern and Walther Gerlach performed the original experiment in 1922 using silver atoms. Our description will follow the 1927 version of the experiment performed with hydrogen atoms by T. E. Phipps and J. B. Taylor. We give the modern interpretation of the experiment, which is not the one given by Stern and Gerlach, since their work predates the introduction of modern quantum theory.

A hydrogen atom, with the electron in its ground state (the lowest energy state), is passed through a magnetic field. The field points along the z-axis (the middle portion of Figure 1) and varies in intensity along that axis. Now, since a hydrogen atom as a whole is electrically neutral, it does not experience the force exerted by a magnetic field on a charged particle. If, however, the

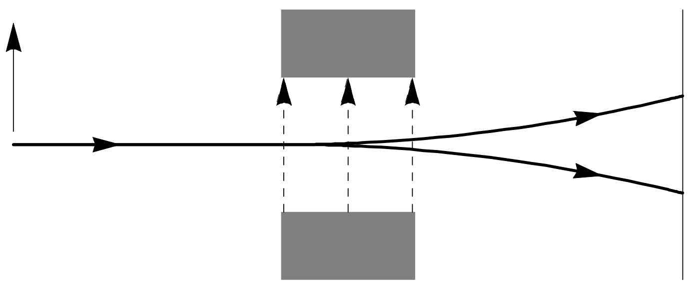

Figure 1: The set-up of the Stern–Gerlach experiment

electron is rotating around the nucleus of the hydrogen atom—that is, if it has angular momentum—it behaves like a tiny magnet, with the north and south poles oriented along the axis of rotation. Since the applied magnetic field is varying in intensity, it will exert a stronger force on one end of the magnet than the other, resulting in a net force along the z-axis. The magnitude of the force will be proportional to the z-component of the angular momentum of the electron.

The upshot of the preceding discussion is this: The Stern–Gerlach experiment effectively *measures the z-component of the angular momentum* of the electron. Now, the ground state of the electron in a hydrogen atom is rotationally invariant and therefore has zero angular momentum. This means not only that the *expected* angular momentum is zero, but that the ground state is an eigenvector for each of the angular momentum operators with eigenvalue zero. Thus, any measurement of the angular momentum should yield a value of zero, with no uncertainty. (See Proposition 3.11.) In the setting of the Stern–Gerlach experiment, then, we might expect there to be no deflection of the hydrogen atom by the magnetic field.

In the actual experiment, however, the atoms *are* deflected. Furthermore, only two different angles of deflection are observed, of equal magnitude but in opposite directions, as on the right side of Figure 1. If one works backward from the deflection to angular momentum, the conclusion is that the z-component of the angular momentum has the value $\hbar/2$ or $-\hbar/2$. This conclusion holds even though the ordinary angular momentum (in the sense of the angular momentum operators $\hat{J}_1$, $\hat{J}_2$, and $\hat{J}_3$) of an electron in the ground state of hydrogen is zero. Of course, one can orient the magnetic field in the Stern–Gerlach experiment along any axis, and the conclusion is the same: The measured values of any component of the angular momentum of an electron in the ground state of hydrogen are $\hbar/2$ and $-\hbar/2$.

We conclude, then, that there must be some other sort of angular momentum present in an electron, of a sort that takes only the values $\pm\hbar/2$. By the usual principles of quantum mechanics, this means that there should be some other angular momentum *operators* S_1, S_2, and S_3, in addition to the operators $\hat{J}_1$, $\hat{J}_2$, and $\hat{J}_3$, where the eigenvalues of each S_j are $\hbar/2$ and $-\hbar/2$. (By contrast, the analysis in Sections 17.3–17.6 indicates that the eigenvalues of the ordinary angular momentum operators $\hat{J}_j$ are of the form $j\hbar$, where j is an integer.) We expect that these operators would have properties similar to those of the $\hat{J}_j$'s. In particular, we may hope that the S_j's would have the same commutation relations as the $\hat{J}_j$'s, which are given in Exercise 10 of Chapter 3. These commutation relations are, up to a factor of $i\hbar$, the same as those of the Lie algebra $\mathsf{so}(3)$; compare Proposition 17.3.

Now, since the S_j's are distinct from the $\hat{J}_j$'s, they must act not on the ordinary complex-valued wave function of the electron, but on some other space. Since, also, we are looking for a space in which each S_j has two distinct eigenvalues, we may guess that this other space has dimension two. We can then find operators of the desired sort by taking the S_j's to be multiples of the generators of the $\mathsf{su}(2)$ Lie algebra:

$$S_j = i\hbar E_j,$$

where E_j is as in (16.4) and acts on $\mathbb{C}^2$. Since $\mathsf{su}(2)$ is isomorphic to $\mathsf{so}(3)$ (Example 16.32), the S_j's indeed have the same commutation relations as the ordinary angular momentum operators. Furthermore, each E_j has eigenvalues $i/2$ and $-i/2$, so that each S_j will have eigenvalues $\hbar/2$ and $-\hbar/2$.

Physicists describe the two-dimensional space on which the S_j's act as describing "internal degrees of freedom" of an electron. The new type of angular momentum that an electron possesses is then referred to as its "spin." The full Hilbert space for an electron, taking into account its spin, is then obtained by combining the space $\mathbb{C}^2$ on which the operators S_j act with the original Hilbert space $L^2(\mathbb{R}^3)$ as follows:

$$L^2(\mathbb{R}^3) \otimes \mathbb{C}^2.$$

(As discussed in Section 19.5, the Hilbert space for a composite system is generally taken to be the tensor product of the individual Hilbert spaces.)

We can think of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ more concretely as $L^2(\mathbb{R}^3; \mathbb{C}^2)$, the space of square-integrable function on $\mathbb{R}^3$ with values in $\mathbb{C}^2$, as follows. If $\{e_1, e_2\}$ is the standard basis for $\mathbb{C}^2$, then every element of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ has the form

$$\psi_1(\mathbf{x}) \otimes e_1 + \psi_2(\mathbf{x}) \otimes e_2.$$

We then identify the preceding element with the function $\psi : \mathbb{R}^3 \to \mathbb{C}^2$ given by

$$\psi(\mathbf{x}) = \psi_1(\mathbf{x})e_1 + \psi_2(\mathbf{x})e_2.$$

Now, since the Lie algebra $\mathsf{so}(3)$ is isomorphic to the Lie algebra $\mathsf{su}(2)$, we may define a representation π of $\mathsf{so}(3)$ on $\mathbb{C}^2$ by setting $\pi(F_j) = E_j$, and this representation is easily seen to be irreducible. Indeed, this representation is just the unique irreducible representation of $\mathsf{so}(3)$ of dimension 2, denoted as $\pi_{1/2}$. That is to say, π corresponds to $l = 1/2$ in the notation of Definition 17.5. Because of the occurrence of the "spin 1/2" representation here, we say that "the spin of the electron is 1/2" or "an electron is a spin 1/2 particle."

We may then define a representation of $\mathsf{so}(3)$ on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ by following Definition 16.48. After putting in a factor of $i\hbar$ (as in Proposition 17.3), we obtain still another set of "angular momentum operators" $\hat{L}_1$, $\hat{L}_2$, and $\hat{L}_3$, acting on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ and given by

$$\hat{L}_j = \hat{J}_j \otimes I + I \otimes S_j.$$

In the physics literature, the tensor product with the identity is usually omitted, and the formula is written as

$$\hat{L}_j = \hat{J}_j + S_j,$$

where it is understood that $\hat{J}_j$ acts only on the $L^2(\mathbb{R}^3)$ factor and S_j acts only on the $\mathbb{C}^2$ factor. These operators are often referred to as the **total angular momentum** operators, since they combine the ordinary angular momentum operators $\hat{J}_j$ and the spin angular momentum operators S_j. Note that since $(A\otimes B)(C\otimes D) = (AB)\otimes(CD)$, the operator $\hat{J}_j$ (which really means $\hat{J}_j\otimes I$) commutes with the operator S_k (which really means $I\otimes S_k$), for all j and k.

Let us now consider what the preceding proposal means for the concept of rotational symmetry. Following Proposition 17.3, we think of the operators $\hat{L}_j$ as describing (modulo a factor of $i\hbar$) the action of the Lie algebra $\mathsf{so}(3)$ on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$. It is then natural to ask whether there is an associated action of the *group* $\mathsf{SO}(3)$ on this space. Now, as noted at the beginning of this section, the S_j's—describing the "spin 1/2" representation of $\mathsf{so}(3)$—do not correspond to an ordinary representation of $\mathsf{SO}(3)$. Nevertheless, we noted there that there is an associated *projective* representation of $\mathsf{SO}(3)$ acting on $\mathbb{C}^2$. The full Hilbert space $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ also carries a projective unitary action of $\mathsf{SO}(3)$, in which $\mathsf{SO}(3)$ acts on $L^2(\mathbb{R}^3)$ in the usual way and where $\mathsf{SO}(3)$ acts projectively on $\mathbb{C}^2$. The fact that the natural action of $\mathsf{SO}(3)$ on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ is only a projective representation and not an ordinary one is closely related to the statements one will find in the physics literature to this effect: "If you rotate the wave function of an electron by 360 degrees, you obtain the negative of the original wave function."

The just-described projective action of $\mathsf{SO}(3)$ on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ is the natural manifestation of rotational symmetry for an electron. Thus, given an operator A on this Hilbert space, we say that A "commutes with rotations" if it commutes with this projective action. This condition is equivalent (modulo technicalities about domains of unbounded operators) to the condition that A commutes with each of the total angular momentum operators $\hat{L}_j$.

We may consider not only electrons, but any type of particle. Various experimental and theoretical arguments indicate that each type of particle has a spin l, where the possible values of l are $0, 1/2, 1, 3/2, \ldots$. A particle with spin l has a space V_l of "internal degrees of freedom" corresponding to the projective representation Π_l of $\mathsf{SO}(3)$. We may summarize this idea as follows.

Summary 1 (Spin) *Each type of particle has a "spin" l, which is a non-negative integer or half integer. The Hilbert space for a single particle of this type is $L^2(\mathbb{R}^3)\otimes V_l$, where V_l is an irreducible* projective *representation of* $\mathsf{SO}(3)$ *of dimension* $2l+1$.

As it turns out, there is a fundamental difference between particles whose spin is an integer and those whose spin is a half integer.

Definition 2 *A particle whose spin is an integer is called a* ***boson*** *and a particle whose spin is a half-integer is called a* ***fermion****.*

To see the significance of the distinction between integer and half-integer spin, one needs to look at the structure of the Hilbert space describing *multiple* particles of a given type, such as the Hilbert space for five electrons. This topic is discussed in Chapter 19.