# 1 Revision to Section 9.2, Adjoint and Closure of an Unbounded Operator

Add the following example just before Definition 9.2 on p. 171.

**Example 1** *Take* $\mathbf{H} = L^2([0,1])$*, take* $\mathrm{Dom}(A)$ *to be the space of continuously differentiable functions* $\psi \in L^2([0,1])$ *such that*

$$\psi(0) = \psi(1),$$

*and define* $A$ *on this domain by*

$$A\psi = i\frac{d\psi}{dx}.$$

*Now suppose* $\phi \in L^2([0,1])$ *is a continuously differentiable. Then* $\phi$ *belongs to* $\mathrm{Dom}(A^*)$ *if and only if* $\phi(0) = \phi(1)$*, and if this condition holds,* $A^*\phi = i\ d\phi/dx$*.*

We are *not* claiming that every function in the domain of $A^*$ is continuously differentiable; we are merely determining which continuously differentiable functions belong to $\mathrm{Dom}(A^*)$. Thus, the example does not completely determine the domain of $A^*$.

Note that $\mathrm{Dom}(A^*)$ contains $\mathrm{Dom}(A)$ and that $A^*$ agrees with $A$ on $\mathrm{Dom}(A)$ inside $\mathrm{Dom}(A^*)$. It then follows directly from the definition of the adjoint that $A$ is a **symmetric operator**, meaning that

$$\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$$

for all $\phi$ and $\psi$ in $\mathrm{Dom}(A)$. Indeed, the motivation for the condition $\psi(0) = \psi(1)$ in $\mathrm{Dom}(A)$ is to ensure that $A$ is symmetric.

**Proof.** Fix a continuously differentiable function $\phi$. Then for all $\psi$ in $\mathrm{Dom}(A)$, we may apply integration by parts to obtain

$$\begin{aligned} \langle \phi, A\psi \rangle &= \int_0^1 \overline{\phi(x)} i \frac{d\psi}{dx}\ dx \\ &= i\overline{\phi(x)}\psi(x)\Big|_0^1 - i\int_0^1 \overline{\frac{d\phi}{dx}}\psi(x)\ dx \\ &= i[\overline{\phi(1)} - \overline{\phi(0)}]\psi(0) + \langle i\ d\phi/dx, \psi \rangle, \end{aligned} \tag{1}$$

where we have used that $\psi(0) = \psi(1)$ in the second equality. Now, since $\phi$ is continuously differentiable, $d\phi/dx$ is in $L^2([0,1])$, from which it follows that the linear functional

$$\psi \mapsto \langle i\ d\phi/dx, \psi \rangle$$

is bounded.

On the other hand, the linear functional $\psi \mapsto \psi(0)$ is easily seen to be *unbounded*: One can produce $\psi$'s in $\mathrm{Dom}(A)$ for which $\psi(0)$ is large but $\|\psi\|$ is small. Thus, the only way the linear functional $\psi \mapsto \langle \phi, A\psi \rangle$ can be bounded is if the coefficient $\psi(0)$ in the last line of (1) is zero, that is, if $\phi(0) = \phi(1)$. Conversely, if $\phi(0)$ is equal to $\phi(1)$, then we are left with

$$\langle \phi, A\psi \rangle = \langle i\ d\phi/dx, \psi \rangle \,,$$

the right-hand side of which is a continuous linear functional in $\psi$. In that case, the unique vector $A^*\phi$ such that $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ for all $\psi \in \mathrm{Dom}(A)$ is, evidently, $A^*\phi = i\ d\phi/dx$, as claimed. ∎