Joint work with **Bruce Driver** and **Todd Kemp** of UCSD and **Franck Gabriel** of Warwick:


The electromagnetic field on $\mathbb{R}^4$ can be thought of as a 2-form $F$:

$$F = B_1 \ dy \wedge dz + B_2 \ dz \wedge dx + B_3 \ dx \wedge dy$$

$$+ \ E_1 \ dx \wedge dt + E_2 \ dy \wedge dt + E_3 \ dz \wedge dt,$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{E}$ is the magnetic field.

Maxwell’s equations imply that $F$ is closed, hence

$$F = dA,$$

where $A$ is the vector potential.
The $A$ is non-unique up to gauge transformation

$$A \mapsto A + df$$

Geometrically, interpret $F$ as the curvature of a connection for a principal $U(1)$ bundle over $\mathbb{R}^4$

Interpret $A$ as the connection 1-form
Physical significance of $A$?

- **Classical particle** moving in an E-M field only “sees” $F$
- Schrödinger equation for **quantum particle** involves $A$:
  \[
  \frac{1}{i\hbar} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} (\nabla - A)^2 \psi + V\psi
  \]
  where $\nabla - A$ is the **covariant derivative**.
- **Aharonov–Bohm effect** says that in a nonsimply connected domain, two different $A$’s with same $F$ have different results
Aharonov–Bohm effect
Quantum interference effects from paths on either side of tube
- Particle moves only outside the tube, where $F = 0$ but $A \neq 0$
- Connection $A$ (not just $F$!) affects the phase of the wavefunction
- Only the holonomy of $A$ around the tube matters:

$$\text{hol}(L) = e^{i \int_L A}$$

- Idea: in quantum world, $A$ matters, but only gauge-invariant functions of $A$ arise
Let $K$ be a connected compact Lie group, e.g. $U(N)$
Let $P$ be a principal $K$-bundle over a manifold $M$
Let $A$ be a connection on $P$, let $F$ be the curvature of $A$
If $P = M \times K$, then $A$ is Lie-algebra valued 1-form on $M$ and

$$F = dA + A \wedge A$$

(quadratic function of $A$)
Holonomies and gauge transformations

- $A$ is a Lie-algebra valued 1-form
- In **commutative case**, 
  \[ \text{hol}_L(A) = e^{\int_L A} \]
  (as in Arahanov–Bohm!)
- In **noncommutative case**
  \[ \text{hol}_L(A) = \lim_{n \to \infty} e^{\int_{L(t_0)} L(t_1)} A \ e^{\int_{L(t_1)} L(t_2)} A \ldots e^{\int_{L(t_{n-1})} L(t_n)} A \]
- Holonomy is (almost) invariant under **gauge transformations** (like $A \mapsto A + df$)
Quantum Yang–Mills theory

- If $K = U(1)$, have E-M force: quantum theory involves **photons**
- If $K = SU(3)$, then $A$ fields represent the **strong nuclear force**
- Associated particles called **gluons** bind quarks together

![Proton and Neutron Diagram]

Matt Strassler 2013
Let $\mathcal{A}$ be the **space of all connections**

In (Euclidean) quantum YM, we consider a heuristic **path-integral**

$$C \int_{\mathcal{A}} f(A) e^{-\|F_A\|^2} DA,$$

**describing a random connection**

- $f$ is gauge-invariant function of connection (e.g. trace of holonomy)
- $\|F_A\|^2$ is square of $L^2$ norm—**quartic function** of $A$
- $DA$ is the (formal) Lebesgue measure on the space of connections
- **Clay Millenium Prize** for rigorous interpretation in 4 dimensions!
• Make a **discrete approximation** to space-time
• Discretized path integral becomes **classical statistical mechanics** model
• Attempt to take the continuum limit with **renormalization**
• Very complicated and difficult!
Rigorous version on plane (Gross–King–Sengupta [1989] and Driver [1989])

- Take $M = \mathbb{R}^2$ (interpreted as space-time)
- So
  \[ A = \{ A_1(x, y) \ dx + A_2(x, y) \ dy \} \]
- Use a **gauge fixing** to subspace $A_0$ of $A$:
  \[ A_0 = \{ A_1(x, y) \ dx \mid A_1(x, 0) = 0 \} \]
- Every connection can be **gauge transformed** into $A_0$
- For $A \in A_0$, the $A \wedge A$ term vanishes, so
  \[ F_A = dA = -\frac{\partial A_1}{\partial y} dx \wedge dy \]
  is a **linear** function of $A$.  

Brian C. Hall
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For gauge-invariant functions $f$, integral reduces to

$$C \int_{A_0} f(A) e^{-\|F_A\|^2} DA$$

Since $F_A$ is linear in $A$, this is a **Gaussian measure**, which has mathematically precise interpretation.

The curvature $F$ can be thought of as a Lie-algebra-valued **white noise**.

Equation of parallel transport is **stochastic differential equation** ($A$ is not smooth).

Theory is invariant under **area-preserving diffeomorphisms**.
PART 3: MAKEENKO–MIGDAL EQUATION AND LARGE-\( N \) LIMIT ON PLANE
Take $K = U(N)$ (unitary group)
Take loop $L$ with simple crossings

Use normalized trace $\text{tr}(X) := \frac{1}{N} \text{trace}(X)$ and define
$$f(A) = \text{tr}(\text{hol}_A(L)),$$

Compute **Wilson loop functional**
$$\mathbb{E}\{\text{tr}(\text{hol}(L))\} := C \int_{A_0} \text{tr}(\text{hol}_A(L)) e^{-\|F_A\|^2} DA$$
Answer depends only on **areas of faces of graph**

Fix one crossing and let \( t_1, t_2, t_3, \) and \( t_4 \) be the areas of the adjacent faces

MM equation reads

\[
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \mathbb{E}\{\text{tr}(\text{hol}(L))\} = \mathbb{E}\{\text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2))\}
\]

where \( L_1 \) and \( L_2 \) are:
- Four faces don’t have to be distinct
- If $F_i$ is the unbounded face, omit the $t_i$-derivative
- **Good news**: the loops $L_1$ and $L_2$ are simpler than $L$
- **Bad news**: RHS is the *expectation of the product* of traces not product of expectations
For Yang–Mills for $U(N)$ in any dimension, expect theory to simplify in large-$N$ limit.

Path integral gives *random connection*; physicists predict that connection becomes *nonrandom* in large-$N$ limit.

In limit, there is only one connection (modulo gauge transformations), called the *master field*.

Rigorous results by M. Anshelevic–A. Sengupta and Th. Lévy in the plane case.
Since connection becomes nonrandom in the limit, all covariances vanish.

No distinction between expectation of a product and product of the expectations.

Large-$N$ version of MME reads:

\[
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \tau(\text{hol}(L)) = \tau(\text{hol}(L_1)) \tau(\text{hol}(L_2))
\]

where $\tau(\cdot)$ is the limiting value of $\mathbb{E}\{\text{tr}(\cdot)\}$. 
“Master field”: Characterizing the large-$N$ limit on plane

- Lévy shows that large-$N$ limit of YM on plane is characterized by:
  1) The large-$N$ MM equation
  2) The unbounded face condition:

$$\frac{d}{d|F|}\tau(\text{hol}(L)) = -\frac{1}{2}\tau(\text{hol}(L)) \quad (F \text{ adjoins unbounded face})$$
Above loop satisfies

\[ \tau(\text{hol}(L)) = e^{-t_3/2} e^{-t_1} e^{-t_2} (1 - t_1)(1 - t_2) \]
Arguments of Makeenko–Migdal and Kazakov–Kostov use formal manipulations with path integral

**Goal:** Prove MME for $U(N)$ rigorously (then take a limit)

Use **Driver’s formula** [1989]

One variable in $U(N)$ for each edge

Put heat kernel (distribution of Brownian motion) for each bounded component

Evaluate with **time = area** and **space variable = holonomy** around component

Integrate over all edge variables
Driver’s formula: example

- Need $\rho_{t_1}(a)$ for inner region, $\rho_{t_2}(ba^{-1})$ for crescent region
- $t_1, t_2 =$ areas
- Holonomy around whole loop is $ab$
- Integrate:

$$\mathbb{E}\{\text{tr}(\text{hol}(L))\} = \int_{U(N)} \int_{U(N)} \text{tr}(ab)\rho_{t_1}(a)\rho_{t_2}(ba^{-1}) \ da \ db$$
Lévy’s proof of planar MME

- **General strategy:** Differentiate under the integral, use heat equation, integrate by parts
- Integrate by parts on a **sequence** of faces, ending at **unbounded face**
- **Cancellation** after alternating sum
- Later proof by **Dahlqvist** using loop variables; broadly similar strategy
New proof on plane (with Driver, Kemp)

- **Local:** Only use **four faces** surrounding the crossing
- Main part of proof is **less than two pages**
- Local nature of proof allows extension to compact surfaces! (Unbounded face is not needed.)
New result: MME on compact surfaces (with Driver, Gabriel, Kemp)

- **Sengupta's formula**: Products of heat kernels on holonomies, with normalization factor
- Example on $S^2$:

$$E\{\text{tr(}\text{hol}(L)\text{)}\} = \frac{1}{Z} \int_{U(N)} \int_{U(N)} \text{tr}(ab) \rho_{t_1}(a) \rho_{t_2}(b^{-1}) \rho_{t_3}(b) \ da \ db$$

where $t_3$ is area of “outside” and $Z$ is a normalizing constant.
• Statement is exactly same as on plane:

\[
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \mathbb{E}\{\text{tr}(\text{hol}(L))\} \\
= \mathbb{E}\{\text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2))\}
\]

• Our proofs for \( \mathbb{R}^2 \) go through almost without change, because proofs are local

• One extra heat kernel does not affect the argument

• Proofs on \( \mathbb{R}^2 \) by Lévy and Dahlqvist both required unbounded face
PART 5: LARGE-N LIMIT ON A SPHERE
Existence of large-$N$ limit on arbitrary surface is **unknown**

If large-$N$ limit exists, **no unbounded face condition**, so MME won’t completely characterize theory

Best we can hope for: Use MME to **reduce arbitrary curve to simple closed curves**
Large-$N$ limit on 2-sphere

- Two phases to analysis
- **Phase 1**: Try to use MME to reduce arbitrary loop to simple closed curve
- **Phase 2**: Compute Wilson loop for simple closed curve (in $N \to \infty$ limit)
Daul and Kazakov claim reduction is always possible.

They only analyze two examples—need a systematic procedure!
Not so easy to simplify more complicated loops!
Systematic procedure: Shrink all but two of faces to zero!
Phase 1 on sphere

- Reduce arbitrary loop to one that winds $n$ times around a simple closed curve
- Not hard to reduce *that* to simple closed curve
- Similar method developed independently by Dahlqvist and Norris
Phase 2 on sphere: Direct method

- Wilson loop for simple closed curve described by Brownian bridge in $U(N)$
- Lifetime of bridge = area of sphere; time-parameter = area enclosed
- Eigenvalue process described by nonintersecting Brownian bridges on circle
- Liechty and Wang have studied large-$N$ limit
Phase 2 on sphere: Method of “most probable representation”

- Write Wilson loop in terms of **heat kernels**, expand in terms of **characters** of representations
- See which representations contribute in limit—optimization problem
- Rigorous treatment using “β-ensembles” by Dahlqvist and Norris
- **Phase transition** when \( \text{area}(S^2) = \pi^2 \)
- Result: **Rigorous description** of large-\(N\) limit on \(S^2\)!
Thank you for your attention!