Eigenvalues of random matrices in the general linear group in the large-$N$ limit

Brian C. Hall
University of Notre Dame

April 13, 2019
Joint work with Bruce Driver and Todd Kemp
Web page: www.nd.edu/~bhall/
Results in physics literature:
Outlines:

- Part 1: Random matrix theory
- Part 2: Brown measure and the large-$N$ limit
- Part 3: Random matrices in $GL(N; \mathbb{C})$
- Part 4: The PDE and its solution
- Part 5: Letting $\epsilon$ tend to zero
Random matrix theory: History

- Introduced by Wigner in 1955 to model energy levels of large nuclei
- Interested in large-$N$ behavior
- “Quantum chaos”: quantum mechanics of systems that are classically “chaotic”
- Spacing between energy levels conjectured to behave like spacing between consecutive eigenvalues of certain Hermitian random matrices
A basic example: Ginibre ensemble

- Simple way to choose a random matrix (not necessarily Hermitian)
- Choose $N \times N$ matrix randomly with independent, identically distributed entries
- **Ginibre ensemble**: entries are complex Gaussian with variance $1/N$
Circular law

- **Circular law**: For large $N$, eigenvalues will (with high probability) be mostly in unit disk and almost uniformly distributed.
- Bulk distribution of eigenvalues becomes *nonrandom* in limit.
PART 2: BROWN MEASURE AND THE LARGE-$N$ LIMIT
Notion of a large-$N$ limit

- "*-moments" of matrix-valued random variable $X$: e.g.

\[ \mathbb{E}\{\text{tr}[X^2(X^*)^3X(X^*)^7X^5]\} \]

where $\text{tr}$ is the normalized trace

- If $X$ depends on $N$, consider large-$N$ limits

- Seek operator algebra $\mathcal{A}$ with "trace" $\tau : \mathcal{A} \to \mathbb{C}$ and $x \in \mathcal{A}$ with

\[ \tau[\text{word in } x \text{ and } x^*] = \lim_{N \to \infty} \mathbb{E}\{\text{tr}[\text{word in } X^N \text{ and } (X^N)^*]\} \]
• **Note:** $x$ is a *single* operator in $\mathcal{A}$, not a random variable
• Want to define something like “eigenvalue distribution” of $x$
• If $x$ is normal ($x^*x = xx^*$) can use spectral theorem
• If $x$ is non-normal, use “Brown measure” (L. Brown, 1986)
Motivating computation

- For an $N \times N$ matrix $A$, set
  \[ s(\lambda) = \log(|\det(A - \lambda)|^{2/N}), \quad \lambda \in \mathbb{C} \]
- Or
  \[ s(\lambda) = \frac{2}{N} \sum_{j=1}^{N} \log|\lambda - \lambda_j| \]
- Then Laplacian $\Delta$ of $s$ is zero except at the eigenvalues
Motivating computation

Indeed

\[
\frac{1}{4\pi} \Delta s(\lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}(\lambda)
\]

(probability measure with mass $1/N$ at each eigenvalue)

Can also write $s$ as

\[
s(\lambda) = \frac{1}{N} \text{trace}\{\log[(A - \lambda)^*(A - \lambda)]\}
\]
Motivating example

- Plot of $-s(\lambda)$
Given $x$ in $(\mathcal{A}, \tau)$, define

$$S(\lambda, \varepsilon) := \tau\{\log[(x - \lambda)^*(x - \lambda) + \varepsilon]\}, \quad \lambda \in \mathbb{C}$$

Limit $\varepsilon \to 0$ exists as subharmonic function

$$s(\lambda) := \lim_{\varepsilon \to 0^+} S(\lambda, \varepsilon)$$

If $x$ is $N \times N$ matrix and $\tau = \frac{1}{N}\text{trace}$, then $s$ is as in previous slide

Define Brown measure of $x$:

$$\mu^x = \frac{1}{4\pi} \Delta s(\lambda)$$
Properties of Brown measure

- $\mu^x$ is a probability measure
- Supported on spectrum $\sigma(x)$ of $x$
- Satisfies
  \[ \int_{\sigma(x)} \lambda^n \, d\mu^x(\lambda) = \tau(x^n) \]
  but this \textit{does not uniquely determine} $\mu^x$. 
Let $X^N$ be distributed as $N \times N$ Ginibre ensemble.

Then $X^N$ converges to a “circular element” $\times$.

Brown measure of $\times$ is uniform prob. measure on unit disk.
General strategy

- **Step 1**: Identify large-$N$ limit and find its Brown measure
- **Step 2**: Prove that eigenvalue distribution converges to Brown measure
- Strategy pioneered by Girko for general circular law (entries independent and identically distributed but not normal)
- Strategy used by Tao and Vu in strongest version of circular law
Brownian motion in $\text{GL}(N;\mathbb{C})$

- General linear group $= \text{group of all } N \times N \text{ invertible matrices}$
- Consider Brownian motion $b^N_t$ in $\text{GL}(N;\mathbb{C})$, starting at $I$
- Compute as
  \[ b^N_t \approx \prod_{j=1}^k \left( I + \frac{\sqrt{t}}{\sqrt{k}} \text{Gin}_j \right), \quad k \text{ large} \]
  where $\{\text{Gin}_j\}$ are independent $N \times N$ Ginibre-distributed matrices
- Distribution of $b^N_t$ is heat kernel measure
Comparison to Brownian motion in $U(N)$

- Brownian motion $u^N_t$ in the unitary group $U(N)$
- Arises in 2D Yang–Mills theory on plane
- Brown measure $\nu_t$ of large-$N$ limit found by Philippe Biane
- $\nu_t$ supported on proper subset of $S^1$ until $t = 4$
In \( \text{GL}(N; \mathbb{C}) \), problem has been mostly open until now.

For small \( t \), have \( b_t^N \approx I + \sqrt{t} \text{ Gin} \).

For small \( t \) and large \( N \), eigenvalues approximately uniform on disk of radius \( \sqrt{t} \) centered at 1 (\( t = 0.12 \) and \( N = 2,000 \)).
Brownian motion in GL(N;C)

- For larger $t$, eigenvalues of $b_t^N$ cluster as $N \to \infty$ into domain $\Sigma_t$
- Nonrigorous argument for this in the cited papers in physics literature
- $N = 2,000$ with $t = 2$ and $t = 3.9$
Brownian motion in $GL(N;\mathbb{C})$

- For larger $t$, eigenvalues of $b_t^N$ cluster as $N \to \infty$ into domain $\Sigma_t$
- $N = 2,000$ with $t = 4$ and $t = 4.1$
Large-\(N\) limit

- \(b_t^N\) converges as \(N \to \infty\) to Biane’s free multiplicative Brownian motion \(b_t\)
- Full convergence result proved by Kemp
- **Goal**: Show that Brown measure of \(b_t\) is supported on \(\Sigma_t\) and compute the Brown measure
Domains introduced by Philippe Biane in 1997
Also appear in cited papers in physics literature
Set
\[ T(\lambda) = |\lambda - 1|^2 \frac{\log(|\lambda|^2)}{|\lambda|^2 - 1} \]
Define
\[ \Sigma_t = \{ \lambda \in \mathbb{C} | T(\lambda) < t \} \]
Properties of $\Sigma_t$

- Simply connected for $t \leq 4$, doubly connected for $t > 4$
- Shown for $t = 4$: 

![Graph showing the properties of $\Sigma_t$ for $t = 4$.]
Properties of $\Sigma_t$

- Plot of $T(\lambda)$ for values from 0 to 5
How does the Brown measure of $b_t$ know about $\Sigma_t$?

Or: how does $T(\lambda)$ arise?
One answer

- Use conformal map $f_t$ of $\Sigma^c_t$ to $(\text{support}(\nu_t))^c$:
  \[ f_t(\lambda) = \lambda e^{\frac{t}{2} \frac{1+\lambda}{1-\lambda}} \]

- Use large-$N$ “Segal–Bargmann transform” to show Brown measure is zero outside $\Sigma_t$ [H–Kemp, 2018]
Following def. of Brown measure, set

\[ S(t, \lambda, \epsilon) := \tau \left[ \log((b_t - \lambda)^*(b_t - \lambda) + \epsilon) \right] \]

and

\[ s_t(\lambda) := \lim_{\epsilon \to 0^+} S(t, \lambda, \epsilon). \]

Brown measure is then

\[ \frac{1}{4\pi} \Delta s_t(\lambda). \]
The function $S(t, \lambda, \varepsilon)$ satisfies the following PDE:

$$
\frac{\partial S}{\partial t} = \varepsilon \frac{\partial S}{\partial \varepsilon} \left( 1 + (|\lambda|^2 - \varepsilon) \frac{\partial S}{\partial \varepsilon} - a \frac{\partial S}{\partial a} - b \frac{\partial S}{\partial b} \right), \quad \lambda = a + ib,
$$

subject to the initial condition

$$
S(0, \lambda, \varepsilon) = \log(|\lambda - 1|^2 + \varepsilon).
$$

- Want $S(t, \lambda, 0)$ but PDE involves derivatives with respect to $\varepsilon$
- *Not* correct (in general) to say $\partial S / \partial t = 0$ when $\varepsilon = 0$
“Hamiltonian” $H$ read off from PDE:

$$H(a, b, \varepsilon, p_a, p_b, p_\varepsilon) = -\varepsilon p_\varepsilon \left(1 + (|\lambda|^2 - \varepsilon)p_\varepsilon - ap_a - bp_b\right)$$

Consider Hamilton’s equations

$$\frac{da}{dt} = \frac{\partial H}{\partial p_a}, \quad \frac{dp_a}{dt} = -\frac{\partial H}{\partial a}$$

and similarly for $b, p_b, \varepsilon, p_\varepsilon$.

**Miracle**: In our case, these ODE’s can be solved explicitly!
Then

\[ S(t, \lambda(t), \varepsilon(t)) = \log(|\lambda_0 - 1|^2 + \varepsilon_0) \]

\[ + \varepsilon_0 t \left( \frac{1}{|\lambda_0 - 1|^2 + \varepsilon_0} \right)^2 + \log|\lambda(t)| - \log|\lambda_0| \]

RHS mostly in terms of initial conditions \( \lambda_0 \) and \( \varepsilon_0 \).

Variant of “method of characteristics”: find special curves along which the solution can be computed.
Given an arbitrary $t$ and $\lambda$, try to find $\varepsilon_0$ and $\lambda_0$ so that

$$\varepsilon(t) = 0; \quad \lambda(t) = \lambda.$$ 

Then get $s_t(\lambda) := S(t, \lambda, 0) = \text{function of } \varepsilon_0 \text{ and } \lambda_0$

Then take Laplacian in $\lambda$
**Ground assault:** To get $\varepsilon(t) \approx 0$, try $\varepsilon_0 \approx 0$

Then $\lambda(t) \approx \lambda_0$

$s_t(\lambda) = \log |\lambda - 1|^2$

$s_t(\lambda)$ is independent of $t$

**Result:** If this works, Brown measure is zero!
**The catch:** Only works if \( \lambda \) is outside \( \Sigma_t \)!

If \( \varepsilon_0 \approx 0 \) and \( \lambda_0 = \lambda \) then \( \varepsilon(t) \) is small and positive until time \( T(\lambda) \), when \( \varepsilon(t) \) ceases to exist.
Same $T(\lambda)$ as in definition of $\Sigma_t$!

Answer to our “mystery”:

$T(\lambda)$ is the **lifetime of a path** with $\varepsilon_0 \approx 0$ and $\lambda_0 \approx \lambda$. 
Ground assault

- Region $\{\lambda | t > T(\lambda)\} = \Sigma_t$ is inaccessible to this simple approach.
- Reproduces result that Brown measure is zero outside $\Sigma_t$
- Ground assault cannot get into $\Sigma_t$!
• **Aerial assult:** Must really embrace $\varepsilon > 0$!
• Choose $\varepsilon_0 > 0$ and try to “land” at $\varepsilon = 0$ at time $t$ with $\lambda(t) = \lambda$
For \((t, \lambda)\) with \(\lambda \in \Sigma_t\), exist (in principle) \(\epsilon_0\) and \(\lambda_0\) with \(\epsilon(t) = 0\) and \(\lambda(t) = \lambda\).
Aerial assault

- Formula is for \((t, \lambda)\) in terms of \((\varepsilon_0, \lambda_0)\)
- Have to invert this relationship to get \(S(t, \lambda, 0)\)
- Still have to take Laplacian w.r.t. \(\lambda\)
Theorem (Driver–H–Kemp)

For points in $\Sigma_t$, the density $W_t$ of the Brown measure has the following form in polar coordinates:

$$W_t(r, \theta) = \frac{1}{r^2} w_t(\theta)$$

for a certain function $w_t$.

- Amazingly simple dependence on $r$!
The formula for $w_t$

- $w_t$ is determined by the geometry of $\Sigma_t$ as follows:

$$w_t(\theta) = \frac{1}{4\pi} \left( \frac{2}{t} + \frac{\partial}{\partial \theta} \frac{2r_t(\theta) \sin \theta}{r_t(\theta)^2 + 1 - 2r_t(\theta) \cos \theta} \right)$$

where $r_t(\theta)$ is “outer radius” of $\Sigma_t$: 

![Diagram showing $r_t(\theta)$ and $\theta$]
The formula for $w_t$

- Use implicit differentiation on $T(r, \theta) = t$ to get:

$$ w_t(\theta) = \frac{1}{2\pi t} \omega(r_t(\theta), \theta) $$

for some explicit function $\omega(r, \theta)$
The formula for $w_t$

The formula for $\omega$:

$$\omega(r, \theta) = 1 + h(r) \frac{\alpha(r) + \beta(r) \cos \theta}{\beta(r) + \alpha(r) \cos \theta},$$

where

$$h(r) = r \frac{\log(r^2)}{r^2 - 1};$$

$$\alpha(r) = (r^2 + 1) h(r) - 2r;$$

$$\beta(r) = r^2 + 1 - 2rh(r).$$
Plots of $W_t$

- $t = 1$
Plots of $W_t$

- $t = 4$
Plots of $W_t$

- $t = 4$ detail
Simulation of eigenvalues

- $t = 2$, Brown measure and eigenvalues
**Corollary**

The push-forward of the Brown measure under the complex log map

\[ \lambda \mapsto \log |\lambda| + i\theta \]

has density in \( \log(\Sigma_t) \) given by

\[ \omega_t(\theta) \]

independent of \( \log |\lambda| \).
More simulations of eigenvalues

- Eigenvalues for $N = 2,000$ and their logarithms for $t = 4.1$
Recall $\nu_t$—limiting eigenvalue distribution for Br. motion in $U(N)$

Recall map $f_t(\lambda) := \lambda e^{t \frac{1+\lambda}{1-\lambda}}$—takes boundary of $\Sigma_t$ to support of $\nu_t$

Define $F_t : \overline{\Sigma_t} \to S^1$ as $f_t$ on boundary, constant along radial segments
Connection to Biane’s measure

Theorem

The push-forward of the Brown measure of $b_t$ under $F_t$ is $\nu_t$!

- Plot of $\nu_t$ versus histogram of $\{F_t(\lambda_j)\}_{j=1}^N$ for $t = 2$
Brown measure of $b_t$ is actually the unique measure on $\Sigma_t$ with:
(1) this push-forward property,
(2) a density of the form $\frac{1}{r^2} g(\theta)$. 
THANK YOU FOR YOUR ATTENTION!